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# SOME PROPERTIES OF I-CLOSED MODULES

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ABSTRACT. Let R be a commutative Noetherian ring, I an ideal of R and M an arbitrary R-module. We introduce and study some properties of the concept of I-closed and I-semiclosed modules. A relation between the I semiclosed property of local cohomologies  $H_I^i(M)$  ( $i \geq 0$ ) and that of M will be investigated.

### 1. Introduction

Throughout, R is a commutative Noetherian ring, I an ideal of R and M is an arbitrary R-module. For a non-negative integer i, the i-th local cohomology of M with respect to I is defined by:

$$H_I^i(M) := \varinjlim_{n \in \mathbb{N}} \operatorname{Ext}_R^i(R/I^n, M).$$

Here, we define the concept of *I*-closed and *I*-semiclosed modules, and present some properties of these modules. We prove that for an *R*-module *M*, if  $H_I^i(M)$  is *I*-semiclosed for all  $i \ge 0$ , then *M* is an *I*-semiclosed, *R*-module; i.e.,  $\text{Supp}_R \text{Ext}_R^i(R/I, M)$  is a finite set, for all  $i \ge 0$ .

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### 2. Main results

**Definition 2.1.** i) We say that M is a *closed* R-module if  $\text{Supp}_R M$  is a finite set.

ii) We say that M is closed with respect to I or I-closed if the set  $V(I) \cap \text{Supp}_R M$  is finite, where V(I) is the set of all prime ideals of R containing I.

iii) We say that M is I-semiclosed if the set  $\text{Supp}_R \text{Ext}_R^i(R/I, M)$  is a finite set for all  $i \ge 0$ .

It is obvious from the definitions that for I = 0, the *I*-closed modules are just closed modules and any *I*-closed module is *I*- semiclosed.

The subcategory C of the category of R-modules and R-homomorphisms are said to be a *Serre subcategory*, if for any exact sequence of R-modules,

 $0 \to X \to Y \to Z \to 0,$ 

the *R*-module Y belongs to  $\mathcal{C}$  if and only if each of X and Z belongs to  $\mathcal{C}$ .

**Proposition 2.2.** i) The class of I-closed modules is a Serre subcategory of the category of R-modules. As a consequence, any finite direct sum of I-closed modules is I-closed.

ii) If N is a finitely generated R-module and M is an I-closed module, then  $\operatorname{Ext}_{R}^{i}(N, M)$  and  $\operatorname{Tor}_{i}^{R}(N, M)$  are I-closed for all  $i \geq 0$ .

iii) Let J be an ideal of R,  $\overline{R} = R/J$  and  $\overline{I} = (I+J)/J$ . Let M be an  $\overline{R}$ -module. Then, M is I-closed as an R-module if and only if M is  $\overline{I}$ -closed as an  $\overline{R}$ -module.

iv) Let N and L be two finitely generated R-modules such that  $\operatorname{Supp}_R L \subseteq \operatorname{Supp}_R N$ . Then, for any non-negative integer t, if  $\operatorname{Ext}^i_R(N,M)$  is Iclosed for all  $i \leq t$ , then so is  $\operatorname{Ext}^i_R(L,M)$ . In particular, M is Isemiclosed if and only if for any finitely generated R-module L with  $\operatorname{Supp}_R L = V(I)$ , the R-module  $\operatorname{Ext}^i_R(L,M)$  is I-closed for all  $i \geq 0$ . Some properties of I-closed modules

v) Let M be an R-module and P be a pure submodule of M. Then, M is I-semiclosed if and only if both P and M/P are I-semiclosed (for the definition of a pure submodule, see [4], p. 94).

vi) Let J be an ideal of R,  $\overline{R} = R/J$  and  $\overline{I} = (I+J)/J$ . Let M be an  $\overline{R}$ -module. Then, M is I-semiclosed as an R-module if and only if M is  $\overline{I}$ -semiclosed as an  $\overline{R}$ -module.

vii) If M is I-semiclosed R-module, then M/IM is I-closed.

viii) If  $0 \to X \to Y \to Z \to 0$  is an exact sequence and two of the modules in the sequence is I-semiclosed, then so is the third one. Consequently, if  $f: X \to Y$  is a homomorphism between two I-semiclosed modules and one of the three modules Kerf, Imf and Cokerf is I-semiclosed, then all three of them are I-semiclosed.

ix) If M is an R-module such that  $\operatorname{Ass}_R M \subseteq V(I)$  and  $\operatorname{Hom}_R(R/I, M)$  is (I-)closed, then the set  $\operatorname{Ass}_R M$  is finite.

**Proof.** i) This is obvious.

ii) Let

$$\mathbf{F}_{\bullet}:\cdots \to F_1 \to F_0 \to 0,$$

be a finite free resolution of N. Then,  $\operatorname{Ext}_{R}^{i}(N, M) = H^{i}(\operatorname{Hom}_{R}(\mathbf{F}_{\bullet}, M))$ , as a subquitent of a direct sum of finitely many copies of M, is *I*-closed by i). The proof of the *I*-closed property of  $\operatorname{Tor}_{i}^{R}(N, M)$  is similar.

iii) We note that for an R-module X,

$$V(I) \cap \operatorname{Supp}_{\bar{R}} X = \{\mathfrak{p}/J : \mathfrak{p} \in V(I) \cap \operatorname{Supp}_{R} X\}.$$

So, the result follows.

iv) Since  $\operatorname{Supp}_R L \subseteq \operatorname{Supp}_R N$ , we get by Grusons's Theorem (Theorem 4.1 in [5]), that there exists a finite filtration,

$$0 = L_0 \subset L_1 \subset \cdots \subset L_{s-1} \subset L_s = L,$$

such that for any i = 1, ..., s, the factor module  $L_i/L_{i-1}$  is a homomorphic image of  $N^{n_i}$ , for some integer  $n_i > 0$ . Using the short exact sequences  $0 \to L_{i-1} \to L_i \to L_i/L_{i-1} \to 0$ , for i = 1, ..., s, we can reduce to the case s = 1. Therefore, there is an exact sequence  $0 \to U \to N^n \to L \to 0$ , for some n > 0 and some finitely generated submodule U of N. Now, using the long exact sequence,

$$0 \to \operatorname{Hom}_R(L, M) \to \operatorname{Hom}_R(N^n, M) \to \cdots$$

 $\cdots \to \operatorname{Ext}_R^{i-1}(U,M) \to \operatorname{Ext}_R^i(L,M) \to \operatorname{Ext}_R^i(N^n,M) \to \cdots,$ 

the first claim follows by induction argument on i and part i). The second claim now is obvious.

v) We note that Cohen's Characterization of purity (Theorem 3.65 in [4]) implies that the sequence,

$$0 \to \operatorname{Ext}^{i}_{R}(R/I, P) \to \operatorname{Ext}^{i}_{R}(R/I, M) \to \operatorname{Ext}^{i}_{R}(R/I, M/P) \to 0,$$

is exact for all  $i \ge 0$  (see the proof of Proposition 2.7 in [3]). Hence, the result follows by part i).

vi) The proof of this part is similar to Proposition 2 in [2]. We present its proof for the reader's convenience.

We note that  $\operatorname{Supp}_R M \subseteq V(\mathfrak{a})$  if and only if  $\operatorname{Supp}_{\bar{R}} M \subseteq V(\bar{\mathfrak{a}})$ . Now, we consider the Change of Rings spectral sequences,

$$E_2^{p,q} = \operatorname{Ext}_{\bar{R}}^p(\operatorname{Tor}_q^R(\bar{R}, R/\mathfrak{a}), M) \underset{p}{\Rightarrow} \operatorname{Ext}_R^{p+q}(R/\mathfrak{a}, M).$$

Suppose first that M is  $\bar{\mathfrak{a}}$ -semiclosed. For each  $t \geq 0$ , there is a finite filtration,

$$0 = \phi^{t+1} H^t \subseteq \phi^t H^t \subseteq \ldots \subseteq \phi^1 H^t \subseteq \phi^0 H^t = H^t = \operatorname{Ext}_R^t(R/\mathfrak{a}, N),$$

such that  $E_{\infty}^{p,t-p} \cong \phi^p H^t / \phi^{p+1} H^t$ , for all  $0 \le p \le t$ .

Since  $\operatorname{Supp}_{\bar{R}}(\operatorname{Tor}_{q}^{R}(\bar{R}, R/\mathfrak{a})) \subseteq V(\bar{\mathfrak{a}})$ , for all q, by Proposition 2.4 iii), it follows that  $E_{2}^{p,q}$  is  $\bar{\mathfrak{a}}$ -closed for all p and q. So,  $E_{r}^{p,q}$ , as a subquotient of  $E_{2}^{p,q}$ , is  $\bar{\mathfrak{a}}$ -closed  $\bar{R}$ -module for all r and all  $p, q \geq 0$  by Lemma 2.3 i). As  $E_{2}^{p,q} = E_{r}^{p,q}$  for all  $p, q \geq 0$ , and all large values of r, we can therefore deduce using Proposition 2.3 i), successively, that  $H^{t} = \operatorname{Ext}_{R}^{t}(R/\mathfrak{a}, N)$ is an  $\bar{\mathfrak{a}}$ -closed  $\bar{R}$ -module for all  $t \geq 0$ . Hence, M is an  $\mathfrak{a}$ -semiclosed Rmodule by Proposition 2.4 ii).

Conversely, suppose that M is  $\mathfrak{a}$ -closed. We prove that

$$E_2^{t,0} = \operatorname{Ext}_{\bar{R}}^t(\bar{R}/\mathfrak{a}\bar{R},M)$$

is an  $\bar{\mathfrak{a}}$ -closed *R*-module for all  $t \ge 0$ . We proceed by induction on *t*. For t = 0,

$$E_2^{0,0} = \operatorname{Hom}_{\bar{R}}(\bar{R}/\mathfrak{a}\bar{R}, M) \cong \operatorname{Hom}_R(R/\mathfrak{a}, M),$$

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is  $\bar{\mathfrak{a}}$ -closed by Proposition 2.4 ii). So, let t > 0 and assume that the claim is true for p < t. By Proposition 2.4 iii), it follows that  $E^{p,q}$  is  $\bar{\mathfrak{a}}$ -closed for all p < t and all  $q \ge 0$ . We have  $E_r^{t,0} \cong E_{\infty}^{t,0}$ , for sufficiently large r. Since  $H^t = \operatorname{Ext}_R^t(R/\mathfrak{a}, N)$  is  $\bar{\mathfrak{a}}$ -closed, then it follows that  $E_{\infty}^{p,t-p}$  is  $\bar{\mathfrak{a}}$ -closed for  $0 \le p \le t$ . Thus, by induction hypothesis, we deduce that  $\operatorname{Im}(E_r^{t-r,r-1} \to E_r^{t,0})$  is  $\bar{\mathfrak{a}}$ -closed. Hence,  $\operatorname{Ker}(E_r^{t,0} \to E_{r+1}^{t,0})$  is  $\bar{\mathfrak{a}}$ -closed. We can continue to work this way to see that  $E_2^{t,0} = \operatorname{Ker}(E_2^{t,0} \to 0)$  is  $\bar{\mathfrak{a}}$ -closed.

vii) Let  $\{x_1, ..., x_n\}$  be a generating set for I.

Let  $\phi: S = R[X_1, ..., X_n] \to R$  be the natural ring epimorphism such that  $\phi(X_i) = x_i$  for i = 1, ..., n. We may assume that R = S/J, where  $J = (X_1, \dots, X_n)$ . Then, by assumption and part vi), M is J-semiclosed. That is,  $\operatorname{Ext}_S^i(S/J, M)$  is J-closed, for all  $i \ge 0$ . In particular, the set  $\operatorname{Supp}_S \operatorname{Ext}_S^n(S/J, M)$  is finite. Let  $K_{\bullet}(X, M)$  denote the Koszul complex of  $X = X_1, ..., X_n$  with coefficients in M and with Koszul cohomologies  $H^i(X, M)$ . Then, since  $X_1, ..., X_n$  is a regular sequence on S, we have,

$$M/IM = M/JM = H^n(X, M) = \operatorname{Ext}^n_S(S/J, M).$$

That is, M/IM is J-closed. Now, by part iii), it should be I-closed.

viii) The first claim follows from the long exact sequence,

$$0 \to \operatorname{Hom}_{R}(R/I, X) \to \operatorname{Hom}_{R}(R/I, Y) \to \operatorname{Hom}_{R}(R/I, Z) \to$$
$$\operatorname{Ext}_{R}^{1}(R/I, X) \to \cdots \to \operatorname{Ext}_{R}^{i}(R/I, X) \to \operatorname{Ext}_{R}^{i}(R/I, Y)$$
$$\to \operatorname{Ext}_{R}^{i}(R/I, Z) \to \operatorname{Ext}_{R}^{i+1}(R/I, X) \to \cdots.$$

The second claim follows using the exact sequences,

$$0 \longrightarrow \operatorname{Ker} f \to X \to \operatorname{Im} f \to 0,$$

and

$$0 \to \operatorname{Im} f \to Y \to \operatorname{Coker} f \to 0,$$

and the corresponding long exact sequences of the Ext-modules as in the first part.

ix) By assumption, the set  $\operatorname{Ass}_R\operatorname{Hom}_R(R/I, M) \subseteq \operatorname{Supp}_R\operatorname{Hom}_R(R/I, M)$  is a finite set. So, the result follows from Exercise 1.2.27 in [1].  $\Box$ 

**Theorem 2.3.** Let t be a non-negative integer and let M be an R-module such that  $H_I^i(M)$  is I-semiclosed for all  $i \leq t$ . Then,  $\operatorname{Ext}_R^i(R/I, M)$  is closed for all  $i \leq t$ .

**Proof.** Since  $\operatorname{Supp}_R \operatorname{Ext}_R^i(R/I, M) \subseteq V(I)$ , then it is enough to prove that  $\operatorname{Ext}_R^i(R/I, M)$  is *I*-closed for all  $i \leq t$ . We proceed by induction on  $i \geq 0$ . In the case i = 0, since  $\operatorname{Hom}_R(R/I, M) \cong \operatorname{Hom}_R(R/I, H_I^0(M))$ , then the result follows by assumption. So, let 0 < i < t and set  $\overline{M} = M/H_I^0(M)$ . The short exact sequence,

$$0 \to H^0_I(M) \to M \to \bar{M} \to 0,$$

gives the long exact sequence,

$$\cdots \to \operatorname{Ext}^{i}_{R}(R/I, H^{0}_{I}(M)) \to \operatorname{Ext}^{i}_{R}(R/I, M) \to \operatorname{Ext}^{i}_{R}(R/I, \bar{M}) \to \cdots,$$

of Ext-modules and the isomorphism  $H_I^i(M) \cong H_I^i(\overline{M})$ , for all i > 0. So, in view of Proposition 2.2 i), we may assume that  $H_I^0(M) = 0$ . Now, let E(M) be the injective hull of M and put L = E(M)/M. Then,  $H_I^0(E(M)) = 0 = \operatorname{Hom}_R(R/I, E(M))$ . Therefore, using the exact sequence,

$$0 \to M \to E(M) \to L \to 0,$$

we get  $H_I^{i+1}(M) \cong H_I^i(L)$  and  $\operatorname{Ext}_R^{i+1}(R/I, M) \cong \operatorname{Ext}_R^i(R/I, L)$ , for all  $i \ge 0$ , and the result follows by induction.

**Corollary 2.4.** Let M be an R-module. If  $H_I^i(M)$  is I-semiclosed for all i, then M is I-semiclosed.

**Proof.** This follows from Theorem 2.3.

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