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LIE-TYPE HIGHER DERIVATIONS ON OPERATOR ALGEBRAS

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ABSTRACT. Motivated by the intensive and powerful works concerning additive mappings of operator algebras, we mainly study Lie-type higher derivations on operator algebras in the current work. It is shown that every Lie (triple-)higher derivation on some classical operator algebras is of standard form. The definition of Lie n-higher derivations on operator algebras and related potential research topics are properly-posed at the end of this article.
Keywords: Lie higher derivation, Lie triple higher derivation, operator algebra.
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1. Introduction

For many years, there has been increasing interest on the study of Lie-type mappings of associative rings and operator algebras, such as Lie isomorphisms, Lie derivations and Lie triple derivations. Many works are contributed to describe the structures of the aforementioned mappings and tremendous progress has been made over the past few years. In particular, Lie derivations and Lie triple derivations on operator algebras draw related researchers' attention and they can be regarded to some extent as one class of more entangled problems. Let $\mathcal{A}$ be a unital algebra over a commutative ring $\mathcal{R}$. An $\mathcal{R}$-linear mapping $D : \mathcal{A} \longrightarrow \mathcal{A}$ is called a *derivation* if $D(theconvexy) = D(x)y+xD(y)$ for all $x, y \in \mathcal{A}$. An $\mathcal{R}$-linear mapping $D : \mathcal{A} \longrightarrow \mathcal{A}$ is called a *Lie derivation* if $D([x, y]) = [D(x), y] + [x, D(y)]$ for all $x, y \in \mathcal{A}$. An $\mathcal{R}$-linear

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mapping $D : \mathcal{A} \longrightarrow \mathcal{A}$ is called a *Lie triple derivation* if $D([[x, y], z]) = [[D(x), y], z] + [[x, D(y)], z] + [[x, y], D(z)]$ for all $x, y, z \in \mathcal{A}$. Obviously, every derivation on $\mathcal{A}$ is a Lie derivation and every Lie derivation is a Lie triple derivation. But the converse statements are usually not true (see the counterexamples in [40]).

Miers initially studied Lie-type mappings of operator algebras in his seminal works [28–30]. He proved that every Lie derivation D on a von Neumann algebra $\mathcal{A}$ can be uniquely written as the sum $D = d + \tau$, where d is an inner derivation of $\mathcal{A}$ and τ is a linear mapping from $\mathcal{A}$ into its center $\mathcal{Z}(\mathcal{A})$ vanishing on each commutator [29]. Furthermore, Miers obtained an analogous decomposition for Lie triple derivations of von Neumann algebras with no abelian summands [30]. Johnson showed in [15] that every continuous Lie derivation from a symmetrically amenable Banach algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{X}$ decomposes into a sum of an ordinary derivation from $\mathcal{A}$ into $\mathcal{X}$ and a linear mapping from $\mathcal{A}$ into the center of $\mathcal{X}$. Alaminos et al [2] jointly proved that every Lie derivation on a symmetrically amenable semisimple Banach algebra also has the same decomposition. Mathieu and Villena obtained that every (not necessarily bounded) Lie derivation D on a C^*-algebra $\mathcal{A}$ can be uniquely decomposed into the sum of a derivation d of $\mathcal{A}$ and a linear mapping τ from $\mathcal{A}$ into its center $\mathcal{Z}(\mathcal{A})$ vanishing on all commutators [27]. In addition, the decomposition problem of Lie triple derivations on various triangular operator algebras are considered in [13, 22, 36, 40, 41]. The aforesaid Miers' result about Lie triple derivations was extended to different contexts, such as triangular uniformly hyperfinite algebras, nest algebras, triangular matrix algebras. However, people pay much less attention to the structure of Lie-type higher derivations on operator algebras. To the best of our knowledge, there are no other articles dealing with Lie-type higher derivations of operator algebras except for [35]. The objective of this article is to describe the structure of Lie-type higher derivations on some classical operator algebras.

Let us first recall some basic facts related to Lie higher derivations of associative algebras. Various higher derivations, which consist of a family of some additive mappings, have been widely studied in commutative and noncommutative contexts (see [8, 10–12, 16, 20, 31–33, 35, 37, 38] and the references therein). Let $\mathcal{A}$ be a uintal associative algebra over a commutative ring $\mathcal{R}$. Let $\mathbb{N}$ be the set of all non-negative integers and $D = \{d_n\}_{n\in\mathbb{N}}$ be a family of $\mathcal{R}$-linear mappings of $\mathcal{A}$ such that $d_0 = id_{\mathcal{A}}$. D is called:

(i) a *higher derivation* if

$$d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)$$

for all $x, y \in \mathcal{A}$ and for each $n \in \mathbb{N}$;

(ii) a *Lie higher derivation* if

$$d_n([x,y]) = \sum_{i+j=n} [d_i(x), d_j(y)]$$

for all $x, y \in \mathcal{A}$ and for each $n \in \mathbb{N}$;

(iii) a *Lie triple higher derivation* if

$$d_n([[x,y],z]) = \sum_{i+j+k=n} [[d_i(x), d_j(y)], d_k(z)]$$

for all $x, y, z \in \mathcal{A}$ and for each $n \in \mathbb{N}$.

Note that d_1 is always a Lie derivation (respectively, Lie triple derivation) if $D = \{d_n\}_{n\in\mathbb{N}}$ is a Lie higher derivation (respectively, Lie triple higher derivation). Obviously, every higher derivation is a Lie higher derivation and every Lie higher derivation is a Lie triple higher derivation. But the converse statements are generaly not true. Assume that $G = \{g_n\}_{n\in\mathbb{N}}$ is a higher derivation of $\mathcal{A}$. We can construct a sequence of $\mathcal{R}$-linear mappings

$$d_n = g_n + f_n, \ \forall n \in \mathbb{N}, \tag{1.1}$$

where $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of $\mathcal{R}$-linear mappings from $\mathcal{A}$ into its center $\mathcal{Z}(\mathcal{A})$ and each f_n vanishes on all commutators of $\mathcal{A}$. It is not difficult to see that $\{d_n\}_{n\in\mathbb{N}}$ is a Lie higher derivation of $\mathcal{A}$, but not a higher derivation of $\mathcal{A}$ if $f_n \neq 0$ for some $n \in \mathbb{N}$. A Lie higher derivation $D = \{d_n\}_{n\in\mathbb{N}}$ is said to be *standard* if it has the property 1.1. Likewise, if we have a sequence $\{f'_n\}_{n\in\mathbb{N}}$ of $\mathcal{R}$-linear mappings from $\mathcal{A}$ into its center $\mathcal{Z}(\mathcal{A})$ and each f'_n vanishes on all second commutator of $\mathcal{A}$, then we can establish a sequence of $\mathcal{R}$-linear mappings

$$d'_n = g_n + f'_n, \ \forall n \in \mathbb{N}. \tag{1.2}$$

It is easy to check $\{d'_n\}_{n\in\mathbb{N}}$ is a Lie triple higher derivation of $\mathcal{A}$, but not a higher derivation of $\mathcal{A}$ if $f'_n \neq 0$ for some $n \in \mathbb{N}$. A Lie triple higher derivation $D' = \{d'_n\}_{n\in\mathbb{N}}$ is said to be *standard* if it has the decomposition 1.2.

This article is devoted to the study of Lie-type higher derivations on some classical operator algebras and its framework is as follows. In the second section we give a new characterization of Lie (triple-)higher derivation on associative algebras, which admit us to transfer the problems of Lie (triple-)higher derivations into the same problems concerning Lie (triple-)derivations. We establish a one-to-one correspondence between the set of all Lie (triple-)higher derivations and the set of all sequences of Lie (triple-)derivations. Then we apply the corresponding relation to study Lie (triple-)higher derivations on some classical operator algebras in the third section. The involved operator algebras include the algebras of bounded linear operators, amenable semisimple Banach algebras, C^*-algebras, von Neumann algebras, reflexive algebras, $\mathcal{J}$-subspace lattice algebras, CSL algebras, triangular operator algebras.

2. Lie (triple-)higher derivations on associative algebras

In this section we will give a new characterization concerning Lie (triple-)higher derivations of associative algebras. These new properties easily enable us to transfer the problems of Lie higher derivations (respectively, Lie triple higher derivations) into the same problems related to Lie derivations (respectively, Lie triple derivations). We establish a one to one correspondence between the set of all Lie higher derivations (respectively, Lie triple higher derivations) and the set of all sequences of Lie derivations (respectively, Lie triple derivations).

A Lie higher derivation $D = \{d_i\}_{i=0}^m$ (respectively, $D = \{d_i\}_{i=0}^\infty$) on $\mathcal{A}$ is said to be *order* m (respectively, *infinite order*) if

$$d_n([x,y]) = \sum_{i+j=n} [d_i(x), d_j(y)]$$

for all $x, y \in \mathcal{A}$ and for each $n = 0, 1, 2, \cdots, m$ (respectively, $n = 0, 1, 2, \cdots$). Similarly, a Lie triple higher derivation $D = \{d_i\}_{i=0}^m$ (respectively, $D = \{d_i\}_{i=0}^\infty$) on $\mathcal{A}$ is said to be *order* m (respectively, *infinite order*) if

$$d_n([[x,y],z]) = \sum_{i+j+k=n} [[d_i(x), d_j(x)], d_k(z)]$$

for all $x \in \mathcal{A}$ and for each $n = 0, 1, 2, \cdots, m$ (respectively, $n = 0, 1, 2, \cdots$). Throughout this paper, all involved Lie higher derivations and Lie triple higher derivations are of infinite order and all obtained results still hold for the case of finite order.

Proposition 2.1. *Let $\mathcal{A}$ be an associative algebra over a field of characteristic zero and $D = \{d_n\}_{n=0}^\infty$ be a Lie higher derivation of $\mathcal{A}$. Then* *there is a sequence* $\Delta = \{\delta_n\}_{n=0}^{\infty}$ *of Lie derivations of* $\mathcal{A}$ *such that*

$$(n+1)d_{n+1} = \sum_{k=0}^{n} \delta_{k+1} d_{n-k} \tag{2.1}$$

for each non-negative integer n.

Proof. Let us take an inductive approach for the index n. If $n = 0$, then

$$d_1([x, y]) = [d_1(x), d_0(y)] + [d_0(x), d_1(y)] = [d_1(x), y] + [x, d_1(y)]$$

for all $x, y \in \mathcal{A}$. If we put $\delta_1 = d_1$, then δ_1 is a Lie derivation of $\mathcal{A}$.

We now suppose that δ_k is a well-established linear mapping of $\mathcal{A}$ and a Lie derivation of $\mathcal{A}$ for each $k \leq n$. Let us define

$$\delta_{n+1} = (n+1)d_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1} d_{n-k}.$$

It is sufficient for us to show that δ_{n+1} is a Lie derivation of $\mathcal{A}$.

For arbitrary elements $x, y \in \mathcal{A}$, we have

$$\begin{aligned}\delta_{n+1}([x, y]) &= (n+1)d_{n+1}([x, y]) - \sum_{k=0}^{n-1} \delta_{k+1} d_{n-k}([x, y]) \\ &= (n+1) \sum_{k=0}^{n+1} [d_k(x), d_{n+1-k}(y)] - \sum_{k=0}^{n-1} \delta_{k+1} \left(\sum_{i=0}^{n-k} [d_i(x), d_{n-k-i}(y)] \right).\end{aligned}$$

Therefore

$$\begin{aligned}\delta_{n+1}([x, y]) = &\sum_{k=0}^{n+1} (k + n + 1 - k)[d_k(x), d_{n+1-k}(y)] - \\ &\sum_{k=0}^{n-1} \delta_{k+1} \left(\sum_{i=0}^{n-k} [d_i(x), d_{n-k-i}(y)] \right) \\ = &\sum_{k=0}^{n+1} k[d_k(x), d_{n+1-k}(y)] - \sum_{k=0}^{n-1} \sum_{i=0}^{n-k} [\delta_{k+1}(d_i(x)), d_{n-k-i}(y)] \\ &+ \sum_{k=0}^{n+1} (n+1-k)[d_k(x), d_{n+1-k}(y)] - \sum_{k=0}^{n-1} \sum_{i=0}^{n-k} [d_i(x), \delta_{k+1}(d_{n-k-i}(y))]\end{aligned}$$

for all $x, y \in \mathcal{A}$. For convenience, let us write

$$P = \sum_{k=0}^{n+1} k[d_k(x), d_{n+1-k}(y)] - \sum_{k=0}^{n-1}\sum_{i=0}^{n-k}[\delta_{k+1}(d_i(x)), d_{n-k-i}(y)],$$

$$Q = \sum_{k=0}^{n+1}(n+1-k)[d_k(x), d_{n+1-k}(y)] - \sum_{k=0}^{n-1}\sum_{i=0}^{n-k}[d_i(x), \delta_{k+1}(d_{n-k-i}(y))].$$

Thus $\delta_{n+1}([x,y]) = P+Q$. In the expression of summation $\sum_{k=0}^{n-1}\sum_{i=0}^{n-k}$, we know that $k \neq n$ and $0 \leq k+i \leq n$. If we set $r = k+i$, then

$$\begin{aligned}
P &= \sum_{k=0}^{n+1} k[d_k(x), d_{n+1-k}(y)] - \sum_{r=0}^{n}\sum_{0\leq k\leq r, k\neq n}[\delta_{k+1}(d_{r-k}(x)), d_{n-r}(y)] \\
&= \sum_{k=0}^{n+1} k[d_k(x), d_{n+1-k}(y)] - \sum_{r=0}^{n-1}\sum_{k=0}^{r}[\delta_{k+1}(d_{r-k}(x)), d_{n-r}(y)] \\
&\quad - \sum_{k=0}^{n-1}[\delta_{k+1}(d_{n-k}(x)), y] \\
&= \sum_{r=0}^{n}(r+1)[d_{r+1}(x), d_{n-r}(y)] - \sum_{r=0}^{n-1}\sum_{k=0}^{r}[\delta_{k+1}(d_{r-k}(x)), d_{n-r}(y)] \\
&\quad - \sum_{k=0}^{n-1}[\delta_{k+1}(d_{n-k}(x)), y] \\
&= \sum_{r=0}^{n-1}[(r+1)d_{r+1}(x) - \sum_{k=0}^{r}\delta_{k+1}(d_{r-k}(x)), d_{n-r}(y)] \\
&\quad + (n+1)[d_{n+1}(x), y] - \sum_{k=0}^{n-1}[\delta_{k+1}(d_{n-k}(x)), y].
\end{aligned}$$

By the induction hypothesis we obtain

$$(r+1)d_{r+1}(x) = \sum_{k=0}^{r}\delta_{k+1}(d_{r-k}(x))$$

for each $r = 0, \cdots, n-1$. This shows that

$$P = [(n+1)d_{n+1}(x) - \sum_{k=0}^{n-1}\delta_{k+1}(d_{n-k}(x)), y] = [\delta_{n+1}(x), y].$$

On the other hand, we can compute that

$$\begin{aligned}
Q &= \sum_{k=0}^{n+1}[d_k(x), (n+1-k)d_{n+1-k}(y)] \\
&\quad - \sum_{k=0}^{n-1}\sum_{i=0}^{n-k}[d_i(x), \delta_{k+1}(d_{n-k-i}(y))] \\
&= \sum_{k=0}^{n+1}[d_k(x), (n+1-k)d_{n+1-k}(y)] \\
&\quad - \sum_{i=0}^{n}\sum_{k=0}^{n-i}[d_i(x), \delta_{k+1}(d_{n-k-i}(y))] + [x, \delta_{n+1}(y)] \\
&= \sum_{i=0}^{n}[d_i(x), (n+1-i)d_{n+1-i}(y)] \\
&\quad - \sum_{i=0}^{n}\sum_{k=0}^{n-i}[d_i(x), \delta_{k+1}(d_{n-k-i}(y))] + [x, \delta_{n+1}(y)] \\
&= \sum_{i=0}^{n}[d_i(x), (n+1-i)d_{n+1-i}(y) - \sum_{k=0}^{n-i}\delta_{k+1}(d_{n-k-i}(y))] + [x, \delta_{n+1}(y)].
\end{aligned}$$

By the induction hypothesis we get $(n+1-i)d_{n+1-i}(y) = \sum_{k=0}^{n-i}\delta_{k+1}(d_{n-k-i}(y))$ for $i = 1, \cdots, n$. Thus we immediately arrive at

$$\begin{aligned}
Q &= \sum_{i=0}^{n}[d_i(x), (n+1-i)d_{n+1-i}(y) - \sum_{k=0}^{n-i}\delta_{k+1}(d_{n-k-i}(y))] + [x, \delta_{n+1}(y)] \\
&= [x, (n+1)d_{n+1}(y) - \sum_{k=0}^{n}\delta_{k+1}(d_{n-k}(y))] + [x, \delta_{n+1}(y)] \\
&= [x, (n+1)d_{n+1}(y) - \sum_{k=0}^{n-1}\delta_{k+1}(d_{n-k}(y)) - \delta_{n+1}(d_0(y))] + [x, \delta_{n+1}(y)] \\
&= [x, \delta_{n+1}(y) - \delta_{n+1}(y)] + [x, \delta_{n+1}(y)] \\
&= [x, \delta_{n+1}(y)].
\end{aligned}$$

Finally, we conclude

$$\delta_{n+1}([x, y]) = P + Q = [\delta_{n+1}(x), y] + [x, \delta_{n+1}(y)]$$

for all $x, y \in \mathcal{A}$. This implies that δ_{n+1} is a Lie derivation of $\mathcal{A}$, which is the admired result. $\square$

We can extract a tedious but intuitive algorithm from the above proposition, which will be used in the sequel.

Algorithm 1. *By the formula 2.1 we can compute each component of* $\{d_n\}$.

$d_0 = I,$

$d_1 = \delta_1,$

$d_2 = \frac{1}{2}\delta_1^2 + \frac{1}{2}\delta_2,$

$d_3 = \frac{1}{6}\delta_1^3 + \frac{1}{6}\delta_1\delta_2 + \frac{1}{3}\delta_2\delta_1 + \frac{1}{3}\delta_3,$

$d_4 = \frac{1}{24}\delta_1^4 + \frac{1}{24}\delta_1^2\delta_2 + \frac{1}{12}\delta_1\delta_2\delta_1 + \frac{1}{12}\delta_1\delta_3 + \frac{1}{8}\delta_2\delta_1^2 + \frac{1}{8}\delta_2^2 + \frac{1}{4}\delta_3\delta_1 + \frac{1}{4}\delta_4,$

$$\begin{aligned} d_5 = {} & \frac{1}{120}\delta_1^5 + \frac{1}{120}\delta_1^3\delta_2 + \frac{1}{60}\delta_1^2\delta_2\delta_1 + \frac{1}{60}\delta_1^2\delta_3 + \frac{1}{40}\delta_1\delta_2\delta_1^2 + \frac{1}{40}\delta_1\delta_2^2 + \frac{1}{20}\delta_1\delta_3\delta_1 \\ & + \frac{1}{20}\delta_1\delta_4 + \frac{1}{30}\delta_2\delta_1^3 + \frac{1}{30}\delta_2\delta_1\delta_2 + \frac{1}{15}\delta_2^2\delta_1 + \frac{1}{15}\delta_2\delta_3 + \frac{1}{10}\delta_3\delta_1^2 + \frac{1}{10}\delta_3\delta_2 \\ & + \frac{1}{5}\delta_4\delta_1 + \frac{1}{5}\delta_5. \end{aligned}$$

By an inductive approach we obtain(see [33])

$$d_n = \sum_{r_1+r_2+\cdots+r_m=n(r_j\in\mathbb{N}, r_j\neq 0)} c_{r_1,r_2,\cdots,r_m}\delta_{r_1}\delta_{r_2}\cdots\delta_{r_m},$$

where

$$c_{r_1,r_2,\cdots,r_m} = \frac{1}{r_1+r_2+\cdots+r_m}\cdot\frac{1}{r_2+\cdots+r_m}\cdot\cdots\cdot\frac{1}{r_{m-1}+r_m}\cdot\frac{1}{r_m}.$$

Recall that an algebra $\mathcal{A}$ is said to be *torsion free* if $nx = 0$ implies that $x = 0$ for all $x \in \mathcal{A}$ and for each positive integer n. Clearly, an arbitrary associative algebra over a field of characteristic 0 is always torsion free. Now we are in a position to state the main theorem of this section.

Theorem 2.2. *Let $\mathcal{A}$ be a torsion free algebra, $\mathcal{D}$ be the set of all Lie higher derivations on $\mathcal{A}$ and $\mathcal{H}$ be the set of all consequences of Lie derivations on $\mathcal{A}$ with first component zero. Then there is a one-to-one correspondence between $\mathcal{D}$ and $\mathcal{H}$.*

Proof. It follows from Proposition 2.1 that for arbitrary Lie higher derivation $D = \{d_n\}_{n=0}^{\infty} \in \mathcal{D}$, there is a sequence $\Delta = \{\delta_n\}_{n=0}^{\infty} \in \mathcal{H}$ of Lie derivations with $\delta_0 = 0$ such that

$$(n+1)d_{n+1} = \sum_{k=0}^{n} \delta_{k+1} d_{n-k}$$

for each non-negative integer n. Hence the following mapping

$$\begin{gathered} \varphi : \mathcal{D} \longrightarrow \mathcal{H} \\ \{d_n\}_{n=0}^{\infty} = D \longmapsto \Delta = \{\delta_n\}_{n=0}^{\infty} \end{gathered}$$

is well-defined, where $(n+1)d_{n+1} = \sum_{k=0}^{n} \delta_{k+1} d_{n-k}$. Note that the solution of the recursive relation of Proposition 2.1 is unique. Therefore φ is injective.

We next prove that φ is also surjective. For a given sequence $\Delta = \{\delta_n\}_{n=0}^{\infty}$ of Lie derivations with $\delta_0 = 0$, we define $d_0 = I$ and

$$(n+1)d_{n+1} = \sum_{k=0}^{n} \delta_{k+1} d_{n-k}$$

for each no-negative integer n. It is sufficient for us to prove that $D = \{d_n\}_{n=0}^{\infty}$ is a Lie higher derivation of $\mathcal{A}$. Obviously, $d_1 = \delta_1$ is a Lie derivation of $\mathcal{A}$. Assume that $d_k([x,y]) = \sum_{i=0}^{k} [d_i(x), d_{k-i}(y)]$ for all $x, y \in \mathcal{A}$ and for each $k \leq n$. Note that

$$\begin{aligned} (n+1)d_{n+1}([x,y]) &= \sum_{k=0}^{n} \delta_{k+1} d_{n-k}([x,y]) \\ &= \sum_{k=0}^{n} \delta_{k+1} \left(\sum_{i=0}^{n-k} [d_i(x), d_{n-k-i}(y)] \right). \end{aligned}$$

Applying the induction hypothesis we get

$$
\begin{aligned}
(n+1)d_{n+1}([x,y]) &= \sum_{k=0}^{n}\sum_{i=0}^{n-k}\{[\delta_{k+1}(d_i(x)), d_{n-k-i}(y)] \\
&\quad +[d_i(x), \delta_{k+1}(d_{n-k-i}(y))]\} \\
&= \sum_{i=0}^{n}[\sum_{k=0}^{n-i}\delta_{k+1}d_{n-i-k}(x), d_i(y)] \\
&\quad + \sum_{i=0}^{n}[d_i(x), \sum_{k=0}^{n-i}\delta_{k+1}d_{n-i-k}(y)] \\
&= \sum_{i=0}^{n}[(n-i+1)d_{n-i+1}(x), d_i(y)] \\
&\quad + \sum_{i=0}^{n}[d_i(x), (n-i+1)d_{n-i+1}(y)] \\
&= \sum_{i=1}^{n+1} i[d_i(x), d_{n+1-i}(y)] \\
&\quad + \sum_{i=0}^{n}(n+1-i)[d_i(x), d_{n-i+1}(y)] \\
&= (n+1)\sum_{k=0}^{n+1}[d_k(x), d_{n+1-k}(y)]
\end{aligned}
$$

for all $x, y \in \mathcal{A}$. Since $\mathcal{A}$ is torsion free, we get

$$
d_{n+1}([x,y]) = \sum_{k=0}^{n+1}[d_k(x), d_{n+1-k}(y)]
$$

for all $x, y \in \mathcal{A}$. This shows that $D = \{d_n\}_{n=0}^{\infty}$ is a Lie higher derivation of $\mathcal{A}$. □

With the same proving techniques in Proposition 2.1 we have

Proposition 2.3. *Let $\mathcal{A}$ be an associative algebra over a field of characteristic zero and $G = \{g_n\}_{n=0}^{\infty}$ be a higher derivation (respectively, Lie triple higher derivation) of $\mathcal{A}$. Then there is a sequence $\Gamma = \{\gamma_n\}_{n=0}^{\infty}$ of* *derivations* (*respectively, Lie triple derivations*) *on* $\mathcal{A}$ *such that*

$$(n+1)g_{n+1} = \sum_{k=0}^{n} \gamma_{k+1} g_{n-k}$$

for each non-negative integer n*. Moreover, the Algorithm 1 still holds for the higher derivation* (*respectively, Lie triple higher derivation*) $G = \{g_n\}_{n=0}^{\infty}$ *and the sequence* $\Gamma = \{\gamma_n\}_{n=0}^{\infty}$ *of derivations* (*respectively, Lie triple derivations*).

In view of Proposition 2.3, we can get another one-to-one correspondence, which is completely parallel to Theorem 2.2.

Theorem 2.4. *Let* $\mathcal{A}$ *be a torsion free algebra,* $\mathcal{G}$ *be the set of all higher derivations* (*respectively, Lie triple higher derivations*) *on* $\mathcal{A}$*, and* $\mathcal{E}$ *be the set of all sequences of derivations* (*respectively, Lie triple derivations*) *on* $\mathcal{A}$ *with first component zero. Then there is a one-to-one correspondence between* $\mathcal{G}$ *and* $\mathcal{E}$*.*

Remark 2.5. *We must point out that the above one-to-one correspondence between higher derivations and derivations on other algebras was also obtained in [11, 33].*

3. Lie-type higher derivations on operator algebras

Let $\mathcal{A}$ be a unital associative algebra over a commutative ring $\mathcal{R}$ and δ be a Lie derivation of $\mathcal{A}$. We shall say that δ is of *standard form* if it can be expressed as the sum

$$\delta = h + \tau, \tag{3.1}$$

where h is a derivation of $\mathcal{A}$ and τ is a linear mapping from $\mathcal{A}$ into its center $\mathcal{Z}(\mathcal{A})$ vanishing on all commutators. Similarly, a Lie triple derivation δ' of $\mathcal{A}$ is called *standard* if it has the decomposition

$$\delta' = h + \tau', \tag{3.2}$$

where h is a derivation of $\mathcal{A}$ and τ' is a linear mapping from $\mathcal{A}$ into its center $\mathcal{Z}(\mathcal{A})$ vanishing on all second commutators. It has been proved that every Lie derivation (respectively, Lie triple derivation) on certain operator algebras has the standard form 3.1 (respectively, 3.2). It is natural to ask whether Lie (triple-)higher derivations on these operator algebras are of the standard form 1.1 or 1.2. We will study Lie-type higher derivations on some classical operator algebras in this section. The involved operator algebras include the algebras of bounded linear operators, C^*-algebras, von Neumann algebras, amenable semisimple Banach algebras, reflexive algebras, $\mathcal{J}$-subspace lattice algebras, CSL algebras and triangular operator algebras.

Let us first give a general result which admits us to transfer the standard form problem of Lie higher derivations to that of Lie derivations.

Proposition 3.1. *Let $\mathcal{A}$ be an associative algebra over a field of characteristic zero. If every Lie derivation of $\mathcal{A}$ has the standard form 3.1, then every Lie higher derivation of $\mathcal{A}$ has the standard form* 1.1.

Proof. We will give the proof of this proposition via the Algorithm 1. Let $D = \{d_n\}_{n=0}^{\infty}$ be a Lie higher derivation of $\mathcal{A}$. By the formula 2.1 there exists a sequence $\{\delta_n\}_{n=0}^{\infty}$ of Lie derivations such that

$$d_n = \sum_{r_1+r_2+\cdots+r_m=n(r_j\in\mathbb{N},r_j\neq 0)} c_{r_1,r_2,\cdots,r_m}\delta_{r_1}\delta_{r_2}\cdots\delta_{r_m},$$

$$c_{r_1,r_2,\cdots,r_m} = \frac{1}{r_1+r_2+\cdots+r_m}\cdot\frac{1}{r_2+\cdots+r_m}\cdot\cdots\cdot\frac{1}{r_{m-1}+r_m}\cdot\frac{1}{r_m}.$$

According to the assumption we know that each Lie derivation $\delta_i(i\in\mathbb{N})$ of the sequence $\{\delta_n\}_{n=0}^{\infty}$ is of standard form 3.1. This implies that each $\delta_i(i\in\mathbb{N})$ can be written as $\delta_i = h_i+\tau_i(i\in\mathbb{N})$, where $h_i(i\in\mathbb{N})$ is a derivation of $\mathcal{A}$ and $\tau_i(i\in\mathbb{N})$ is a linear mapping from $\mathcal{A}$ into its center $\mathcal{Z}(\mathcal{A})$ vanishing on all commutators. Therefore

$$\begin{aligned}
d_1 &= h_1+\tau_1\\
&\triangleq H_1+S_1,\\
d_2 &= \frac{1}{2}\delta_1^2+\frac{1}{2}\delta_2\\
&= \frac{1}{2}(h_1+\tau_1)^2+\frac{1}{2}(h_2+\tau_2)\\
&= (\frac{1}{2}h_1^2+\frac{1}{2}h_2)+(\frac{1}{2}h_1\tau_1+\frac{1}{2}\tau_1h_1+\frac{1}{2}\tau_1^2+\frac{1}{2}\tau_2)\\
&\triangleq H_2+S_2,\\
d_3 &= \frac{1}{6}\delta_1^3+\frac{1}{6}\delta_1\delta_2+\frac{1}{3}\delta_2\delta_1+\frac{1}{3}\delta_3\\
&= \frac{1}{6}(h_1+\tau_1)^3+\frac{1}{6}(h_1+\tau_1)(h_2+\tau_2)+\frac{1}{3}(h_2+\tau_2)(h_1+\tau_1)+\frac{1}{3}(h_3+\tau_3)\\
&= (\frac{1}{6}h_1^3+\frac{1}{6}h_1h_2+\frac{1}{3}h_2h_1+\frac{1}{3}h_3)+\frac{1}{6}(h_1^2\tau_1+h_1\tau_1h_1+h_1\tau_1^2+\tau_1h_1^2+\tau_1h_1\tau_1\\
&\quad +\tau_1^2h_1+\tau_1^3+h_1\tau_2+\tau_1h_2+\tau_1\tau_2)+\frac{1}{3}(h_2\tau_1+\tau_2h_1+\tau_2\tau_1)+\frac{1}{3}\tau_3\\
&\triangleq H_3+S_3.
\end{aligned}$$

Now suppose that H_k and S_k are well-established. Then

$$\begin{aligned} d_{k+1} &= \frac{1}{k+1}(\delta_1 d_k + \delta_2 d_{k-1} + \cdots + \delta_k d_1 + \delta_{k+1} d_0) \\ &= \frac{1}{k+1}[(h_1+\tau_1)(H_k+S_k) + (h_2+\tau_2)(H_{k-1}+S_{k-1}) + \cdots \\ &\quad + (h_k+\tau_k)(H_1+S_1) + (h_{k+1}+\tau_{k+1})] \\ &= \frac{1}{k+1}[h_1 H_k + h_2 H_{k-1} + \cdots + h_k H_1 + h_{k+1} + S'_{k+1}] \\ &\triangleq H_{k+1} + S_{k+1}. \end{aligned}$$

By Proposition 2.3 it is easy to verify that each $H_i (i = 1, 2, \cdots, k+1)$ in the above collections is a higher derivation of $\mathcal{A}$ and that each $S_i (i = 1, 2, \cdots, k+1)$ in the above collections is a linear mapping from $\mathcal{A}$ into its center $\mathcal{Z}(\mathcal{A})$ vanishing on all commutators. This shows that the Lie higher derivation $D = \{d_n\}_{n=0}^{\infty}$ has the standard form 1.1. $\square$

Similarly, we obtain

Proposition 3.2. *Let $\mathcal{A}$ be an associative algebra over a field of characteristic zero. If every Lie triple derivation of $\mathcal{A}$ is of standard form 3.2, then every Lie triple higher derivation of $\mathcal{A}$ is of standard form 1.2.*

We next apply Proposition 3.1 and Proposition 3.2 to some classical operator algebras. It turns out that every Lie higher derivation on these operator algebras is of standard form 1.1 and that every Lie triple higher derivation on these operator algebras is of standard form 1.2. Let us first see the best common operator algebras.

3.1. **Algebras of bounded linear operators.** Let X be a Banach space over the real or complex field $\mathbb{F}$ with $\dim X > 1$. We denote the algebra of all bounded linear operators on X by $\mathcal{B}(X)$. Lu and Liu in [25] proved that every Lie derivation on $\mathcal{B}(X)$ is of standard form 3.1. By Proposition 3.1 we have

Corollary 3.3. *Let X be a Banach space over the real or complex field $\mathbb{F}$ with $\dim X > 1$ and $\mathcal{B}(X)$ be the algebra of all bounded linear operators on X. Then every Lie higher derivation on $\mathcal{B}(X)$ has the standard form* 1.1.

In view of the structure of algebras of bounded linear operators, the following conjecture is at our hand.

Conjecture 3.4. *Let X be a Banach space over the real or complex field $\mathbb{F}$ with $\dim X > 1$ and $\mathcal{B}(X)$ be the algebra of all bounded linear operators on X. Then every Lie triple derivation on $\mathcal{B}(X)$ has the standard 3.2 and every Lie triple higher derivation on $\mathcal{B}(X)$ has the standard form 1.2.*

3.2. **Symmetrically amenable Banach algebras.** Let $\mathcal{A}$ be a Banach algebra, X be a Banach $\mathcal{A}$-bimodule and X^* be the dual Banach $\mathcal{A}$-bimodule of X. Recall that a Banach algebra $\mathcal{A}$ is said to be *amenable* if every continuous derivation from $\mathcal{A}$ into X^* is inner, whenever X is a Banach $\mathcal{A}$-bimodule. It was shown in [14] that a Banach algebra is amenable if and only if $\mathcal{A}$ has a bounded approximate diagonal. The flip mapping on the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is defined by $(a\otimes b)^\circ = b\otimes a$ for all $a, b \in \mathcal{A}$. An element $\mathbf{t}$ of $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is called *symmetric* if $\mathbf{t}^\circ = \mathbf{t}$. Suppose that $\mathcal{A}$ has a bounded approximate diagonal consisting of symmetric tensors; then it is called *symmetrically amenable.* Let $\mathcal{A}$ be a symmetrically amenable semisimple Banach algebra, then every Lie derivation on $\mathcal{A}$ is of standard form 3.1 [2]. Applying Proposition 3.1 yields

Corollary 3.5. *Let $\mathcal{A}$ be a symmetrically amenable semisimple Banach algebra. Then every Lie higher derivation on $\mathcal{A}$ has the standard form* 1.1.

3.3. **C^*-algebras.** Let us consider Lie-type higher derivations on C^*-algebras. Let $\mathcal{A}$ be a C^*-algebra. Mathieu and Villena showed us that every Lie derivation on $\mathcal{A}$ is of standard form 3.1 [27]. It follows from Proposition 3.1 that

Corollary 3.6. *Let $\mathcal{A}$ be a C^*-algebra. Then every Lie higher derivation on $\mathcal{A}$ has the standard form* 1.1.

Furthermore, it was proved that if $\mathcal{A}$ is a unital C^*-algebra without tracial states, then each Lie derivation on $\mathcal{A}$ is a derivation [27]. Consequently, we obtain the following corollary.

Corollary 3.7. *Let $\mathcal{A}$ be a unital C^*-algebra without tracial states, then every Lie higher derivation on $\mathcal{A}$ is a higher derivation.*

One embarrassing question is the lack of our knowledge about lie triple derivations of C^*-algebras. To the best of our knowledge there is no any article dealing with lie triple derivations of C^*-algebras. This suggests the following properly-posed question.

Question 3.8. *Let $\mathcal{A}$ be a C^*-algebra. Is any Lie triple derivation on $\mathcal{A}$ of standard form 3.2? Does any Lie triple higher derivation has the standard form 1.2?*

3.4. **von Neumann algebras.** Let $\mathcal{A}$ be a von Neumann algebra. Then every Lie derivation on $\mathcal{A}$ is of standard form 3.1. This result belongs to Miers [29] and was also obtained by Mathieu and Villena in a far more conceptual approach [27]. In particular, it shows that every Lie derivation on a properly infinite von Neumann algebra is a derivation indeed [27]. By Proposition 3.1 we arrive at

Corollary 3.9. *Let $\mathcal{A}$ be a von Neumann algebra. Then every Lie higher derivation on $\mathcal{A}$ has the standard form* 1.1*. In particular, every Lie higher derivation on a properly infinite von Neumann algebra is a higher derivation indeed.*

In addition, we know that if $\mathcal{A}$ is a von Neumann algebra with no abelian summands, then every Lie triple derivation on $\mathcal{A}$ has the standard form 3.2 [30]. Combining this result with Proposition 3.2 leads to

Corollary 3.10. *Let $\mathcal{A}$ be a von Neumann algebra with no abelian summands. Then every Lie triple higher derivation on $\mathcal{A}$ is of standard form 1.2.*

3.5. **Reflexive algebras.** Given a Banach space X with topological dual X^*, by $\mathcal{B}(X)$ we denote the algebra of all bounded linear operators on X. The terms *operator* on X and *subspace* of X will mean "bounded linear mapping of X into itself" and "norm closed linear manifold of X", respectively. For any $A \in \mathcal{B}(X)$, A^* is the adjoint of A. For any non-empty subset $L \subseteq X$, $L^\perp$ denotes its annihilator, that is, $L^\perp = \{f \in X^* | f(x) = 0 \text{ for all } x \in L\}$. A family $\mathcal{L}$ of subspaces of X is a *subspace lattice* if it contains $\{0\}$ and X, and is complete in the sense that it is closed under the formation of arbitrary closed linear spans (denoted by $\bigvee$) and intersections (denoted by $\bigwedge$).

For a given subspace lattice $\mathcal{L}$ on X, the associated *subspace lattice algebra* $\operatorname{Alg}\mathcal{L}$ is the set of operators on X leaving every subspace in $\mathcal{L}$ invariant, that is

$$\operatorname{Alg}\mathcal{L} = \{\, A \in \mathcal{B}(X) \mid A(x) \in E \ \text{ for all } x \in E \ \text{ and for every } \ E \in \mathcal{L} \,\}.$$

Dually, if $\mathcal{A}$ is a subalgebra of $\mathcal{B}(X)$, by $\operatorname{Lat}\mathcal{A}$ we denote the lattice of subspaces of X that are left invariant by each operator in $\mathcal{A}$. An algebra $\mathcal{A}$ is said to be *reflexive* if $\mathcal{A} = \mathrm{AlgLat}\mathcal{A}$, and a lattice $\mathcal{L}$ is called *reflexive* if $\mathcal{L} = \mathrm{LatAlg}\mathcal{L}$. Clearly, every reflexive algebra is of the form $\mathrm{Alg}\mathcal{L}$ for some subspace lattice $\mathcal{L}$ and vice versa.

For arbitrary subspace E in $\mathcal{L}$, Longstaff [18] defined the following related subspaces:

$$E_- = \bigvee\{\, F \in \mathcal{L} \mid F \nsupseteq E \,\}, E \neq 0,$$

$$E_+ = \bigwedge\{\, F \in \mathcal{L} \mid F \nsubseteq E \,\}, E \neq X.$$

Although Lu and Liu [24] pointed that the problem of describing Lie derivations of algebras on Banach spaces is much more difficult than that of characterizing Lie derivations of algebras on Hilbert spaces, they proved that

Theorem 3.11. *[24, Theorem 2.1] Let X be a Banach space over the real or complex field $\mathbb{F}$ with $\dim X > 1$ and $\mathcal{L}$ be a subspace lattice of X with $X_- \neq X$. If $\delta : \mathrm{Alg}\,\mathcal{L} \to \mathcal{B}(X)$ is a Lie derivation, then δ is of standard form 3.1.*

In view of Proposition 3.1 we get the corresponding higher version of the above theorem.

Corollary 3.12. *Let X be a Banach space over the real or complex field $\mathbb{F}$ with $\dim X > 1$ and $\mathcal{L}$ be a subspace lattice of X with $X_- \neq X$. Then every Lie higher derivation from $\mathrm{Alg}\,\mathcal{L}$ into $\mathcal{B}(X)$ has the standard form 1.1.*

3.6. **$\mathcal{J}$-subspace lattice algebras.** Let X be a Banach space over the real or complex field $\mathbb{F}$. A family $\mathcal{L}$ of subspaces of X is a subspace lattice of X which contains $\{0\}$ and X, and is closed under the operations closed linear span $\bigvee$ and intersection $\bigwedge$ in the sense that $\bigvee_{\gamma\in\Gamma} L_\gamma \in \mathcal{L}$ and $\bigwedge_{\gamma\in\Gamma} L_\gamma \in \mathcal{L}$ for every family $\{L_\gamma \colon \gamma \in \Gamma\}$ of elements in $\mathcal{L}$. For a subspace lattice $\mathcal{L}$ of X, the associated subspace lattice algebra $\mathrm{Alg}\mathcal{L}$ is the collection of all bounded operators leaving each subspace in $\mathcal{L}$ invariant. As the above definition, for arbitrary subspace K in $\mathcal{L}$ we establish

$$K_- = \bigvee\{L \in \mathcal{L} : K \nsubseteq L\}.$$

The class of $\mathcal{J}$-subspace lattices was defined by Panaia in his dissertation [34] and covers atomic Boolean subspace lattices and pentagon subspace lattices. $\mathcal{J}$-subspace lattices are a particular sort of complemented lattice, satisfying certain other criteria. To be precise, define

$$\mathcal{J}(\mathcal{L}) = \{K \in \mathcal{L} \colon K \neq 0 \text{ and } K_- \neq X\}.$$

Then $\mathcal{L}$ is called a *$\mathcal{J}$-subspace lattice* on X, provided all of the following conditions are satisfied:

(1)$\bigvee\{K \colon K \in \mathcal{J}(L)\} = X$;
(2)$\bigcap\{K_- \colon K \in \mathcal{J}(L)\} = 0$;
(3)$K \bigvee K_- = X$ for each K in $\mathcal{J}(L)$;
(4)$K \bigcap K_- = 0$ for each K in $\mathcal{J}(\mathcal{L})$.

If $\mathcal{L}$ is a $\mathcal{J}$-subspace lattice, the associated subspace lattice algebra Alg$\mathcal{L}$ is called a *$\mathcal{J}$-subspace lattice algebra*. It should be remarked that both atomic Boolean subspace lattices and pentagon subspace lattices are members of the class of $\mathcal{J}$-subspace lattices [19]. Lu in [23] described the structure of Lie derivations of $\mathcal{J}$-subspace lattice algebras and observed that every Lie derivation on Alg$\mathcal{L}$ has the standard form 3.1. In light of Proposition 3.1 we have

Corollary 3.13. *Let X be a Banach space over the real or complex field $\mathbb{F}$ and* Alg$\mathcal{L}$ *be the $\mathcal{J}$-subspace lattice algebra associated with X. Then every Lie higher derivation on* Alg$\mathcal{L}$ *is of standard form* 1.1.

3.7. **CSL algebras.** Let $\mathbf{H}$ be a complex Hilbert space, $\mathcal{B}(\mathbf{H})$ be the algebra of all bounded linear operators on $\mathbf{H}$ and I be the identity operator. The term *projection* on $\mathbf{H}$ means that "self-adjoint idempotent operator on $\mathbf{H}$". A *subspace lattice* $\mathcal{L}$ of $\mathbf{H}$ is a strongly closed collection of projections on $\mathbf{H}$ that is closed under the usual lattice operations $\bigvee$ and $\bigwedge$, and contains the zero operator 0 and the identity operator I. A *commutative subspace lattice* (in brief, CSL) is a subspace lattice in which each pair of projections in $\mathcal{L}$ commute. A subspace lattice $\mathcal{L}$ is called *completely distributive* if $P = \bigvee\{\, Q \in \mathcal{L} \mid Q_- \not\supseteq P \,\}$ for all $P \in \mathcal{L}$ with $P \neq 0$, where $Q_- = \bigvee\{\, E \in \mathcal{L} \mid E \not\supseteq Q \,\}$. Lu [21] investigated Lie derivations of certain CSL algebras and proved that if Alg$\mathcal{L}$ is a reflexive algebra on a separable Hilbert space $\mathbf{H}$ with completely distributive and commutative lattice, then every Lie derivation on Alg$\mathcal{L}$ is of standard form 3.1. While Zhang and Du [42] studied Lie derivation on an independent finite-width CSL algebra of a complex separable Hilbert space $\mathbf{H}$ (we refer the readers to [9] about the basic properties of finite-width CSL algebras). They proved that if $\mathbf{H}$ is a complex separable Hilbert space with $\dim\mathbf{H} \geq 3$ and $\mathrm{Alg}\mathcal{L}$ is an independent finite-width CSL algebra on $\mathbf{H}$, then every Lie derivation on $\mathrm{Alg}\mathcal{L}$ has the standard form 3.1. By Proposition 3.1 we get the following two results.

Corollary 3.14. *Let* $\mathrm{Alg}\mathcal{L}$ *be a reflexive algebra on a separable Hilbert space* $\mathbf{H}$ *with completely distributive and commutative lattice. Then every Lie higher derivation on* $\mathrm{Alg}\mathcal{L}$ *has the standard form* 1.1.

Corollary 3.15. *Let* $\mathbf{H}$ *be a complex separable Hilbert space with* $\dim\mathbf{H} \geq 3$ *and* $\mathrm{Alg}\mathcal{L}$ *be an independent finite-width CSL algebra on* $\mathbf{H}$. *Then every Lie higher derivation on* $\mathrm{Alg}\mathcal{L}$ *is of standard form* 1.1.

3.8. **Triangular uniformly hyperfinite algebras.** Let $\mathcal{A}$ be an associative algebra over the complex field $\mathbb{C}$ and $\{p_n\}$ be an increasing sequence of positive integers such that $p_n | p_{n+1}$ for each $n \geq 1$. Consider a sequence of C^*-algebras $\mathcal{A}_n \cong^* M_{p_n \times p_n}(\mathbb{C})$ and $*$-homorphisms $\phi_n : \mathcal{A}_n \longrightarrow \mathcal{A}_{n+1}$. The C^*-algebra inductive limit $\mathcal{A}$ of the system $\{(\mathcal{A}_n, \phi_n)\}$ is called a *uniformly hyperfinite* or UHF algebra. Alternatively, $\mathcal{A}$ is a UHF algebra if there exists an increasing sequence $\{\mathcal{A}_n\}_n$ of full matrix algebras whose union is dense in $\mathcal{A}$. Since $\mathcal{A}_n$ is simple and finite-dimensional, it is $*$-isomorphic to some full matrix algebra $M_{p_n \times p_n}(\mathbb{C})$. Let $\mathfrak{D}$ be a maximal abelian self-adjoint subalgebra (i.e. a masa) of a UHF algebra $\mathcal{A}$, and let $\mathcal{C}$ be any subset of $\mathcal{A}$. The *normalizer* of $\mathfrak{D}$ in $\mathcal{C}$ is the set

$$\mathcal{N}_{\mathfrak{D}}(\mathcal{C}) = \{w \in \mathcal{C} \mid w \text{ is a partial isometry such that } w^*\mathfrak{D}w \subseteq \mathfrak{D} \text{ and } w\mathfrak{D}w^* \subseteq \mathfrak{D}\}.$$

A *triangular UHF (TUHF) algebra* $\mathcal{Q}$ is the Banach algebra direct limit of a system

$$\mathcal{Q}_1 \xrightarrow{\varphi_1} \mathcal{Q}_2 \xrightarrow{\varphi_2} \mathcal{Q}_3 \xrightarrow{\varphi_3} \mathcal{Q}_4 \xrightarrow{\varphi_4} \cdots,$$

where $\mathcal{Q}_n$ is isometrically isomorphic to some full upper triangular matrix algebra $\mathcal{T}_{p_n}$ and $\varphi_n : \mathcal{Q}_n \longrightarrow \mathcal{Q}_{n+1}$ is an embedding, i.e. the restriction of a C^*-isomorphism, so that the extension of φ_n carries $\mathcal{N}_{\mathfrak{D}_n}(\mathcal{A}_n)$ into $\mathcal{N}_{\mathfrak{D}_{n+1}}(\mathcal{A}_{n+1})$. It was shown in [13] that every continuous Lie triple derivation on a triangular UHF algebra is of standard form 3.2. So we have the following

Corollary 3.16. *Every continuous Lie triple higher derivation on a triangular UHF algebra has the standard form 1.2.*

3.9. **Nest algebras.** Let X be a Banach space over the real or complex field $\mathbb{F}$ and $\mathcal{B}(X)$ be the algebra of all bounded linear operators on X. Let I be an index set. A *nest* is a set $\mathcal{N}$ of closed subspaces of X satisfying the following conditions:

(a) $0, X \in \mathcal{N}$;
(b) If $N_1, N_2 \in \mathcal{N}$, then either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$;
(c) If $\{N_i\}_{i\in I} \subseteq \mathcal{N}$, then $\bigcap_{i\in I} N_i \in \mathcal{N}$;
(d) If $\{N_i\}_{i\in I} \subseteq \mathcal{N}$, then the norm closure of the linear span of $\bigcup_{i\in I} N_i$ also lies in $\mathcal{N}$.

If $\mathcal{N} = \{0, X\}$, then $\mathcal{N}$ is called a trivial nest, otherwise it is called a non-trivial nest.

The nest algebra associated with $\mathcal{N}$, denoted by $\mathcal{T}(\mathcal{N})$, is the weakly closed operator algebra consisting of all bounded linear operators that leave $\mathcal{N}$ invariant, i.e.,

$$\mathcal{T}(\mathcal{N}) = \{\, T \in \mathcal{B}(X) \mid T(N) \subseteq N \text{ for all } N \in \mathcal{N}\}.$$

If X is a Hilbert space, then every nontrivial nest algebra is a triangular algebra. Indeed, if $N \in \mathcal{N}\backslash\{0, X\}$ and E is the orthogonal projection onto N, then $\mathcal{N}_1 = E(\mathcal{N})$ and $\mathcal{N}_2 = (1-E)(\mathcal{N})$ are nests of N and $N^{\perp}$, respectively. Moreover, $\mathcal{T}(\mathcal{N}_1) = E\mathcal{T}(\mathcal{N})E, \mathcal{T}(\mathcal{N}_2) = (1-E)\mathcal{T}(\mathcal{N})(1-E)$ are nest algebras and

$$\mathcal{T}(\mathcal{N}) = \left[\begin{array}{cc} \mathcal{T}(\mathcal{N}_1) & E\mathcal{T}(\mathcal{N})(1-E) \\ O & \mathcal{T}(\mathcal{N}_2) \end{array}\right] \left(\text{or } \mathcal{T}'(\mathcal{N}) = \left[\begin{array}{cc} \mathcal{T}(\mathcal{N}_1) & O \\ (1-E)\mathcal{T}(\mathcal{N})E & \mathcal{T}(\mathcal{N}_2) \end{array}\right]\right).$$

However, it is not always the case for a nest $\mathcal{N}$ on a general Banach space X, since $N \in \mathcal{N}$ may be not complemented. We refer the readers to [7] for the theory of nest algebras.

It is clear that every nontrivial nest algebra on a finite dimensional Banach space is isomorphic to a complex (or real) block upper (or lower) triangular matrix algebra. Let X be an infinite dimensional Banach space over the real or complex field $\mathbb{F}$ and $\mathcal{N}$ be a nest on X. Suppose that there exists a non-trivial element $N \in \mathcal{N}$ which is complemented in X. Then $X = N \dotplus M$ for some closed subspace M. Let $\mathcal{N}_1 = \{N' \cap N | N' \in \mathcal{N}\}$ and $\mathcal{N}_2 = \{N' \cap M | N' \in \mathcal{N}\}$. It follows that

$$\mathcal{T}(\mathcal{N}) = \left[\begin{array}{cc} \mathcal{T}(\mathcal{N}_1) & \mathcal{B}(M, N) \\ O & \mathcal{T}(\mathcal{N}_2) \end{array}\right]$$

is an upper triangular algebra and that

$$\mathcal{T}'(\mathcal{N}) = \left[\begin{array}{cc} \mathcal{T}(\mathcal{N}_1) & O \\ \mathcal{B}(N, M) & \mathcal{T}(\mathcal{N}_2) \end{array}\right]$$

is a lower triangular algebra. Benkovič in [4] obtained the following results. Let X be an infinite dimensional Banach space over the real or complex field $\mathbb{F}$, $\mathcal{N}$ be a nest on X and $\mathcal{T}(\mathcal{N})$ the nest algebra associated with $\mathcal{N}$. If $\mathcal{N} = \{0, X\}$ or if there exists $N \in \mathcal{N}\backslash\{0, X\}$ which is complemented in X, then every Lie derivation on $\mathcal{T}(\mathcal{N})$ is of standard form 3.1. In particular, if $X = \mathbf{H}$ is a Hilbert space over the real or complex field $\mathbb{F}$ and $\mathcal{T}(\mathcal{N})$ is a nest algebra associated with the nontrivial nest $\mathcal{N}$, then every Lie derivation on $\mathcal{T}(\mathcal{N})$ has the standard form 3.1. The corresponding higher version of the two results are easily obtained by Proposition 3.1.

Corollary 3.17. *Let X be an infinite dimensional Banach space over the real or complex field $\mathbb{F}$, $\mathcal{N}$ be a nest on X and $\mathcal{T}(\mathcal{N})$ the nest algebra associated with $\mathcal{N}$. If $\mathcal{N} = \{0, X\}$ or if there exists $N \in \mathcal{N}\backslash\{0, X\}$ which is complemented in X, then every Lie higher derivation on $\mathcal{T}(\mathcal{N})$ is of standard form* 1.1*.*

Corollary 3.18. *Let $\mathbf{H}$ be a Hilbert space over the real or complex field $\mathbb{F}$, $\mathcal{N}$ be a nontrivial nest on $\mathbf{H}$ and $\mathcal{T}(\mathcal{N})$ be the nest algebra associated with $\mathcal{N}$. Then every Lie higher derivation on $\mathcal{T}(\mathcal{N})$ has the standard form* 1.1*.*

The following corollary is due to Lu [22], Sun and Ma [36] and Zhang et al [41]. They independently characterized Lie triple derivations of nest algebras via completely different approaches. More recently, roughly speaking, they both showed that every Lie triple derivation on the nest algebra $\mathcal{T}(\mathcal{N})$ is of standard form 3.2.

Corollary 3.19. *Let X be a Banach space over the real or complex field $\mathbb{F}$, $\mathcal{N}$ be a nontrivial nest on X and $\mathcal{T}(\mathcal{N})$ be the nest algebra associated with $\mathcal{N}$. Then every Lie triple higher derivation on $\mathcal{T}(\mathcal{N})$ has the standard form 1.2.*

Let $\mathcal{T}_n(\mathbb{C})$ be the algebra consisting of all $n \times n$ upper triangular matrices over the complex field $\mathbb{C}$. Then $\mathcal{T}_n(\mathbb{C})$ can be looked on as a nest algebra. So we assert

Corollary 3.20. *Every Lie higher derivation on $\mathcal{T}_n(\mathbb{C})$ is of standard from* 1.1*, and every Lie triple higher derivation on $\mathcal{T}_n(\mathbb{C})$ has the standard form 1.2.*

3.10. **Full matrix algebras.** Let n be an arbitrary positive integer, $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) = 0$ and $M_n(\mathbb{F})$ be the full matrix algebra consisting of all $n \times n$ matrices over $\mathbb{F}$. Alaminos et al in [3] showed that every Lie derivation on $M_n(\mathbb{F})$ has the standard form 3.1. As a direct consequence of Proposition 3.1 we have

Corollary 3.21. *Every Lie higher derivation on $M_n(\mathbb{F})$ has the standard form* 1.1*.*

However, to the best of our knowledge there is no any article dealing with Lie triple derivations of $M_n(\mathbb{F})$. This suggests the following question.

Question 3.22. *Does any Lie triple higher derivation on $M_n(\mathbb{F})$ is of standard form 1.2?*

3.11. **Block upper and lower triangular matrix algebras.** Let $\mathbb{C}$ be the complex field. Let $\mathbb{N}$ be the set of all positive integers and let $n \in \mathbb{N}$. For any positive integer m with $m \leq n$, we denote by $\bar{d} = (d_1, \cdots, d_i, \cdots, d_m) \in \mathbb{N}^m$ an ordered m-vector of positive integers such that $n = d_1 + \cdots + d_i + \cdots + d_m$. The *block upper triangular matrix algebra* $B_n^{\bar{d}}(\mathbb{C})$ is a subalgebra of $M_n(\mathbb{C})$ with form

$$B_n^{\bar{d}}(\mathbb{C}) = \begin{bmatrix} M_{d_1}(\mathbb{C}) & \cdots & M_{d_1 \times d_i}(\mathbb{C}) & \cdots & M_{d_1 \times d_m}(\mathbb{C}) \\ & \ddots & \vdots & & \vdots \\ & & M_{d_i}(\mathbb{C}) & \cdots & M_{d_i \times d_m}(\mathbb{C}) \\ & O & & \ddots & \vdots \\ & & & & M_{d_m}(\mathbb{C}) \end{bmatrix}.$$

Likewise, the *block lower triangular matrix algebra* $B_n'^{\bar{d}}(\mathbb{C})$ is a subalgebra of $M_n(\mathbb{C})$ with form

$$B_n'^{\bar{d}}(\mathbb{C}) = \begin{bmatrix} M_{d_1}(\mathbb{C}) & & & & \\ \vdots & \ddots & & O & \\ M_{d_i \times d_1}(\mathbb{C}) & \cdots & M_{d_i}(\mathbb{C}) & & \\ \vdots & & \vdots & \ddots & \\ M_{d_m \times d_1}(\mathbb{C}) & \ldots & M_{d_m \times d_i}(\mathbb{C}) & \ldots & M_{d_m}(\mathbb{C}) \end{bmatrix}.$$

Note that the full matrix algebra $M_n(\mathbb{C})$ of all $n \times n$ matrices over $\mathbb{C}$ and the upper(respectively, lower) triangular matrix algebra $T_n(\mathbb{C})$ of all $n \times n$ upper triangular matrices over $\mathbb{C}$ are two special cases of block upper(respectively, lower) triangular matrix algebras. If $n \geq 2$ and $B_n^{\bar{d}}(\mathbb{C}) \neq M_n(\mathbb{C})$, then $B_n^{\bar{d}}(\mathbb{C})$ is an upper triangular algebra and can be written as

$$B_n^{\bar{d}}(\mathbb{C}) = \begin{bmatrix} B_j^{\bar{d}_1}(\mathbb{C}) & M_{j\times(n-j)}(\mathbb{C}) \\ O_{(n-j)\times j} & B_{n-j}^{\bar{d}_2}(\mathbb{C}) \end{bmatrix},$$

where $1 \leq j < m$ and $\bar{d}_1 \in \mathbb{N}^j, \bar{d}_2 \in \mathbb{N}^{m-j}$. Similarly, if $n \geq 2$ and $B_n'^{\bar{d}}(\mathbb{C}) \neq M_n(\mathbb{C})$, then $B_n'^{\bar{d}}(\mathbb{C})$ is a lower triangular algebra and can be represented as

$$B_n'^{\bar{d}}(\mathbb{C}) = \begin{bmatrix} B_j'^{\bar{d}_1}(\mathbb{C}) & O_{j\times(n-j)} \\ M_{(n-j)\times j}(\mathbb{C}) & B_{n-j}'^{\bar{d}_2}(\mathbb{C}) \end{bmatrix},$$

where $1 \leq j < m$ and $\bar{d}_1 \in \mathbb{N}^j, \bar{d}_2 \in \mathbb{N}^{m-j}$. Cheung in [6] proved that every Lie derivation on the block upper (respectively, lower) triangular matrix algebra $B_n^{\bar{d}}(\mathbb{C})$ (respectively, $B_n'^{\bar{d}}(\mathbb{C})$) is of standard form 3.1. Xiao and Wei in [40] extended Cheung's result to the case of Lie triple derivations and showed that every Lie triple derivation on the block upper (respectively, lower) triangular matrix algebra $B_n^{\bar{d}}(\mathbb{C})$ (respectively, $B_n'^{\bar{d}}(\mathbb{C})$) has the standard form 3.2. So we have

Corollary 3.23. *Every Lie higher derivation on $B_n^{\bar{d}}(\mathbb{C})$ or $B_n'^{\bar{d}}(\mathbb{C})$ has the standard form* 1.1*, and every Lie triple higher derivation on $B_n^{\bar{d}}(\mathbb{C})$ or $B_n'^{\bar{d}}(\mathbb{C})$ is of standard form 1.2.*

4. **Future research topics**

We will end this article with a potential future research topic. Taking into account the definitions of Lie derivations and Lie triple derivations, we can extend them in a much more general way. Let n be a positive integer with $n \geq 2$. Let us first see a sequence of polynomials

$$\begin{aligned} p_1(x_1) &= x_1, \\ p_2(x_1, x_2) &= [p_1(x_1), x_2] = [x_1, x_2], \\ p_3(x_1, x_2, x_3) &= [p_2(x_1, x_2), x_3] = [[x_1, x_2], x_3], \\ p_4(x_1, x_2, x_3, x_4) &= [p_3(x_1, x_2, x_3), x_4] = [[[x_1, x_2], x_3], x_4], \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots , \\ p_n(x_1, x_2, \cdots, x_n) &= [p_{n-1}(x_1, x_2, \cdots, x_{n-1}), x_n]. \end{aligned}$$

Let $\mathcal{A}$ be a unital associative algebra over a commutative ring $\mathcal{R}$. An $\mathcal{R}$-linear mapping $D : \mathcal{A} \longrightarrow \mathcal{A}$ is called a *Lie n-derivation* if

$$D(p_n(x_1, x_2, \cdots, x_n)) = \sum_{i=1}^{n} p_n(x_1, \cdots, x_{i-1}, D(x_i), x_{i+1}, \cdots, x_n)$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{A}$. Obviously, arbitrary Lie derivation is a Lie 2-derivation and any Lie triple derivation is a Lie 3-derivation. Let $\mathcal{A}$ be a unital associative algebra over a commutative ring $\mathcal{R}$ and D be a Lie n-derivation on $\mathcal{A}$. D is said to be *standard form* if it can be expressed as the sum

$$D = d + \tau, \tag{4.1}$$

where d is a derivation of $\mathcal{A}$ and τ is a linear mapping from $\mathcal{A}$ into its center $\mathcal{Z}(\mathcal{A})$ vanishing on each $(n-1)$-th commutator of type

$$\underbrace{[\cdots[[}_{n-1} x_1, x_2], x_3], \cdots, x_n].$$

This concept is due to Abdullaev, who proved that every Lie n-derivation on a von Neumann algebra is of standard form 4.1 [1].

Let $\mathbb{N}$ be the set of all non-negative integers and $D = \{d_m\}_{m\in\mathbb{N}}$ be a family of $\mathcal{R}$-linear mappings of $\mathcal{A}$ such that $d_0 = id_{\mathcal{A}}$. D is called a *Lie n-higher derivation* if

$$d_m(p_n(x_1, x_2, \cdots, x_n)) = \sum_{i_1+i_2+\cdots+i_n=m} p_n(d_{i_1}(x_1), d_{i_2}(x_2), \cdots, d_{i_n}(x_n))$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{A}$ and for each $m \in \mathbb{N}$. Thus every Lie higher derivation is a Lie 2-higher derivation, every Lie triple higher derivation is a Lie 3-higher derivation. Like what we have discussed in the Section 3, we should pay much attention to one class of Lie n-higher derivations on $\mathcal{A}$. Suppose that $G = \{g_m\}_{m\in\mathbb{N}}$ is a higher derivation of $\mathcal{A}$. If $\{f''_m\}_{m\in\mathbb{N}}$ is a sequence of $\mathcal{R}$-linear mappings from $\mathcal{A}$ into its center $\mathcal{Z}(\mathcal{A})$ and each f''_m vanishes on each $(n-1)$-th commutator of type $\underbrace{[\cdots[[}_{n-1} a_1, a_2], a_3], \cdots, a_n]$, then we can establish a sequence of $\mathcal{R}$-linear mappings

$$d''_m = g_m + f''_m, \ \forall m \in \mathbb{N}. \tag{4.2}$$

It is easy to verify that $\{d''_m\}_{m\in\mathbb{N}}$ is a Lie n-higher derivation of $\mathcal{A}$, but not a higher derivation of $\mathcal{A}$ if $f''_m \neq 0$ for some $m \in \mathbb{N}$. A Lie n-higher derivation $D = \{d''_m\}_{m\in\mathbb{N}}$ is said to be *standard* if it has the property 4.2.

In light of the systematic works [1–6, 13, 15, 17, 21–27, 29, 30, 35, 36, 38–42], it has considerable interests to study Lie n-derivations and Lie n-higher derivations on operator algebras. Of course, we hope that all existing results about Lie (triple-)derivations could be extended to the case of Lie n-derivations and that the results related to Lie (triple-)higher derivations could be correspondingly generalized to the case of Lie n-higher derivations. This suggests the following questions, which will form a large and long-standing project.

Question 4.1. *Let $\mathcal{A}$ be an associative algebra over a field of characteristic zero. Are the Proposition 3.1 and Proposition 3.2 also true for Lie n-derivations and Lie n-higher derivations of $\mathcal{A}$?*

Question 4.2. *Does any Lie n-derivation on the aforementioned operator algebras has the standard form 4.1? Is any Lie n-higher derivation on the aforementioned operator algebras of standard form 4.2? The involved operator algebras include various operator algebras, such as algebras of bounded linear operator, C^*-algebras, von Neumann algebras, reflexive algebras, $\mathcal{J}$-subspace lattice algebras, CSL algebras, triangular operator algebras.*

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