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GROUPS IN WHICH EVERY SUBGROUP HAS FINITE INDEX IN ITS FRATTINI CLOSURE

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(Communicated by Ali Reza Ashrafi)

Abstract. In 1970, Menegazzo [Gruppi nei quali ogni sottogruppo è intersezione di sottogruppi massimali, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **48** (1970), 559–562.] gave a complete description of the structure of soluble IM-groups, i.e., groups in which every subgroup can be obtained as intersection of maximal subgroups. A group G is said to have the FM-property if every subgroup of G has finite index in the intersection $\hat{X}$ of all maximal subgroups of G containing X. The behaviour of (generalized) soluble FM-groups is studied in this paper. Among other results, it is proved that if G is a (generalized) soluble group for which there exists a positive integer k such that $|\hat{X} : X| \leq k$ for each subgroup X, then G is finite-by-IM-by-finite, i.e., G contains a finite normal subgroup N such that G/N is a finite extension of an IM-group.

Keywords: Maximal subgroup, Frattini closure, FM-group.
MSC(2010): Primary: 20E28; Secondary: 20E15.

1. Introduction

The behaviour of maximal subgroups of a group often has a strong influence on the structure of the group itself. In particular when the group is a finite group or generally when the group is rich in maximal subgroups, as in the case when a suitable finiteness condition is imposed.

Let G be a group, and let X be a subgroup of G. The *Frattini closure* of X in G is the intersection $\hat{X}$ of all maximal subgroups of G

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containing X, with the convention that $\hat{X} = G$ if X is contained in no maximal subgroups of G. A subgroup X such that $\hat{X} = X$ is said to be *Frattini closed* in G. If N is any normal subgroup of a group G, then $\hat{N}/N$ coincides with the Frattini subgroup $\Phi(G/N)$ of the factor group G/N, and hence $\hat{N}$ is likewise normal in G; in particular, N is Frattini closed in G if and only if G/N has trivial Frattini subgroup.

A group G is called an *IM-group* if all its subgroups are Frattini closed, i.e., if every subgroup of G can be obtained as intersection of a collection of maximal subgroups. The structure of soluble IM-groups was completely described by F. Menegazzo [5]. In fact, it turns out that if G is any soluble IM-group, then G is periodic and metabelian, and the normality relation in G is transitive; moreover, the commutator subgroup G' of G is a Hall subgroup and the (non-trivial) primary components of the abelian groups G' and G/G' have prime exponent. Of course, the property IM is a purely lattice-theoretic concept, and it is known that a finite group G is an IM-group if and only if its subgroup lattice is relatively complemented (see [9], Corollary 3.3.11). Moreover, for arbitrary groups the property IM is equivalent to a certain weak complementation property (see [9], Theorem 3.3.12).

Let G be a group. We shall say that a subgroup X of G is *nearly Frattini closed* in G if it has finite index in its Frattini closure $\hat{X}$, and that G is an *FM-group* (or that it has the *FM-property*) if all its subgroups are nearly Frattini closed. As subgroups of finite index can be recognized within the lattice of subgroups (see [10]), it follows that also the property FM is a lattice-theoretic one.

The aim of this paper is to investigate the behaviour of (generalized) soluble FM-groups. General properties of FM-groups are described in Section 2, where in particular it is proved that if G is a hyperabelian FM-group, then the second commutator subgroup G'' of G is finite (so that in particular G is soluble) and every subnormal subgroup of G is normal. In Section 3 the study of locally nilpotent FM-groups is reduced to the abelian case, and among other results it is proved that every abelian group with the property FM is a direct product of a periodic group and a torsion-free group of finite rank.

A group G is said to be a *BFM-group* if there exists a positive integer k such that $|\hat{X} : X| \leq k$ for every subgroup X of G. In Section 4, it is proved that if G is any (generalized) soluble group with the property BFM, then G is finite-by-IM-by-finite, i.e., G contains normal subgroups N and K such that $N \leq K$, N and $|G : K|$ are finite and K/N

is an IM-group. As the Frattini closure can be considered as a natural replacement for the ordinary normal closure of a subgroup, the above result should be compared with a famous theorem of B.H. Neumann [6], proving that if every subgroup of a group G has finite index in its normal closure, then G has finite commutator subgroup.

Most of our notation is standard and can be found in [8].

2. General properties of FM-groups

Our first lemma describes an elementary property of maximal subgroups containing subnormal subgroups of defect 2. It was already remarked by G. Kaplan [4] in the case of finite groups.

Lemma 2.1. *Let G be a group, and let X be a subnormal subgroup of G with defect 2. If X is contained in a maximal subgroup M of G, then also its normal closure X^G is contained in M.*

Proof. Assume for a contradiction that the normal subgroup X^G is not contained in M, so that $G = X^G M$. If g is any element of G, then $g = ab$, where a is an element of X^G and b belongs to M. Moreover, as X is normal in X^G, we have

$$X^g = X^{ab} = X^b \leq M.$$

Therefore X^G lies in M, and this contradiction proves the statement. □

It is now easy to prove that the Frattini closure of a subnormal subgroup of defect 2 coincides with that of its normal closure. In particular, it follows that all subnormal subgroups of an IM-group are normal.

Corollary 2.2. *Let G be a group, and let X be a subnormal subgroup of G with defect 2. Then X and X^G have the same Frattini closure in G, i.e., $X \leq X^G \leq \hat{X}$. In particular, the Frattini closure of X is normal in G.*

Proof. It follows from Lemma 2.1 that a maximal subgroup of G contains X if and only if it contains X^G. Therefore the Frattini closures of X and X^G in G coincide. □

Lemma 2.3. *Let G be a group, and let M be a maximal subgroup of G. If N is a normal subgroup of G such that $M \cap N$ is contained in some maximal subgroup of N, then $M \cap N$ is Frattini closed in N.*

Proof. Since M normalizes both N and $M \cap N$, it also normalizes the Frattini closure K of $M \cap N$ in N. Assume for a contradiction that $M \cap N$ is properly contained in K, so that K is not contained in M and hence $G = KM$. Therefore

$$N = KM \cap N = K(M \cap N) = K,$$

a contradiction, since $M \cap N$ is contained in some maximal subgroup of N. □

Corollary 2.4. *Let G be a group, and let N be a normal subgroup of G. If M is a maximal subgroup of G such that the index $|G : M|$ is finite, then $M \cap N$ is Frattini closed in N.*

Proof. As the index $|N : M \cap N|$ is finite, we have that either N is a subgroup of M or $M \cap N$ is contained in some maximal subgroup of N, and hence $M \cap N$ is Frattini closed in N by Lemma 2.3. □

Our next result is an extension of the well-known fact that if G is a group in which all maximal subgroups have finite index, then its Frattini subgroup $\Phi(G)$ contains the Frattini subgroup of every normal subgroup of G.

Lemma 2.5. *Let G be a group in which every maximal subgroup has finite index, and let N be a normal subgroup of G. If X is any subgroup of N, the Frattini closure of X in N is contained in the Frattini closure of X in G.*

Proof. Let X^* be the Frattini closure of X in N. If M is any maximal subgroup of G containing X, it follows from Corollary 2.4 that $M \cap N$ is the intersection of a (possibly empty) collection of maximal subgroups of N containing X, and so

$$X^* \leq M \cap N \leq M.$$

Therefore X^* lies in the Frattini closure of X in G. □

We will prove that if G is any (generalized) soluble group with the FM-property, then every subnormal subgroup of G has finite index in its normal closure. The structure of groups with this latter property has ben investigated by C. Casolo [1], who proved that it depends on the behaviour of certain automorphism groups. Recall that an automorphism of a group G is called a *power automorphism* if it maps every subgroup of G onto itself. It is known that the set of all power automorphisms of a group G is an abelian residually finite subgroup of the full automorphism group of G (see [2]). We state here the following result, which can be obtained just combining the statements of Lemma 2.7 and Theorem 2.11 of [1].

Lemma 2.6. *Let A be an abelian group, and let T be the subgroup consisting of all elements of finite order of A. If Γ is a group of automorphisms of A such that every subgroup of A has finite index in some Γ-invariant subgroup, then Γ induces on A/T a group of power automorphisms. Moreover, T contains Γ-invariant subgroups E and D such that $E \leq D$, E is finite, D/E is divisible and Γ induces groups of power automorphisms on D/E and T/D.*

Using Lemma 2.6, it is easy to prove the following result on groups in which all subnormal subgroups have finite index in their normal closure.

Corollary 2.7. *Let G be a hyperabelian group in which every subnormal subgroup has finite index in its normal closure. Then G is soluble.*

Proof. The Fitting subgroup F of G is nilpotent (see [1], Lemma 3.1), so that F/F' is the Fitting subgroup of G/F', and hence replacing G by G/F' it can be assumed without loss of generality that F is abelian. Let T be the subgroup consisting of all elements of finite order of F. By Lemma 2.6, there exists in F a series of G-invariant subgroups

$$\{1\} \leq E \leq D \leq T \leq F$$

such that E is finite and G induces groups of power automorphisms on D/E, T/D and F/T. Consider the stabilizer in G of the above series, i.e., put

$$C = C_G(E) \cap C_G(D/E) \cap C_G(T/D) \cap C_G(F/T).$$

It is well-known that $C/C_C(F)$ is nilpotent, and so C is soluble because $C_C(F) = F$. On the other hand, any group of power automorphisms is abelian, and hence the factor group G/C is soluble. Therefore G itself is a soluble group. □

Since the Frattini subgroup $\Phi(G)$ of a group G is the Frattini closure of the identity subgroup of G, it follows that any FM-group has finite Frattini subgroup. Moreover, it is clear that the class of FM-groups is closed under homomorphic images, and hence in particular an abelian group G has the FM-property if and only if all its factor groups have finite Frattini subgroup.

Lemma 2.8. *Let G be a group, and let N be a finite normal subgroup of G. If X is a subgroup of G such that the product XN is nearly Frattini closed in G, then X itself is nearly Frattini closed.*

Proof. As the subgroup XN has finite index in its Frattini closure K in G, also the index $|K : X|$ is finite. On the other hand, the Frattini closure $\hat{X}$ of X in G is obviously contained in K, so that X has finite index in $\hat{X}$, i.e. it is nearly Frattini closed in G. □

As a direct consequence of Lemma 2.8, the following result shows that the obstacle produced by a finite normal subgroup is easily overcome. It proves in particular that all finite-by-IM groups have the FM-property.

Corollary 2.9. *Let G be a group containing a finite normal subgroup N such that the factor group G/N has the FM-property. Then G is an FM group.*

Our next two lemmas deal with the structure of groups in which every subnormal subgroup has finite index in its Frattini closure.

Lemma 2.10. *Let G be a group whose subnormal subgroups are nearly Frattini closed. Then G has no divisible abelian non-trivial normal subgroups.*

Proof. Assume for a contradiction that A is a divisible abelian non-trivial normal subgroup of G. As the Frattini subgroup of G is finite, there exists a maximal subgroup M of G which does not contain A. Let a be an element of $A \setminus M$. As the cyclic subgroup $\langle a \rangle$ is subnormal in G with defect at most 2, it follows from Corollary 2.2 that the normal closure $\langle a \rangle^G$ is contained in the Frattini closure of $\langle a \rangle$. Thus the index $|\langle a \rangle^G : \langle a \rangle|$ is finite, and so $\langle a \rangle^G$ is finitely generated. Moreover, $G = \langle a \rangle^G M$, and so

$$A = \langle a \rangle^G M \cap A = \langle a \rangle^G (A \cap M).$$

It follows that also the non-trivial group $A/A \cap M$ is finitely generated, contradicting the hypothesis that A is divisible. □

Lemma 2.11. *Let G be a group whose subnormal subgroups are nearly Frattini closed. Then every abelian chief factor of G is finite.*

Proof. It is obviously enough to prove that any abelian minimal normal subgroup N of G is finite. Since N cannot be divisible by Lemma 2.10, it must have prime exponent. Let x be a non-trivial element of N. Then $\langle x \rangle$ is a finite subnormal subgroup of G with defect at most 2, so that $N = \langle x \rangle^G$ is contained in the Frattini closure of $\langle x \rangle$ and hence it is finite. □

It is well-known that if G is a hyper-(abelian or finite) group whose chief factors are finite, then every maximal subgroup of G has finite index. Thus our next result is a consequence of Lemma 2.11 and Lemma 2.5. It shows that within the universe of hyper-(abelian or finite) groups the property FM is inherited by normal subgroups.

Corollary 2.12. *Let G be a hyper-(abelian or finite) FM-group, and let N be a normal subgroup of G. Then N has the FM-property.*

Lemma 2.13. *Let G be a hyper-(abelian or finite) FM-group. Then every subnormal subgroup of G has finite index in its normal closure.*

Proof. Let X be any subnormal subgroup of G with defect $s \geq 2$. Then the subgroup $Y = X^{X^G}$ is subnormal in G with defect 2, and hence its normal closure $Y^G = X^G$ is contained in the Frattini closure $\hat{Y}$ of Y in G by Corollary 2.2. It follows that the index $|X^G : Y|$ is finite. Moreover, X^G has the FM-property by Corollary 2.12, and the subnormal defect of X in X^G is $s-1$, so that it can be assumed by induction that X has finite index in Y. Therefore also the index $|X^G : X|$ is finite, and the statement is proved. □

The last result of this section shows in particular that hyperabelian groups with the FM-property are soluble and finite-by-metabelian.

Lemma 2.14. *Let G be a hyperabelian-by-finite FM-group. Then G is soluble-by-finite and its largest soluble normal subgroup is finite-by-metabelian.*

Proof. As G is hyperabelian-by-finite, it contains a largest hyperabelian normal subgroup H, and the index $|G : H|$ is finite. Every subnormal subgroup X of H has finite index in its normal closure in G by Lemma 2.13, and so also the index $|X^H : X|$ is finite. Therefore H is soluble by Corollary 2.7. Moreover, the subgroup H is finite-by-metabelian (see [1], Corollary 3.7). □

It was mentioned in the introduction that every soluble IM-group is periodic. In contrast to this property, observe that there exist metabelian groups with the property FM for which even the commutator subgroup is non-periodic. In fact, if A is any finitely generated torsion-free abelian (non-trivial) group, it is easy to show that the semidirect product $G = \langle x \rangle \ltimes A$, where $x^2 = 1$ and $a^x = a^{-1}$ for all a in A, is an FM-group, but its commutator subgroup $G' = A^2$ is torsion-free.

3. **Locally nilpotent FM-groups**

The first result of this section proves that any locally nilpotent group whose normal subgroups are nearly Frattini closed has the FM-property, so that in particular all extraspecial groups are FM. Moreover, it shows that locally nilpotent FM-groups are nilpotent, and reduces the study of such groups to the abelian case.

Lemma 3.1. *Let G be a locally nilpotent group. The following statements are equivalent:*

(a) *G has the FM-property.*

(b) *Every homomorphic image of G has finite Frattini subgroup.*

(c) *The commutator subgroup G' of G is finite and the abelian group G/G' has the FM-property.*

Proof. As the class of FM-groups is closed under homomorphic images, (b) is an obvious consequence of (a). Suppose now that statement (b) holds. Then in particular the Frattini subgroup $\Phi(G)$ of G is finite, so that also G' is finite, since the factor group $G/\Phi(G)$ is abelian. Moreover, it is also clear that the abelian group G/G' has the FM-property. Finally, if G' is finite and G/G' is an FM-group, it follows from Lemma 2.9 that G likewise has the FM-property. □

The above lemma allows us to describe locally nilpotent FM-groups which are either torsion-free or finitely generated.

Corollary 3.2. *Let G be a torsion-free locally nilpotent group. If G has the FM-property, then it is abelian of finite rank.*

Proof. As the group G is abelian by Lemma 3.1, and it has no divisible non-trivial sections by Lemma 2.10, it follows that G has finite rank. □

Corollary 3.3. *Let G be a finitely generated nilpotent group. Then G has the FM-property if and only if its commutator subgroup is finite.*

Proof. Since any finitely generated abelian group has finite Frattini subgroup, it has the FM-property. Therefore the statement follows from Lemma 3.1. □

Our next result deals with the case of periodic abelian groups.

Theorem 3.4. *Let G be a periodic abelian group. Then G has the FM-property if and only if $G = E \times V$, where E is a finite subgroup and V is a subgroup whose elements have square-free order.*

Proof. Suppose first that G is an FM-group, and for each prime number p, let G_p be the p-component of G. Consider the set π of all primes p such that $(G_p)^p \neq \{1\}$. As

$$\Phi(G) = \operatorname*{Dr}_{p\in\pi}(G_p)^p,$$

the property FM yields that the set π contains only finitely many primes, and the subgroup $(G_p)^p$ is finite for each $p \in \pi$. It follows that

$$G_\pi = \operatorname*{Dr}_{p\in\pi} G_p = E \times U,$$

where E is finite and the (non-trivial) primary components of U have prime exponent. Then $G = E \times V$, where $V = U \times G_{\pi'}$, and it is clear that all elements of V have square-free order.

Conversely, if the group $G = E \times V$ has a direct decomposition as in the statement, every homomorphic image of G has finite Frattini subgroup, and hence G has the FM-property. □

It is well-known that an arbitrary abelian group need not split over the subgroup consisting of all its elements of finite order. On the other hand, it can be proved that such property holds for all abelian FM-groups.

Theorem 3.5. *Let G be an abelian group with the FM-property. Then $G = T \times A$, where T is periodic and A is torsion-free of finite rank.*

Proof. Let T be the subgroup consisting of all elements of finite order of G. As T is an FM-group, it follows from Theorem 3.4 that

$$T = E \times V,$$

where E is finite and all elements of V have square-free order. Moreover, the torsion-free group G/T has finite rank by Corollary 3.2, and so G contains a finitely generated torsion-free subgroup U such that G/U is periodic. Another application of Theorem 3.4 yields that

$$G/U = L/U \times W/U,$$

where L/U is finite and all elements of W/U have square-free order. Put $|E| = m$ and $|L/U| = n$, and let π be the set of all prime divisors of mn. The set G_π, consisting of all π-elements of T, is a pure subgroup of finite exponent of G, and hence $G = G_\pi \times Y$ for some subgroup Y. Write $U_0 = Y \cap G_\pi U$. Then

$$U_0 \cap T = Y \cap G_\pi U \cap T = Y \cap G_\pi = \{1\},$$

and hence U_0 is torsion-free. Moreover, U_0 is finitely generated and Y/U_0 is isomorphic to a section of G/U, so that

$$Y/U_0 = H/U_0 \times K/U_0,$$

where H/U_0 is finite with order dividing n and all elements of K/U_0 have square-free order. The intersection $T_0 = T \cap Y$ is a π'-group, and so it is contained in K. In particular, H is torsion-free and

$$K/U_0 = T_0U_0/U_0 \times R/U_0$$

for some subgroup R. Put $A = HR$. Then

$$Y = HK = HT_0R = (HR)T_0 = AT_0$$

and

$$A \cap T_0 = HR \cap K \cap T_0 = R(H \cap K) \cap T_0 = R \cap T_0 = \{1\}.$$

Therefore $Y = A \times T_0$, and hence

$$G = G_\pi \times Y = G_\pi \times T_0 \times A = T \times A,$$

where the subgroup A is torsion-free, and it follows from Corollary 3.2 that A has finite rank. □

The above theorem reduces the study of abelian groups with the FM-property to the case of torsion-free groups of finite rank. The last result of this section classifies FM-groups of rank 1.

Theorem 3.6. *Let G be a torsion-free abelian group of rank 1. Then G has the FM-property if and only if there exist an infinite set π of prime numbers and a cyclic non-trivial subgroup C of G such that G/C is a π'-group whose non-trivial primary components have prime order.*

Proof. Suppose first that G has the FM-property. Since all homomorphic images of G are likewise FM-groups, it follows from Theorem 3.4 that G contains a cyclic non-trivial subgroup C such that all non-trivial primary components of the periodic group G/C have prime order. Let π be the set of all prime numbers p such that G/C has no elements of order p. Then

$$\Phi(G) = \bigcap_{p \in \pi} C^p,$$

and so the set π is infinite, because G has trivial Frattini subgroup.

Conversely, under the hypotheses of the statement, the group G has trivial Frattini subgroup. Moreover, every proper homomorphic image of G is periodic and has the FM-property by Theorem 3.4. Therefore G is an FM-group. □

Let π be an infinite set of prime numbers such that also the complementary set of π in the set of all prime numbers is infinite. In the additive group of rational numbers consider the subgroup

$$G = \langle 1/p \mid p \in \pi \rangle.$$

Then G is a torsion-free group of rank 1 with the FM-property, and it is not finitely generated.

4. Groups with the bounded FM-property

Let $k \geq 1$ be an integer. We shall say that a group G is an *FM_k-group* (or that it has the *FM_k-property*) if $|\hat{X} : X| \leq k$ for every subgroup X of G. A group satisfying the FM_k-property for some k will be called a *BFM-group*. Clearly, G is an FM_1-group if and only if it is IM. Obviously, an abelian group G is BFM if and only if the Frattini subgroup of every homomorphic image of G is finite with bounded order. In particular, the additive group of integers does not have the BFM-property.

The following result shows that, at least in the case of (generalized) soluble groups, also the property FM_k is inherited by normal subgroups.

Lemma 4.1. *Let G be a hyper-(abelian or finite) FM_k-group, and let N be a normal subgroup of G. Then N has the FM_k-property.*

Proof. Let X be any subgroup of N, and let X^* be the Frattini closure of X in N. Since every maximal subgroup of G has finite index by Lemma 2.11, it follows from Lemma 2.5 that X^* is contained in the Frattini closure of X in G. Therefore $|X^* : X| \leq k$, and hence N is an FM_k-group. □

As a consequence of the above lemma, we obtain that (generalized) soluble groups with the BFM-property are periodic.

Corollary 4.2. *Let G be a hyper-(abelian or finite) group with the BFM-property. Then G is periodic.*

Proof. The statement follows from Lemma 4.1, since the BFM-property does not hold for infinite cyclic groups. □

Recall that a group G is said to be a *T-group* if normality in G is a transitive relation, or equivalently if all subnormal subgroups of G

are normal. Of course, every simple group is a T-group, but the structure of soluble T-groups was successfully investigated by W. Gaschütz [3] in the finite case and by D.J.S. Robinson [7] for arbitrary groups. Among many relevant results, they proved in particular that all soluble T-groups are metabelian; moreover, if G is any periodic soluble T-group and $L = [G', G]$, then there are no odd primes in the set $\pi(L) \cap \pi(G/L)$ and the 2-component L_2 of the abelian group L is divisible (here - for any group X - the symbol $\pi(X)$ denotes the set of all prime numbers p such that the group X has elements of order p).

Consider now a soluble IM-group G. As we mentioned before, an application of Corollary 2.2 yields that G is a T-group. On the other hand, it is clear that nilpotent IM-groups are abelian and that an IM-group cannot contain divisible abelian non-trivial normal subgroups. Therefore $G' = [G', G]$ is the last term of the lower central series of G, and $\pi(G') \cap \pi(G/G') = \emptyset$. Our next result generalizes this remark to metabelian groups with the BFM-property, and proves in particular that in this case the set $\pi(G') \cap \pi(G/G')$ is finite.

Lemma 4.3. *Let k be a positive integer, and let G be a metabelian FM_k-group. Then the intersection $\pi(G') \cap \pi(G/G')$ is contained in the set π_k, consisting of all prime numbers $p \leq k$.*

Proof. Let $q > k$ be a prime number in the set $\pi(G/G')$, and let U be the subgroup consisting of all q'-elements of G'. In order to prove that G' has no elements of order q, we may replace G by the factor group G/U, and so assume that G' is a q-group. Then G contains a unique Sylow q-subgroup Q, and the FM_k-property holds for Q by Lemma 4.1. In particular, the Frattini subgroup of Q has order at most k, and so $\Phi(Q) = \{1\}$ since $q > k$. Therefore Q is abelian of exponent q. If X is any subgroup of Q, it follows from Corollary 2.2 that $|X^G : X| \leq k$, so that $X^G = X$ and all subgroups of Q are normal in G. As the factor group G/Q is abelian, it is known that G is a T-group (see [7], Lemma 5.2.2), and hence the odd prime q cannot belong to $\pi(G')$ (see [7], Theorem 4.2.2). The lemma is proved. □

Lemma 4.4. *Let G be a metabelian BFM-group, and let p be a prime number such that the p-component of G' is infinite. Then the p-component of G/G' is finite.*

Proof. Let U be the p'-component of G'. Replacing G by the factor group G/U, it can be assumed without loss of generality that G' is a p-group. Thus G has a unique Sylow p-subgroup P, and P has the BFM-property by Lemma 4.1. In particular, the Frattini subgroup $\Phi(P)$ is finite, and so P is nilpotent. It follows from Lemma 2.13 that every subgroup of P has finite index in its normal closure in G, and hence there exists a finite G-invariant subgroup N of P containing $\Phi(P)$ such that G induces on P/N a group of power automorphisms (see [1], Lemma 2.9). As $G'N/N$ is infinite, the group G/N contains a subgroup X/N having infinite index in its normal closure (see [6]). Again by Lemma 2.13, the subgroup X cannot be subnormal in G, and so G/N is not nilpotent. Therefore P/N is not contained in the centre of G/N, so that $[P,G]N = P$ and hence the p-component P/G' of G/G' is finite. □

We can finally prove our main result on (generalized) soluble groups with the BFM-property. It shows that they are close to IM-groups, with the obstruction of two finite sections.

Theorem 4.5. *Let G be a hyperabelian-by-finite BFM-group. Then G is finite-by-IM-by-finite.*

Proof. By Lemma 2.14, the group G is soluble-by-finite, and if S is the largest soluble normal subgroup of G, then the second commutator subgroup S'' of S is finite. Moreover, it follows from Lemma 4.1 that S/S'' is a BFM-group. Since it is clearly enough to prove that the group S/S'' if finite-by-IM-by-finite, we may suppose without loss of generality that G is metabelian. Then every subgroup of G' has finite index in its normal closure in G by Lemma 2.13. Since G has no divisible abelian non-trivial normal subgroups by Lemma 2.10, it follows from Lemma 2.7 that G' contains a finite G-invariant subgroup E such that all subgroups of G'/E are G-invariant. Replacing now G by G/E, it can also be assumed that all subgroups of G' are normal in G.

The set of prime numbers $\pi = \pi(G') \cap \pi(G/G')$ is finite by Lemma 4.3. Consider the subset π_0 of π consisting of all primes p for which G' has a finite p-component, and put $\pi_1 = \pi \setminus \pi_0$. Then the direct product K of all primary components of G' relative to primes in π_0 is a finite characteristic subgroup of G'. Moreover, it follows from Lemma 4.4 that

$$G/G' = U/G' \times V/G',$$

where U/G' is a finite π_1-subgroup and the intersection

$$\pi(G'/K) \cap \pi(V/G')$$

is empty. Moreover, the Frattini subgroup of V/G' is finite, so that

$$V/G' = W/G' \times Z/G',$$

where W/G' is finite and all elements of Z/G' have square-free order. Let K^* be the Frattini closure of K in G'. Then K^*/K is the Frattini subgroup of G'/K, so that every (non-trivial) primary component of G'/K^* has prime exponent, and hence Z/K^* is an IM-group (see [9], Theorem 3.3.10). As K^* is finite and Z has finite index in G, the group G is finite-by-IM-by-finite. The proof is complete. □

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