SOME PROPERTIES OF $I$-CLOSED MODULES

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Abstract. Let $R$ be a commutative Noetherian ring, $I$ an ideal of $R$ and $M$ an arbitrary $R$-module. We introduce and study some properties of the concept of $I$-closed and $I$-semiclosed modules. A relation between the $I$-semiclosed property of local cohomologies $H^i_I(M)$ ($i \geq 0$) and that of $M$ will be investigated.

1. Introduction

Throughout, $R$ is a commutative Noetherian ring, $I$ an ideal of $R$ and $M$ is an arbitrary $R$-module. For a non-negative integer $i$, the $i$-th local cohomology of $M$ with respect to $I$ is defined by:

$$H^i_I(M) := \lim_{n \to \infty} \text{Ext}_R^i(R/I^n, M).$$

Here, we define the concept of $I$-closed and $I$-semiclosed modules, and present some properties of these modules. We prove that for an $R$-module $M$, if $H^i_I(M)$ is $I$-semiclosed for all $i \geq 0$, then $M$ is an $I$-semiclosed, $R$-module; i.e., $\text{Supp}_R \text{Ext}_R^i(R/I, M)$ is a finite set, for all $i \geq 0$.

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2. Main results

Definition 2.1. i) We say that $M$ is a closed $R$-module if $\text{Supp}_R M$ is a finite set.

ii) We say that $M$ is closed with respect to $I$ or $I$-closed if the set $V(I) \cap \text{Supp}_R M$ is finite, where $V(I)$ is the set of all prime ideals of $R$ containing $I$.

iii) We say that $M$ is $I$-semiclosed if the set $\text{Supp}_R \text{Ext}^i_R (R/I, M)$ is a finite set for all $i \geq 0$.

It is obvious from the definitions that for $I = 0$, the $I$-closed modules are just closed modules and any $I$-closed module is $I$-semiclosed.

The subcategory $C$ of the category of $R$-modules and $R$-homomorphisms are said to be a Serre subcategory, if for any exact sequence of $R$-modules,

$$0 \to X \to Y \to Z \to 0,$$

the $R$-module $Y$ belongs to $C$ if and only if each of $X$ and $Z$ belongs to $C$.

Proposition 2.2. i) The class of $I$-closed modules is a Serre subcategory of the category of $R$-modules. As a consequence, any finite direct sum of $I$-closed modules is $I$-closed.

ii) If $N$ is a finitely generated $R$-module and $M$ is an $I$-closed module, then $\text{Ext}^i_R (N, M)$ and $\text{Tor}^i_R (N, M)$ are $I$-closed for all $i \geq 0$.

iii) Let $J$ be an ideal of $R$, $\bar{R} = R/J$ and $\bar{I} = (I + J)/J$. Let $M$ be an $R$-module. Then, $M$ is $I$-closed as an $R$-module if and only if $M$ is $\bar{I}$-closed as an $R$-module.

iv) Let $N$ and $L$ be two finitely generated $R$-modules such that $\text{Supp}_R L \subseteq \text{Supp}_R N$. Then, for any non-negative integer $t$, if $\text{Ext}^i_R (N, M)$ is $I$-closed for all $i \leq t$, then so is $\text{Ext}^i_R (L, M)$. In particular, $M$ is $I$-semiclosed if and only if for any finitely generated $R$-module $L$ with $\text{Supp}_R L = V(I)$, the $R$-module $\text{Ext}^i_R (L, M)$ is $I$-closed for all $i \geq 0$. 
Some properties of $I$-closed modules

v) Let $M$ be an $R$-module and $P$ be a pure submodule of $M$. Then, $M$ is $I$-semiclosed if and only if both $P$ and $M/P$ are $I$-semiclosed (for the definition of a pure submodule, see [4], p. 94).

vi) Let $J$ be an ideal of $R$, $\bar{R} = R/J$ and $\bar{I} = (I + J)/J$. Let $M$ be an $\bar{R}$-module. Then, $M$ is $I$-semiclosed as an $R$-module if and only if $M$ is $\bar{I}$-semiclosed as an $\bar{R}$-module.

vii) If $M$ is $I$-semiclosed $R$-module, then $M/IM$ is $I$-closed.

viii) If $0 \to X \to Y \to Z \to 0$ is an exact sequence and two of the modules in the sequence is $I$-semiclosed, then so is the third one. Consequently, if $f : X \to Y$ is a homomorphism between two $I$-semiclosed modules and one of the three modules $\text{Ker} f$, $\text{Im} f$ and $\text{Coker} f$ is $I$-semiclosed, then all three of them are $I$-semiclosed.

ix) If $M$ is an $R$-module such that $\text{Ass}_R M \subseteq V(I)$ and $\text{Hom}_R(R/I, M)$ is $(I-)$closed, then the set $\text{Ass}_R M$ is finite.

**Proof.** i) This is obvious.

ii) Let $$F_\bullet : \cdots \to F_1 \to F_0 \to 0,$$
be a finite free resolution of $N$. Then, $\text{Ext}^i_R(N, M) = H^i(\text{Hom}_R(F_\bullet, M))$, as a subqoutien of a direct sum of finitely many copies of $M$, is $I$-closed by i). The proof of the $I$-closed property of $\text{Tor}^i_R(N, M)$ is similar.

iii) We note that for an $\bar{R}$-module $X$, $$V(\bar{I}) \cap \text{Supp}_R X = \{p/J : p \in V(I) \cap \text{Supp}_R X\}.$$ So, the result follows.

iv) Since $\text{Supp}_R L \subseteq \text{Supp}_R N$, we get by Grusons’s Theorem (Theorem 4.1 in [5]), that there exists a finite filtration, $$0 = L_0 \subset L_1 \subset \cdots \subset L_{s-1} \subset L_s = L,$$ such that for any $i = 1, \ldots, s$, the factor module $L_i/L_{i-1}$ is a homomorphic image of $N^{n_i}$, for some integer $n_i > 0$. Using the short exact sequences $0 \to L_{i-1} \to L_i \to L_i/L_{i-1} \to 0$, for $i = 1, \ldots, s$, we can reduce to the case $s = 1$. Therefore, there is an exact sequence
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\[0 \to U \to N^n \to L \to 0, \text{ for some } n > 0 \text{ and some finitely generated submodule } U \text{ of } N. \] Now, using the long exact sequence,

\[0 \to \text{Hom}_R(L, M) \to \text{Hom}_R(N^n, M) \to \cdots\]

\[\cdots \to \text{Ext}^{i-1}_R(U, M) \to \text{Ext}^i_R(L, M) \to \text{Ext}^i_R(N^n, M) \to \cdots,\]

the first claim follows by induction argument on \(i\) and part i). The second claim now is obvious.

v) We note that Cohen’s Characterization of purity (Theorem 3.65 in [4]) implies that the sequence,

\[0 \to \text{Ext}^i_R(R/I, P) \to \text{Ext}^i_R(R/I, M) \to \text{Ext}^i_R(R/I, M/P) \to 0,\]

is exact for all \(i \geq 0\) (see the proof of Proposition 2.7 in [3]). Hence, the result follows by part i).

vi) The proof of this part is similar to Proposition 2 in [2]. We present its proof for the reader’s convenience.

We note that \(\text{Supp}_R M \subseteq V(\tilde{a})\) if and only if \(\text{Supp}_{\bar{R}} M \subseteq V(\bar{a})\). Now, we consider the Change of Rings spectral sequences,

\[E^{p,q}_2 = \text{Ext}^p_{\bar{R}}(\text{Tor}^R_{\bar{a}}(\bar{R}, R/\bar{a}), M) \Rightarrow \text{Ext}^{p+q}_R(R/\bar{a}, M).\]

Suppose first that \(M\) is \(\tilde{a}\)-semiclosed. For each \(t \geq 0\), there is a finite filtration,

\[0 = \phi^{t+1}H^t \subseteq \phi^t H^t \subseteq \ldots \subseteq \phi^1 H^t \subseteq \phi^0 H^t = H^t = \text{Ext}^t_R(R/\bar{a}, N),\]

such that \(E^{p,p+q}_2 = \phi^{p+q}H^t/\phi^{p+q}H^t\), for all \(0 \leq p \leq t\).

Since \(\text{Supp}_{\bar{R}}(\text{Tor}^R_{\bar{a}}(\bar{R}, R/\bar{a})) \subseteq V(\bar{a})\), for all \(q\), by Proposition 2.4 iii), it follows that \(E^{p,q}_2\) is \(\tilde{a}\)-closed for all \(p, q\). So, \(E^{p,q}_2\), as a subquotient of \(E^{p,q}_2\), is \(\tilde{a}\)-closed \(\bar{R}\)-module for all \(r\) and all \(p, q \geq 0\) by Lemma 2.3 i). As \(E^{p,q}_2 = E^{p,q}_2\) for all \(p, q \geq 0\), and all large values of \(r\), we can therefore deduce using Proposition 2.3 i), successively, that \(H^t = \text{Ext}^t_R(R/\bar{a}, N)\) is an \(\tilde{a}\)-closed \(\bar{R}\)-module for all \(t \geq 0\). Hence, \(M\) is an \(a\)-semiclosed \(R\)-module by Proposition 2.4 ii).

Conversely, suppose that \(M\) is \(a\)-closed. We prove that

\[E^{t,0}_2 = \text{Ext}^t_R(R/\bar{a}R, M)\]

is an \(\tilde{a}\)-closed \(\bar{R}\)-module for all \(t \geq 0\). We proceed by induction on \(t\).

For \(t = 0\),

\[E^{0,0}_2 = \text{Hom}_{\bar{R}}(\bar{R}/a\bar{R}, M) \cong \text{Hom}_R(R/\bar{a}, M),\]
is $\bar{a}$-closed by Proposition 2.4 ii). So, let $t > 0$ and assume that the claim is true for $p < t$. By Proposition 2.4 iii), it follows that $E^{p,q}$ is $\bar{a}$-closed for all $p < t$ and all $q \geq 0$. We have $E^{t,0}_r \cong E^{t,0}_\infty$, for sufficiently large $r$. Since $H^t = \text{Ext}^t_R(R/a, N)$ is $\bar{a}$-closed, then it follows that $E^{p,t-p}_\infty$ is $\bar{a}$-closed for $0 \leq p \leq t$. Thus, by induction hypothesis, we deduce that $\text{Im}(E^{t,r}_t \to E^{t,0}_r)$ is $\bar{a}$-closed. Hence, $\text{Ker}(E^{t,0}_t \to E^{t,0}_{t+1})$ is $\bar{a}$-closed.

We can continue to work this way to see that $E^{t,0}_2 = \text{Ker}(E^{t,0}_2 \to 0)$ is $\bar{a}$-closed.

vii) Let $\{x_1, ..., x_n\}$ be a generating set for $I$.

Let $\phi : S = R[X_1, ..., X_n] \to R$ be the natural ring epimorphism such that $\phi(X_i) = x_i$ for $i = 1, ..., n$. We may assume that $R = S/J$, where $J = (X_1, ..., X_n)$. Then, by assumption and part vi), $M$ is $J$-semiclosed. That is, $\text{Ext}_S^i(S/J, M)$ is $J$-closed, for all $i \geq 0$. In particular, the set $\text{Supp}_S \text{Ext}_S^0(S/J, M)$ is finite. Let $K_\bullet(X, M)$ denote the Koszul complex of $X = X_1, ..., X_n$ with coefficients in $M$ and with Koszul cohomologies $H^i(X, M)$. Then, since $X_1, ..., X_n$ is a regular sequence on $S$, we have,

$$M/IM = M/JM = H^n(X, M) = \text{Ext}_S^0(S/J, M).$$

That is, $M/IM$ is $J$-closed. Now, by part iii), it should be $I$-closed.

viii) The first claim follows from the long exact sequence,

$$0 \to \text{Hom}_R(R/I, X) \to \text{Hom}_R(R/I, Y) \to \text{Hom}_R(R/I, Z) \to$$

$$\text{Ext}_R^0(R/I, X) \to \text{Ext}_R^1(R/I, X) \to \text{Ext}_R^1(R/I, Y) \to \text{Ext}_R^1(R/I, Z) \to \text{Ext}_R^{1+1}(R/I, X) \to \cdots.$$ 

The second claim follows using the exact sequences,

$$0 \to \text{Ker} f \to X \to \text{Im} f \to 0,$$

and

$$0 \to \text{Im} f \to Y \to \text{Coker} f \to 0,$$

and the corresponding long exact sequences of the Ext-modules as in the first part.

ix) By assumption, the set $\text{Ass}_R \text{Hom}_R(R/I, M) \subseteq \text{Supp}_R \text{Hom}_R(R/I, M)$ is a finite set. So, the result follows from Exercise 1.2.27 in [1]. □
Theorem 2.3. Let $t$ be a non-negative integer and let $M$ be an $R$-module such that $H^i_I(M)$ is $I$-semiclosed for all $i \leq t$. Then, $\text{Ext}^i_R(R/I, M)$ is closed for all $i \leq t$.

Proof. Since $\text{Supp}_R \text{Ext}^i_R(R/I, M) \subseteq V(I)$, then it is enough to prove that $\text{Ext}^i_R(R/I, M)$ is $I$-closed for all $i \leq t$. We proceed by induction on $i \geq 0$. In the case $i = 0$, since $\text{Hom}_R(R/I, M) \cong \text{Hom}_R(R/I, H^0_I(M))$, then the result follows by assumption. So, let $0 < i < t$ and set $\bar{M} = M/H^0_I(M)$. The short exact sequence,

$$0 \to H^0_I(M) \to M \to \bar{M} \to 0,$$

gives the long exact sequence,

$$\cdots \to \text{Ext}^i_R(R/I, H^0_I(M)) \to \text{Ext}^i_R(R/I, M) \to \text{Ext}^i_R(R/I, \bar{M}) \to \cdots,$$

of Ext-modules and the isomorphism $H^i_I(M) \cong H^i_I(\bar{M})$, for all $i > 0$. So, in view of Proposition 2.2 i), we may assume that $H^0_I(M) = 0$.

Now, let $E(M)$ be the injective hull of $M$ and put $L = E(M)/M$. Then, $H^0_I(E(M)) = 0 = \text{Hom}_R(R/I, E(M))$. Therefore, using the exact sequence,

$$0 \to M \to E(M) \to L \to 0,$$

we get $H^{i+1}_I(M) \cong H^{i}_I(L)$ and $\text{Ext}^{i+1}_R(R/I, M) \cong \text{Ext}^{i}_R(R/I, L)$, for all $i \geq 0$, and the result follows by induction.

Corollary 2.4. Let $M$ be an $R$-module. If $H^i_I(M)$ is $I$-semiclosed for all $i$, then $M$ is $I$-semiclosed.

Proof. This follows from Theorem 2.3.

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References
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