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ON FINITE X-DECOMPOSABLE GROUPS FOR $X = \{1, 2, 3, 4\}$

X. GUO* AND R. CHEN

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Abstract. Let $\mathcal{N}_G$ denote the set of all proper normal subgroups of a group G and A be an element of $\mathcal{N}_G$. We use the notation $ncc(A)$ to denote the number of distinct G-conjugacy classes contained in A and also $\mathcal{K}_G$ for the set $\{ncc(A) \mid A \in \mathcal{N}_G\}$. Let X be a non-empty set of positive integers. A group G is said to be X-decomposable, if $\mathcal{K}_G = X$. In this paper we give a classification of finite X-decomposable groups for $X = \{1, 2, 3, 4\}$.
Keywords: n-decomposable, X-decomposable, G-conjugacy classes.
MSC(2010): Primary: 20D10; Secondary: 20D20.

1. Introduction

All groups in this paper are finite. The relation between the structure of a group and the cardinality of its conjugacy classes has already been extensively studied (see, e.g., [7–9, 12, 19]). Let G be a group and N be a normal subgroup of G. Then N is a union of G-conjugacy classes contained in N, and some authors hope to investigate the structure of a normal subgroup if it is a union of a *small* number of G-conjugacy classes (see, e.g., [1, 13, 16]). Furthermore, some authors hope to determine the structure of a group if every non-trivial normal subgroup is a union of a given number of G-conjugacy classes (see, e.g., [2, 3, 10]).

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Let n be a positive integer. Recall that a normal subgroup N of a group G is called n-decomposable if it is a union of n distinct G-conjugacy classes, and a group G is called an n-decomposable group if it is not simple and its every non-trivial normal subgroup is n-decomposable. Up to now, 2-, 3-, 7-, 8-, 9- and 10-decomposable normal subgroups have been investigated (see [5, 6, 17] and [18]) and the authors in [2] give some properties for finite n-decomposable groups. Furthermore, they classify finite n-decomposable groups for $n = 2, 3, 4$ in the same paper.

Let G be a group. For convenience, we use $\mathcal{N}_G$ to denote the set of all proper normal subgroups of G. If A is an element of $\mathcal{N}_G$, then we use $ncc(A)$ to denote the number of distinct G-conjugacy classes contained in A. Furthermore, suppose that X is a non-empty set of positive integers and $\mathcal{K}_G = \{ncc(A) \mid A \in \mathcal{N}_G\}$. A group G is said to be X-decomposable if $\mathcal{K}_G = X$. A. R. Ashrafi in [3] raised the following question:

Question. [3, Question 2.7] *Suppose that X is a finite subset of positive integers containing 1. Is there a finite X-decomposable group G?*

Now X-decomposable groups have been classified for $X = \{1, 2, 3\}$, $\{1, 3, 4\}$ and $\{1, 2, 4\}$. They are as follows:

Theorem A. [4] *Let G be a finite non-perfect $\{1, 2, 3\}$-decomposable group. Then G is isomorphic to Z_6, D_8, Q_8, S_4, SmallGroup(20, 3) or SmallGroup(24, 3).*

Theorem B. [3] *Let G be a finite non-perfect $\{1, 3, 4\}$-decomposable group. Then G is isomorphic to SmallGroup(36, 9), a metabelian group of order $2^n(2^{\frac{n-1}{2}} - 1)$, in which n is an odd positive integer and $2^{\frac{n-1}{2}} - 1$ is a Mersenne prime or a metabelian group of order $2^n(2^{\frac{n}{3}} - 1)$ where $3|n$ and $2^{\frac{n}{3}} - 1$ is a Mersenne prime.*

Theorem C. [10] *Let G be a finite non-perfect $\{1, 2, 4\}$-decomposable group. Then G is isomorphic to Q_{12}, $Z_2 \times A_4$ or $G = \langle a, b, c \mid a^{11} = b^5 = c^2 = 1, b^{-1}ab = a^4, c^{-1}ac = a^{-1}, c^{-1}bc = b^{-1}\rangle$.*

We note here that SmallGroup(n, i) in Theorem A and Theorem B is the i^{th} group of order n in the small group library of GAP (see [15]).

In this paper, we continue to study the above question for the case $X = \{1, 2, 3, 4\}$ and give the classification of non-perfect $\{1, 2, 3, 4\}$-decomposable groups. Our main result is as follows:

Main Theorem. *Let G be a finite non-perfect $\{1, 2, 3, 4\}$-decomposable group. Then G is one of the following groups:*

(1) $|G| = 216$ *and* $G = \langle a, b, c, d, e, f \mid a^3 = d^2 = e^3 = f^3 = 1, b^2 = c^2 = d, b^a = cd, c^a = bc, c^b = cd, e^a = f^2, e^b = e^2f, e^c = f^2, e^d = e^2, f^a = ef^2, f^b = ef, f^c = e, f^d = f^2\rangle$.

(2) $|G| = 600$ *and* $G = \langle a, b, c, d, e, f \mid a^3 = d^2 = e^5 = f^5 = 1, b^2 = c^2 = d, b^a = bc, c^a = b, c^b = cd, e^a = ef^3, e^b = e^3f^3, e^c = e^3, e^d = e^4, f^a = e^4f^3, f^b = f^2, f^c = e^4f^2, f^d = f^4\rangle$.

(3) $|G| = 42$ *and* $G = \langle a, b \mid a^7 = b^6 = 1, b^{-1}ab = a^5\rangle$.

(4) $G = D_{12}$.

Let G be a finite group. Throughout this paper, $G', \Phi(G), Z(G)$ and $exp(G)$ denotes the derived subgroup, the Frattini subgroup, the center and the exponent of G, respectively. A group G is said to be non-perfect if $G' \neq G$. If x is an element in G, then $x^G = \{x^g \mid g \in G\}$ is the G-conjugacy class containing x. Furthermore, Z_n denotes the cyclic group of order n, $E(p^n)$ denotes the elementary abelian group of order p^n and $d(n)$ denotes the set of all positive divisors of n. We always assume that $X = \{1, 2, 3, 4\}$ in the next two sections.

2. **Preliminaries**

In this section, we list some fundamental facts which are useful in the sequel.

Example 2.1. [3, Example 2.5] *Let* $G = \langle a, b \mid a^6 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$ *be the dihedral group of order 12. Then* $\mathcal{N}_G = \{1, H = \langle a^2, b\rangle, K = \langle a^2, ab\rangle, \langle a\rangle, \langle a^2\rangle, \langle a^3\rangle\}$. *It is easy to see that* $\langle a^2\rangle$ *and* $\langle a^3\rangle$ *are 2-decomposable,* H *and* K *are 3-decomposable and* $\langle a\rangle$ *is 4-decomposable. Therefore,* G *is* X*-decomposable.*

Lemma 2.2. [10, Example 2.1] *Let* G *be an abelian group of order* n *and* $Y = d(n) - \{n\}$. *Then* G *is* Y*-decomposable.*

Corollary 2.3. *There is no finite abelian* X*-decomposable group.*

Lemma 2.4. *There is no finite* X*-decomposable group of prime power order.*

Proof. Suppose that there is a prime p such that G is a p-group. Then $p = 2$ by [17, Theorem 1(3)]. Assume that $|G| = 2^n$ for some integer n. There is a chief series

$$1 = G_0 < G_1 < \cdots < G_{n-1} < G_n = G$$

in G such that $|G_i| = 2^i$ for $i = 1, 2, \cdots, n$. As G is X-decomposable, we have $n = 4$.

Since $Z(G) \neq 1$ and G is non-abelian by Corollary 2.3, $Z(G)$ can not be 4-decomposable in G. Furthermore, if $Z(G)$ is 3-decomposable in G, then $|Z(G)| = 3$, contradicting that G is a 2-group. Therefore, $Z(G)$ is 2-decomposable in G, and thus $|Z(G)| = 2$. Let H be a 3-decomposable normal subgroup of G. As $Z(G) \cap H \neq 1$ and $Z(G)$ is a minimal subgroup of G, we have that $Z(G) < H$ and $|H| = 4$. Suppose that $H = Z(G) \cup x^G$. Then $|x^G| = 2$ and thus $|C_G(x)| = 8$. So $C_G(x)$ is normal in G. Since $\langle Z(G), x\rangle \leq Z(C_G(x))$, $C_G(x)$ is abelian. Therefore,

$$1 < Z(G) < H < C_G(x) < G$$

is a chief series of G. Let $C_G(x) = H \cup y^G$. Then $|y^G| = |C_G(x)| - |H| = 4$, and thus $|C_G(y)| = 4$, which contradicts the fact that $C_G(x)$ is abelian and $y \in C_G(x)$. □

Lemma 2.5. *Let G be a finite X-decomposable group such that G' is a Sylow 2-subgroup of G. Suppose that G' is 4-decomposable in G and that $Z(G) = Z(G')$ is of order 2. Then $Z(G)$ is contained in every non-trivial normal subgroup of G.*

Proof. As G' is a Sylow 2-subgroup of G, then it is solvable.

Let N be a non-trivial proper normal subgroup of G. We claim that $N \leq G'$. In fact, by the hypothesis, one can see that G' is a maximal subgroup of G. If $N \nleq G'$, then $G = G'N$ and thus $(G/N)' = G/N$, which contradicts that G/N is solvable.

Now, it is easy to see that $Z(G) = Z(G') \leq N$ since $N \cap Z(G') \neq 1$ and $|Z(G')| = 2$.

□

Lemma 2.6. [10, Lemma 2.1] *Suppose that p and q are primes and n is a positive integer such that $p^n = 1 + 3q$. Then $p = 7, n = 1$, and $q = 2$ or $p = 2, n = 4$, and $q = 5$.*

Lemma 2.7. *If n is a positive integer and $n \geq 2$, then there is no odd prime q such that $q^2 = 2^n - 1$.*

Proof. Suppose that there exists an odd prime q such that $q^2 = 2^n - 1$. Then there exist positive integers l and t such that $q = 2^l \cdot t + 1$. Therefore, $2^n = q^2 + 1 = (2^l \cdot t + 1)^2 + 1 = 2^{2l} \cdot t^2 + 2^{l+1} \cdot t + 2$. It follows that $2^{2l-1} \cdot t^2 + 2^l \cdot t + 1 = 2^{n-1}$, which is a contradiction.

□

Lemma 2.8. [10, Lemma 2.2] *There is no prime p such that $2p + 1$ is also a prime and that $2p^2 + p + 1 = 2^n$ for some positive integer n.*

3. Proof of the main theorem

In this section, we will give the proof of our main theorem. We have shown in Corollary 2.3 and Lemma 2.4 that G is neither an abelian group nor a group of prime power order if G is an X-decomposable group. Also an X-decomposable group must be of even order by [17, Theorem 1(3)], and we will use these facts frequently in the proofs.

We first give the following three theorems.

Theorem 3.1. *Let G be a finite non-perfect X-decomposable group. If G' is 4-decomposable in G, then G is one of the following two groups:*

(1) $|G| = 216$ and $G = \langle a, b, c, d, e, f \mid a^3 = d^2 = e^3 = f^3 = 1, b^2 = c^2 = d, b^a = cd, c^a = bc, c^b = cd, e^a = f^2, e^b = e^2 f, e^c = f^2, e^d = e^2, f^a = ef^2, f^b = ef, f^c = e, f^d = f^2\rangle$.

(2) $|G| = 600$ and $G = \langle a, b, c, d, e, f \mid a^3 = d^2 = e^5 = f^5 = 1, b^2 = c^2 = d, b^a = bc, c^a = b, c^b = cd, e^a = ef^3, e^b = e^3 f^3, e^c = e^3, e^d = e^4, f^a = e^4 f^3, f^b = f^2, f^c = e^4 f^2, f^d = f^4\rangle$.

Proof. Since G' is 4-decomposable in G, G' must be one of the following groups by [13, Theorem 1] and [13, Theorem 2]:

1) $G' \cong A_5$, the alternating group of degree 5, and $G/C_G(G') \cong S_5$.

2) G' is a p-group for some prime p and $G''' = 1$.

3) G' is a group of order $p^n q^b$, where p and q are distinct primes, and n and b are positive integers.

Furthermore, if G' is of type 3), then G' has the following three possibilities.

(A) G' is the direct product of its elementary abelian Sylow p- and q-subgroups.

(B) G' is a Frobenius group with kernel N and $G'/N \cong Z_q$ or Z_{q^2} or Q_8, where N is 2-decomposable in G.

(C) G' is a Frobenius group with kernel N and $G'/N \cong Z_q$, where N is 3-decomposable in G.

Case 1. $G' \cong A_5$ and $G/C_G(G') \cong S_5$.

As A_5 is centerless, $G' \cap C_G(G') = Z(G') \cong Z(A_5) = 1$. If $C_G(G') \neq 1$, then $G = G'C_G(G')$ since G' is a maximal normal subgroup of G. Therefore, $S_5 \cong G/C_G(G') \cong G' \cong A_5$, which is a contradiction. If $C_G(G') = 1$, then $G \cong S_5$. However, S_5 does not contain a 2-decomposable normal subgroup. Hence this case is impossible.

Case 2. G' is a p-group for some prime p and $G''' = 1$.

Assume that $|G'| = p^n$ for some positive integer n. As G' is a maximal subgroup of G and G is not of prime power order, there is a prime $q \neq p$ such that $|G| = p^n q$. It follows that G is solvable. Arguing similarly as in the proof of Lemma 2.5, we see that every proper normal subgroup of G is contained in G'. Recall that G is non-abelian, $Z(G)$ can not be maximal in G, so we conclude that $Z(G) < G'$.

If G' is abelian, then $G' \leq C_G(x)$ for every $x \in G'$, and so $|x^G| = 1$ or q. It follows that $p^n = 1 + 1 + 1 + q$ or $p^n = 1 + 1 + q + q$ or $p^n = 1 + q + q + q$. If $p^n = 1 + 1 + 1 + q$, then $|Z(G)| = 3$ and $p = 3$. Therefore, $q = 2$ as G is of even order, and thus $3^n = 5$, which is impossible. If $p^n = 1 + 1 + q + q$, then $|Z(G)| = 2$ and so $p = 2$. Let Q be a Sylow q-subgroup of G. Then Q acts on the abelian group G', and thus $G' = Z(G) \times [G', Q]$. It is clear that $[G', Q]Q$ is a normal subgroup of G and so $G = Z(G) \times [G', Q]Q$. It follows that $G' = [G', Q]$, leading that $Z(G) = 1$, which is a contradiction. If $p^n = 1 + q + q + q$, then $p = 7, n = 1$, and $q = 2$ or $p = 2, n = 4$, and $q = 5$ by Lemma 2.6. If $p = 7, n = 1$ and $q = 2$, then $|G| = 14$. It is easy to see that G has no 2-decomposable normal subgroup. If $p = 2, n = 4$ and $q = 5$, then we can choose H to be a 2-decomposable normal subgroup of G. As $H \leq G'$, we may assume that $|H| = 2^t$, then $2^t = 1 + q = 6$, which is impossible. Consequently, we conclude that G' is non-abelian.

If $1 < G'' < Z(G') < G'$, then there exist positive integers $1 < s < t$ such that $|G''| = p^s$ and $|Z(G')| = p^t$. Let $Z(G') = G'' \cup x^G$. Then $G' = C_G(x)$, and thus $|x^G| = q$. It follows that $p^t = p^s + q$, which gives the contradiction that p divides q.

If $1 < Z(G') < G'' < G'$, then $G'' = \Phi(G')$ and there exist positive integers $1 < s < t$ such that $|Z(G')| = p^s$ and $|G''| = p^t$. Let $Z(G') = 1 \cup x^G, G'' = Z(G') \cup y^G, G' = G'' \cup z^G$. Assume that $Z(G) = 1$. If $p = 2$, then $|x^G| = 2^s - 1 = q$ and $|z^G| = 2^t(2^{n-t} - 1)$. As G' is non-abelian and $|G'/\Phi(G')| = 2^{n-t}$, we have $n - t > 1$. Therefore, $q = 2^{n-t} - 1$ and $n - t = s$. It follows that $|C_G(z)| = 2^{n-t} = 2^s$, which is contrary to that $Z(G') < \langle Z(G'), z \rangle \leq C_G(z)$. Hence $q = 2$. Then $p^s = 1 + |x^G| = 3$, $|y^G| = 3(3^{t-1} - 1)$ and $|z^G| = 3^t(3^{n-t} - 1)$. Therefore, $t = 2, n = 3$ and $|G| = 54$. Recall that $G' = G'' \cup z^G$, then $|C_G(z)| = 3$, which contradicts $Z(G') \neq 1$. So $Z(G) \neq 1$. Note that $Z(G) \leq Z(G')$, implies $Z(G) = Z(G')$ is 2-decomposable in G, and thus $Z(G) = Z(G')$ is of order 2. Now consider the factor group $\overline{G} = G/Z(G)$. It is easy to see that $\overline{G}$ is $\{1, 2, 3\}$-decomposable. Then $|G| = 40$ or 48 by Theorem A, Corollary 2.3 and Lemma 2.4. First, suppose that

$|G| = 40$, then $\Phi(G') = G''$. It follows that $|G'/\Phi(G')| = 2$, which implies the contradiction that G' is abelian. Suppose that $|G| = 48$. By arguing similarly as in the former case, we see that $|G'/G''| \neq 2$, hence $|G'/G''| = 4$. Suppose that $G'' = Z(G') \cup u^G$. Then $|u^G| = 2$, and thus $C_G(u) \trianglelefteq G$. It follows that $G' \leq C_G(u)$, which contradicts that $u \notin Z(G')$.

From the above two paragraphs we conclude that if $Z(G') \neq G''$, then $Z(G')G''$ is a 4-decomposable normal subgroup of G, and thus $G' = Z(G')G''$, leading the contradiction that $G'' = G''' = 1$. Hence $Z(G') = G''$.

If $|Z(G)| = 3$, then $p = 3, q = 2$ and $Z(G) = Z(G')$. Let T be a 2-decomposable normal subgroup of G. Keeping in mind that $T \leq G'$, we have $T \cap Z(G') \neq 1$. However, as we have seen that $|Z(G')| = 3$, then $Z(G') \leq T$, which is a contradiction.

If $|Z(G)| = 2$, then $p = 2$. If $Z(G) \neq Z(G')$, then $Z(G') = G''$ is 3-decomposable in G. Set $|Z(G')| = 2^s$. Then $2 + q = 2^s$ and therefore $q = 2 = p$, which is a contradiction. So $Z(G) = Z(G') = G''$ and thus $Z(G') = G'' = \Phi(G')$. Therefore, G' is an extraspecial 2-group and $|G'| = 2^n = 2^{2m+1}$ for some positive integer m. Consider the factor group $\overline{G} = G/Z(G)$. Then $\overline{G}$ is $\{1, 2, 3\}$-decomposable and $|G| = 40$ or 48 by Theorem A, Corollary 2.3 and Lemma 2.4. If $|G| = 40$, then $|G'| = 8$. Let K be a 3-decomposable normal subgroup of G. Then $Z(G') \leq K \leq G'$. Suppose $K = Z(G') \cup u^G$. Then $|u^G| = 2$ and thus $G' \leq C_G(u)$, which contradicts that $u \notin Z(G')$. If $|G| = 48$, then $|G'| = 16 = 2^4 = 2^{2m+1}$, another contradiction.

Therefore, $Z(G) = 1$. Suppose $|G''| = p^s$ for some positive integer s. If $G'' = Z(G')$ is 3-decomposable in G, then $1 + 2q = p^s$. Recall that $p = 2$ or $q = 2$, we have $q = 2$ and $p^s = 5$. Since $|G'| - |Z(G')| = 5^n - 5$ divides $|G| = 5^n \cdot 2$, we see that $5^{n-1} - 1$ divides 2, which is impossible. Hence, $G'' = Z(G')$ is 2-decomposable in G. Let $1 \neq g_1 \in G''$. Then $C_G(g_1) = G'$, and so $q = p^s - 1$. Suppose that $G' = Z(G') \cup g_2^G \cup g_3^G$. Then $C_G(g_i) \geq \langle g_i, Z(G') \rangle$ for $i = 2$ and 3. Therefore, $|C_G(g_2)| = p^{s+t_1} \geq p^{s+1}$ and $|C_G(g_3)| = p^{s+t_2} \geq p^{s+1}$. Hence, $p^n - p^s = \frac{p^n q}{p^{s+t_1}} + \frac{p^n q}{p^{s+t_2}}$. It follows that

$$p^s(p^{n-s} - 1) = q(p^{n-s-t_1} + p^{n-s-t_2}).$$

As G/G'' is non-abelian, the length of G/G''-conjugacy classes of each non-trivial element in G'/G'' is q. Hence $q + q = p^{n-s} - 1$ or $q = p^{n-s} - 1$. If $q + q = p^{n-s} - 1$, then by $p^s(p^{n-s} - 1) = q(p^{n-s-t_1} + p^{n-s-t_2})$, we have $2p^s q = q(p^{n-s-t_1} + p^{n-s-t_2})$. Therefore, $p^{n-2s-t_1} + p^{n-2s-t_2} = 2$, whence

$p^{n-2s-t_1} = p^{n-2s-t_2} = 1$. It follows that $t_1 = t_2$ and $n = 2s + t_1$. Since $q + q$ is even, we see that $p \neq 2$, and thus $q = 2$, $p = 5$ and $n - s = 1$. Therefore, $s + t_1 = n - s = 1$, which is a contradiction. If $q = p^{n-s} - 1$, then $n = 2s$. It follows from $p^s(p^{n-s} - 1) = q(p^{n-s-t_1} + p^{n-s-t_2})$ that $p^{n-2s-t_1} + p^{n-2s-t_2} = 1$. Hence $p = 2$ and $t_1 = t_2 = 1$. In this case, we have $2^n = |G'| = 2^s + 2^{s+1} + 2^{s+1} = 2^s \cdot 5$, which is impossible.

Case 3. G' is a group of order $p^n q^b$.

In this case, there exists a prime r such that $|G| = p^n q^b r$. It follows that G is solvable as G' is solvable. Arguing similarly as in the proof of Lemma 2.5, we have that every proper normal of G is contained in G'. We discuss the three possibilities (A), (B) and (C) described in the beginning of the proof of this theorem.

(A) $G' = P_1 \times Q_1$ with P_1 and Q_1 its elementary abelian Sylow p- and q-subgroups, respectively.

If $r = p$, then $Z(P) \cap P_1 \neq 1$ for every Sylow p-subgroup P of G. Write $K = Z(P) \cap P_1$. Then $K \leq Z(G)$ and thus $K \trianglelefteq G$. If P_1 or Q_1 is 3-decomposable in G, then $P_1 \times Q_1$ has at least 5 G-conjugacy classes, which is a contradiction. Therefore, P_1 is 2-decomposable in G, and thus $P_1 = K \leq Z(G)$. So $|P_1| = 2$ and $|G| = 4q^b$. It follows that G/Q_1 is of order 4 and thus it is abelian, which implies the contradiction that $G' \leq Q_1$. By arguing similarly, we have $r \neq q$. Therefore, $q \neq r \neq p$. If P_1 is 3-decomposable in G, then there are more that 4 G-conjugacy classes in $P_1 \times Q_1$, which is a contradiction. Therefore, P_1 is 2-decomposable in G. Similarly, we have that Q_1 is 2-decomposable in G. If $P_1 \leq Z(G)$, then $|P_1| = 2$ and $|G| = 2q^b r$. Let K be a subgroup of G with order $q^b r$. Then $K \trianglelefteq G$ and thus $G' \leq K$ since G/K is abelian of order 2, which is a contradiction. Therefore, $P_1 \nleq Z(G)$. Similarly, we have that $Q_1 \nleq Z(G)$. Therefore, $p^n - 1 = r = q^b - 1$, and thus $p = q$, another contradiction.

(B) $G' = N \rtimes H$ is a Frobenius group with kernel N and $G'/N \cong Z_q$ or Z_{q^2} or Q_8, where N is 2-decomposable in G.

(i) $G'/N \cong Z_{q^2}$.

If $r = p$, then $|G| = p^{n+1}q^2$. Let $P \in \mathrm{Syl}_p(G)$ and $N = \{1\} \cup x^G$. As $Z(P) \cap N \neq 1$, without loss of generality we may assume that $x \in Z(P)$. Therefore, $C_G(x) = P$ and $q^2 = p^n - 1$. Since, $p = 2$ or $q = 2$, we have $q = 2$ and $p^n = 5$ by Lemma 2.7. It follows that $|G| = 100$. By Sylow's Theorem, a Sylow 5-subgroup of G is normal in G, so $P \trianglelefteq G$. Since G/P

is abelian of order 4, we have that $G' \le P$, which is a contradiction by order consideration.

If $r = q$, then $|G| = p^n q^3$. As $|G/N| = q^3$, we may choose K/N to be a normal subgroup of G/N such that $|K/N| = q$. Then $|K| = p^n q$ and $|G/K| = q^2$. It follows that G/K is abelian, and thus $G' \le K$, which is again a contradiction by order consideration.

Now, we conclude that $p \ne r \ne q$. By the fact that N is 2-decomposable in G and G' is a Frobenius group, we have $|N| - 1 = p^n - 1 = q^2$ or $q^2 r$. If $p^n - 1 = q^2$, then $p = 2$ or $q = 2$. By Lemma 2.7, we have $q = 2$ and $p^n = 5$. Therefore, $|G| = 5{\cdot}2{\cdot}r$. As $r \ne 2$, there exists a normal subgroup K of G of order $5r$. It follows that G/K is abelian of order 2, leading to the contradiction that $G' \le K$. Now, suppose that $p^n - 1 = q^2 r$. Let K be a 3-decomposable normal subgroup of G. Since $K \le G'$, $N \le K$ by [14, Exercise 8.5.7]. It follows that $|K| = p^n q$. If $q \ne 2$, then $q = 3$ and $r = 2$ as $|G'| - |K| = p^n q(q-1)$ divides $p^n q^2 r$. Therefore, $|G| = 342$. Let $G' = K \cup w^G$. Then $|w^G| = 114$ and thus $|C_G(w)| = 3$. The fact that G' has abelian Sylow 3-subgroups gives $|C_{G'}(w)| \ge 9$, which is a contradiction. If $q = 2$, then $r \ne 2$ and $|G/K| = 2r$. Let T/K be a subgroup of G/K of order r. Then T is a normal subgroup of G of index 2, and thus $G' \le T$, contrary to that $|G'| = p^n 2^b$ and $|T| = p^n 2^{b-1} r$.

(ii) $G'/N \cong Z_q$.

Let $Q \in \mathrm{Syl}_q(G')$. Then $G = G' N_G(Q) = N N_G(Q)$ by the Frattini's argument. As G' is a Frobenius group, $N \cap N_G(Q) = 1$. Therefore, $G/N \cong N_G(Q)$ and $N_G(Q)$ is non-abelian. Hence $r \ne q$. If $r = p$ and $P \in \mathrm{Syl}_p(G)$, then $N \le P$ since N is a normal p-subgroup of G. It follows that $P = P \cap G = P \cap N N_G(Q) = N(P \cap N_G(Q))$ and thus $|P \cap N_G(Q)| = p$. If $1 \ne x \in Z(P) \cap G'$, then $C_G(x) = P$ since G' is a Frobenius group and thus $|G : C_G(x)| = q$. If $N \not\le Z(P)$, then there exists $y \in N - Z(P)$. In this case, $|G : C_G(y)| > q$, and therefore x and y are not conjugate in G, which contradicts the fact that N is 2-decomposable in G. Hence $N \le Z(P)$ and P is abelian. Since N is 2-decomposable in G, Q acts transitively on $N - \{1\}$. So $p^n - 1 = q$. For every $1 \ne x \in Q$, we have $C_G(x) = Q$ and $|x^G| = p^{n+1}$. As G' is 4-decomposable, $p^{n+1} + p^{n+1} = qp^n - p^n$. It follows that $q = 2p + 1$. Therefore, $p^n = 2(p+1)$, and it is easy to see that $p = 2$ and $2^n = 6$, which is impossible. Hence, $q \ne r \ne p$.

Let $R \in \mathrm{Syl}_r(G)$. If R acts trivially on G', then $R \le Z(G) \le G'$, which is a contradiction. If R acts on N trivially, then $p^n - 1 = q$ and $p^n r + p^n r = qp^n - p^n$. It follows that $p = 2, q = 2^n - 1$ and $r = 2^{n-1} - 1$.

If $n \geq 4$, then either n or $n-1$ is even. Without loss of generality we may assume that $n = 2k$ for some positive integer $k > 1$. Then $q = 2^n - 1 = (2^k - 1)(2^k + 1)$, which is a contradiction since neither $2^k - 1$ nor $2^k + 1$ is equal to 1. Therefore, $n \leq 3$. It follows that $q = 7$ and $r = 3$. So $|C_G(N)| = 24$ and $G/C_G(N)$ is abelian. It follows that $G' \leq C_G(N)$ and $N \leq Z(G') = 1$, which is a contradiction. Therefore, R does not act trivially on N. As N is 2-decomposable in G, we have $p^n - 1 = qr$. By arguing similarly we have that $p^n r + p^n r = qp^n - p^n$. In this case, $q = 2r + 1$ and $p^n = 2r^2 + r + 1$. If $r \neq 2$, then $p = 2$. However, the equation $2^n = 2r^2 + r + 1$ does not have any solution by Lemma 2.8. So $r = 2$. In this case, $q = 5, p = 11$, $n = 1$ and $|G| = 110$. Let H be a 3-decomposable subgroup of G. As $H \leq G'$, we have $N \leq H$ by [14, Exercise 8.5.7], which contradicts the fact that N is a maximal subgroup of G'.

(iii) $G'/N \cong Q_8$

In this case, $N < G''$ by [14, Exercise 8.5.7]. Therefore, $|G'| = 8p^n$ and $|G''/N| = |Q_8'| = 2$, and thus $|G''| = 2p^n$. Notice that $1 < N < G'' < G' < G$, then both $|N| - 1 = p^n - 1$ and $|G'| - |G''| = 6p^n$ divide $|G| = 8p^n r$. It follows that $r = 3$ and that $p^n - 1$ divides 24. Recall that G' is a Frobenius group, we have $p \neq 2$ and write $2 \nmid |C_G(x)|$ for every $1 \neq x \in N$. Consequently, $|G| = 600$ or 216. In both cases, we write $\overline{G} = G/N$. Then $\overline{G}$ is a $\{1, 2, 3\}$-decomposable group. It follows from Theorem A that $\overline{G}$ is isomorphic S_4 or $SmallGroup(24, 3)$. Since the derived subgroup of S_4 has index 2 in S_4, $\overline{G}$ is isomorphic to $SmallGroup(24, 3)$. Noticing that $N = F(G)$ is a minimal normal subgroup of G and that G is solvable, we have $\Phi(G) < F(G)$, whence $\Phi(G) = 1$. So there exists $H \leq G$ such that $G = NH$ and $N \cap H = 1$. Therefore, $H \cong G/N$ is isomorphic to $SmallGroup(24, 3)$. If $|G| = 216$, then we may assume that $H = \langle a, b, c, d \mid a^3 = d^2 = 1, b^2 = c^2 = d, b^a = cd, c^a = bc, c^b = cd\rangle$ and that $N = \langle e, f \mid e^3 = f^3 = 1, [e, f] = 1\rangle$. As N is normal in G, we may assume that $e^a = e^i f^j, f^a = e^k f^l, e^b = e^s f^t, f^b = e^m f^n, e^c = e^u f^v, f^c = e^w f^x$, where $i, j, k, l, s, t, m, n, u, v, w, x \in \{0, 1, 2\}$. Then $e^{a^3} = e^{b^4} = e^{c^4} = e, f^{a^3} = f^{b^4} = f^{c^4} = f, e^{b^2} = e^{c^2} = e^d \neq e, f^{b^2} = f^{c^2} = f^d \neq f, e^{ba} = e^{acd}, f^{ba} = f^{acd}, e^{ca} = e^{abc}, f^{ca} = f^{abc}, e^{cb} = e^{bcd}$ and $f^{cb} = f^{bcd}$. Therefore, the integers $i, j, k, l, s, t, m, n, u, v, w, x$ must satisfy all of the following congruence equations:

$s^2 + mt - u^2 - vw \equiv 0 \ (mod\ 3)$

$st + tn - uv - vx \equiv 0 \ (mod\ 3)$

$sm + mn - uw - wx \equiv 0 \ (mod\ 3)$

$tm+n^2-wv-x^2\equiv 0\ (mod\ 3)$
$i^3+2ijk+jkl\equiv 1\ (mod\ 3)$
$ijk+2jkl+l^3\equiv 1\ (mod\ 3)$
$i^2j+kj^2+ijl+l^2j\equiv 0\ (mod\ 3)$
$i^2k+ikl+kl^2\equiv 0\ (mod\ 3)$
$s^4+2stmn+m^2t^2+mn^2t\equiv 1\ (mod\ 3)$
$s^2tm+2smnt+t^2m^2+n^4\equiv 1\ (mod\ 3)$
$u^4+2uvwx+v^2w^2+wx^2v\equiv 1\ (mod\ 3)$
$u^2vw+2uvwx+v^2w^2+x^4\equiv 1\ (mod\ 3)$
$usi+umj+wti+wnj-iu-kv\equiv 0\ (mod\ 3)$
$vsi+vmj+xti+xnj-ju-lv\equiv 0\ (mod\ 3)$
$usi+umj+wti+wnj-iw-kx\equiv 0\ (mod\ 3)$
$vsi+vmj+xti+xnj-jw-lx\equiv 0\ (mod\ 3)$
$u^3v+2uv^2w+2v^2wx+u^2xv+uvx^2+x^3v\equiv 0\ (mod\ 3)$
$u^3w+u^2wx+2uvw^2+uwx^2+2vw^2x+wx^3\equiv 0\ (mod\ 3)$
$s^3t+2st^2m+2t^2mn+s^2nt+stn^2+n^3t\equiv 0\ (mod\ 3)$
$s^3m+s^2mn+2stm^2+smn^2+2tm^2n+mn^3\equiv 0\ (mod\ 3)$
$s^3u+s^2wt+mtus+mwt^2+s^2mv+smxt+mnvs+mnst-su-mv\equiv 0\ (mod\ 3)$
$s^2um+s^2wn+tum^2+mtwn+sm^2v+smxn+m^2vn+mxn^2-sw-mx\equiv 0\ (mod\ 3)$
$stum+stwn+tnum+twn^2+tvm^2+tmxn+n^2vm+xn^3-tw-nx\equiv 0\ (mod\ 3)$
$s^2ut+swt^2+tnus+t^2wn+tmvs+mxt^2+n^2vs+n^2xt-tu-nv\equiv 0\ (mod\ 3)$
$s^2ui+s^2wj+mtui+mtwj+smvi+smxj+mnvi+mnxj-is-lt\equiv 0\ (mod\ 3)$
$stui+stwj+tnui+tnwj+tmvi+tmxj+n^2vj+n^2xj-js-lt\equiv 0\ (mod\ 3)$
$s^2uk+s^2wl+mtuk+mtwl+smvk+smxl+mnvk+mnxl-im-kn\equiv 0\ (mod\ 3)$
$stuk+stwl+tnuk+tnwl+tmvk+tmxl+n^2vk+n^2xl-jm-ln\equiv 0\ (mod\ 3)$.

Noting that $e^{b^2}\neq e$ and $f^{b^2}\neq f$, we have neither

$$\begin{cases} s^2+mt\equiv 1\ (mod\ 3) \\ st+tn\equiv 0\ (mod\ 3) \end{cases} \tag{3.1}$$

nor

$$(3.2) \qquad \begin{cases} sm+mn \equiv 0 \ (mod\ 3) \\ tm+n^2 \equiv 1 \ (mod\ 3) \end{cases}$$

happens.

According to the calculation, we have the following 8 solutions for the above conditions:

1. $i=0, j=1, k=2, l=2, s=0, t=2, m=1, n=0, u=1, v=2, w=2, x=2$,
2. $i=0, j=2, k=1, l=2, s=0, t=1, m=2, n=0, u=1, v=1, w=1, x=2$,
3. $i=2, j=2, k=1, l=0, s=1, t=1, m=1, n=2, u=2, v=1, w=1, x=1$,
4. $i=2, j=1, k=2, l=0, s=1, t=1, m=1, n=2, u=0, v=1, w=2, x=0$,
5. $i=2, j=2, k=1, l=0, s=1, t=2, m=2, n=2, u=0, v=2, w=1, x=0$,
6. $i=0, j=1, k=2, l=2, s=2, t=1, m=1, n=1, u=1, v=1, w=1, x=2$,
7. $i=0, j=2, k=1, l=2, s=2, t=1, m=1, n=1, u=0, v=2, w=1, x=0$,
8. $i=2, j=2, k=1, l=0, s=0, t=1, m=2, n=0, u=2, v=2, w=2, x=1$.

Let $G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8$ be the groups corresponding to the above solutions, respectively. In group G_2, we may replace f by f^2, then G_2 has the same representation as G_1, and thus $G_2 \cong G_1$. Similarly, in groups $G_3, G_4, G_5, G_6, G_7, G_8$, we may replace e, f by e^2, ef, replace e by ef^2, replace e, f by ef, f^2, replace e, f by f^2, ef, replace e, f by ef^2, e^2, replace e, f by f, e, respectively, then all of them are isomorphic to G_1. Therefore, we may define G as in (1) of this Theorem. If $|G| = 600$, with the help of Program 1 in the Appendix, we have that $G \cong SmallGroup(600, 150)$, and thus we may assume that G is defined as in (2) of this Theorem.

(C) G' is a Frobenius group with kernel N and $G'/N \cong Z_q$, where N is 3-decomposable in G.

As $|G'| - |N| = p^n q - p^n$ divides $p^n qr$, we see that $q-1$ divides qr. So $q=2$, and $r \neq 2$ or $q=3$, and $r=2$.

Suppose $r \neq p$. If $q=2$ and $r \neq 2$, then $|G| = 2p^n r$. We can choose T to be a subgroup of G of order $p^n r$. Then $T \trianglelefteq G$ and $G' \leq T$ as G/T

is abelian of order 2, which is a contradiction by order consideration. If $q = 3$ and $r = 2$, then $|G| = 6p^n$. Suppose that N is abelian. Then $p^n = 1 + 2qr$ or $p^n = 1 + 2q$ or $p^n = 1 + qr + q$. It follows that $|G| = 78$ or 42. Since every normal subgroup of G is contained in G', it is easy to see that there is no 2-decomposable normal subgroup in G in each case. Therefore, N is non-abelian, and thus $Z(N)$ is 2-decomposable in G. Suppose $|Z(N)| = p^t$ for some integer t. Then $p^n = p^t + p^s qr$ or $p^n = p^t + p^s q$ for some integer s. If $p^n = p^t + p^s qr$, then $p = 7$ and $n - t = 1$. Note that $Z(N)$ is 2-decomposable, we have $p^t = 1 + qr$. Therefore, $t = 1, n = 2$ and $|G| = 294$. Suppose $N = Z(N) \cup u^G$. Then $|C_G(u)| = 7$, which is contrary to the fact that $Z(N) \neq 1$. If $p^n = p^t + p^s q$, then $p = 2 = r$, contrary to our assumption.

Therefore, $r = p$. In this case, we have $|G| = p^{n+1}q$. If $q = 2$, we can choose T to be a subgroup of G of order p^{n+1}. Then G/T is abelian and thus $G' \leq T$, which is a contradiction. If $p = 2$, then $q = 3$ and $|G| = 2^{n+1} \cdot 3$. Let K be a 2-decomposable normal subgroup of G. Then $K \leq G'$, and thus $K \leq N$ by [14, Exercise 8.5.7]. If $t = 1$, then $K = Z(G)$, which gives $Z(G') \neq 1$, and this contradicts the fact that G' is a Frobenius group. Since both $|K| - 1 = 2^t - 1$ and $|N| - |K| = 2^n - 2^t$ divide $|G| = 2^{n+1} \cdot 3$, we have $t = 2$, and $n = 3$ or $n = 4$. Therefore, $|G| = 48$ or 96. First suppose that $|G| = 48$. Then $\overline{G} = G/K$ is $\{1, 2, 3\}$-decomposable and $|\overline{G}| = 12$, which contradicts Theorem A. Now, suppose that $|G| = 96$. Then $|N| = 16$. It is easy to see that $N' = Z(N)$ is an elementary abelian 2-group of order 4 and that $exp(N) = 4$. However, by investigating the structures of non-abelian 2-groups of order 16 with exponent 4, we find that there does not exist a group satisfying this condition. Therefore, there is no X-decomposable group in this case.

□

Theorem 3.2. *There is no finite non-perfect X-decomposable group G such that G' is 3-decomposable in G.*

Proof. Since G' is 3-decomposable, G' must be one of the following groups by [16]:

1) $|G'| = p^n$ for some prime p and some integer n and G' is metabelian.

2) $|G'| = p^n$ for some prime p and some integer n and G' is elementary abelian.

3) G' is a Frobenius group and $G' = \{1\} \cup g^G \cup h^G$, with $h^{-1} \in h^G$ and $(|h|, |g|) = 1$.

Furthermore, if G' is of type 3), then $|G'| = 2^n p$, where $p = 2^n - 1$ is a prime by [2, Lemma 1].

We see that in all cases, G is solvable as G' is solvable. Let N be an arbitrary normal subgroup of G. We claim that $G' \leq N$ or $N \leq G'$. For otherwise, since G is X-decomposable and G' is 3-decomposable in G, there are more than 4 G-conjugacy classes in $G'N$. It follows that $G = G'N$, and thus $(G/N)' = G/N$, which is a contradiction.

Case A. $|G'| = p^n$ for some prime p and some integer n and G' is metabelian.

If $Z(G) \nleq G'$, then $G' < Z(G)$ by the above paragraph, which gives the contradiction that G is abelian. Therefore, $Z(G) \leq G'$.

(i) If G has at least two distinct 4-decomposable normal subgroups K_1 and K_2, then $K_1 \cap K_2 = G'$ and $G = K_1K_2$. Furthermore, there exist primes r_1 and r_2 such that $|G/K_1| = |K_2/G'| = r_1$ and $|G/K_2| = |K_1/G'| = r_2$, and thus $|G| = p^n r_1 r_2$. On the other hand, since $|K_1|-|G'|$ and $|K_2| - |G'|$ divide $|G|$, we have that both $r_1 - 1$ and $r_2 - 1$ divide r_1r_2. It is easy to see that $|G| = 4p^n$ or $|G| = 6p^n$.

Suppose $|G| = 4p^n$. Then $p \neq 2$. Let K be a 2-decomposable normal subgroup of G. Then $K \leq G'$, and so there exists a positive integer $t < n$ such that $|K| = p^t$. Then both $|K| - 1 = p^t - 1$ and $|G'| - |K| = p^n - p^t$ divide $4p^n$. It is easy to see that $p^n = 9$ or 25. Suppose that $p^n = 9$ and that $G' = K \cup x^G$ for some $x \in G'$. Then $|x^G| = 9 - 3 = 6$ and $|C_G(x)| = 6$. On the other hand, we have that $G' \leq C_G(x)$ as G' is abelian, and thus $|C_G(x)| \geq 9$, which is a contradiction. If $p^n = 25$, by arguing similarly as for $p^n = 9$, we can get a contradiction.

Now, suppose $|G| = 6p^n$. Let H be a 2-decomposable normal subgroup of G. Then $H \leq G'$. If $H \leq Z(G)$, then $|H| = 2$ and $p = 2$. Therefore, $|G'| - |H| = 2^n - 2$ divides $2^n \cdot 6$. It follows that $|G| = 24$ or 48. First suppose $|G| = 24$. If G has normal Sylow 3-subgroup Q, then $Z(G) \times Q$ is 4-decomposable in G, and thus $G' \leq Z(G) \times Q$, which is a contradiction. Therefore, a Sylow 3-subgroup of G is not normal and $G/Z(G) \cong A_4$ by [11, Theorem 4.3.4], which is a contradiction. Now, suppose $|G| = 48$. Then $|G'| = 8$ and we can choose a 4-decomposable normal subgroup of G, say K_1, such that $|K_1| = 24$. If $K_1 = G' \cup w^G$, then $|w^G| = 16$ and thus $|C_G(w)| = 3$, contrary to that $Z(G) \neq 1$.

Therefore, $H \nleq Z(G)$. Recall that $H \leq G'$, so there exists a positive integer i such that $|H| = p^i$. Then both $|H| - 1 = p^i - 1$ and $|G'| - |H| = p^n - p^i$ divides $6p^n$. Note that $|H| \neq 2$ as $H \nleq Z(G)$. We conclude that $|G|$ =48, 54, 96 or 294. If $|G| = 48$. Let K_1 be a 4-decomposable normal subgroup such that $|K_1| = 24$ and $K_1 = G' \cup w^G$. Then $|C_G(w)| = 3$. On the other hand, write $G' = H \cup v^G$. Then $|C_G(v)| = 12$, which is a contradiction. If $|G| = 54$, then we can choose K_1 to be a 4-decomposable normal subgroup such that $|K_1| = 27$ and that $K_1 = G' \cup u^G$. It follows that K_1 is a Sylow 3-subgroup of G and $Z(K_1) \neq 1$, which contradicts the fact that $|C_G(u)| = 3$. If $|G| = 96$. We can choose K_2 to be a 4-decomposable subgroup of G such that $|K_2| = 32$ and $K_2 = G' \cup k^G$. Then $|C_G(k)| = 6$, which contradicts that $Z(G') \neq 1$. Finally, suppose $|G| = 294$. Let $G' = H \cup h^G$. Then $|C_G(h)| = 7$, contrary to that G' is abelian.

(ii) There is exactly one 4-decomposable normal subgroup in G. Then there exists a prime $q \neq p$ such that G/G' is a cyclic group and $|G/G'| = q^2$. Let H/G' be a normal subgroup of G/G' of order q. Then $|H| = p^n q$ and H is 4-decomposable in G. Therefore, $|H| - |G'| = p^n(q-1)$ divides $|G| = p^n q$. It follows that $q = 2$ and $|G| = 4p^n$. By arguing similarly as in (i), we conclude that there is no X-decomposable group in this case.

Case B. $|G'| = p^n$ for some prime p and some integer n and G' is elementary abelian.

We can similarly have $Z(G) \leq G'$ as in Case A.

(i) There are at least two distinct 4-decomposable normal subgroups in G. By arguing similarly as in Case A(i), we have $|G| = 4p^n$ or $|G| = 6p^n$.

Suppose $|G| = 4p^n$. Then $p \neq 2$. If $Z(G) \neq 1$, then $|Z(G)| = 3$ and $G' = Z(G)$ as G' is 3-decomposable in G. It follows that G is abelian, which is a contradiction. Therefore $Z(G) = 1$, and thus G' is the only minimal normal subgroup of G by [2, Theorem 1(i)], so G does not have a 2-decomposable normal subgroup, which is a contradiction.

Now suppose $|G| = 6p^n$. If $Z(G) = 1$, by arguing similarly as in the above paragraph, we can get a contradiction. Therefore, $Z(G) \neq 1$. If $|Z(G)| = 2$, then $|G'| - |Z(G)| = 2^n - 2$ divides $|G| = 6 \cdot 2^n$. It follows that $|G| = 24$ or 48. If $|G| = 24$, then $|G'| = 4$. Let K_1 be a 4-decomposable normal subgroup of G such that $|K_1| = 12$ and let $K_1 = G' \cup x^G$. Then $|C_G(x)| = 3$, which contradicts the fact that $Z(G) \neq 1$. If $|G| = 48$, by arguing similarly as for $|G| = 24$, we arrive at a contradiction. Therefore, $Z(G) = G'$ is of order 3. Consequently, we conclude that $|G| = 18$ and $G/Z(G) = G/G'$ is a cyclic group of order 6, which gives the contradiction that G is abelian.

(ii) There is exactly one 4-decomposable normal subgroup in G. By arguing similarly as in Case A(ii), we have that $|G| = 4p^n$. By (i) of this case, we see that there is no X-decomposable group in this case.

Case C. G' is a Frobenius group of order $2^n p$, where $p = 2^n - 1$ is a prime and $G' = \{1\} \cup g^G \cup h^G$, with $h^{-1} \in h^G$ and $(|h|, |g|) = 1$.

Let H be a 4-decomposable normal subgroup of G. Then $G' \leq H$ by the beginning of this theorem. We see that H is a Frobenius group by [13, Theorem 2]. Let M be the Frobenius kernel of H. Then M is nilpotent, and thus $M \leq G'$ by [14, Exercise 8.5.7]. It follows that M is the Frobenius kernel of G'. So $|H| = 2^n p^b$ and $|G| = 2^n p^b r$ for some prime r. As $|H| - |G'| = 2^n(p^b - p)$ divides $2^n p^b r$, we have that $p^{b-1} - 1$ divides r. Therefore, $p^{b-1} - 1 = r$ since $p = 2^n - 1$. It is easy to see that $r = 2, p = 3, b = 2$ and $n = 2$. Let $H = G' \cup w^G$. Then $|w^G| = 24$ and thus $|C_G(w)| = 3$, which contradicts the fact that G has abelian Sylow 3-subgroups. □

Theorem 3.3. *Let G be a finite non-perfect X-decomposable group. If G' is 2-decomposable in G, then G is one of the following two groups:*

(1) $|G|$=42 and $G = \langle a, b \mid a^7 = b^6 = 1, b^{-1}ab = a^5 \rangle$.

(2) $G = D_{12}$.

Proof. As G' is 2-decomposable in G, there is a prime p such that G' is an elementary abelian p-group by [17, Theorem 1]. Suppose $|G'| = p^n$ for some positive integer n.

If $G' \leq \Phi(G)$, then G is nilpotent. As $Z(G)$ can not be maximal in G, $Z(G)$ is 2- or 3-decomposable in G. However, $|Z(G)|$ is divided by at least two primes since G is not of prime power order, which is a contradiction. Therefore, $G' \nleq \Phi(G)$. In this case, there exists a maximal subgroup M of G such that $G' \nleq M$. So $G = G'M$ and $G' \cap M = 1$. Moreover, $M \cong G/G'$ is abelian. For $1 \neq x \in M$, the maximality of M implies that $C_G(x) = M$ or $C_G(x) = G$.

If $C_G(x) = M$ for every $1 \neq x \in M$, then G is a Frobenius group with kernel G' and a complement M. By the structure of the Frobenius complements, M is a cyclic group. Take K to be an arbitrary non-trivial subgroup of M. Then $G'K \trianglelefteq G$ and so $G'K$ is 3- or 4-decomposable in G. For every $1 \neq y \in G'K \backslash G'$, y must be a p'-element and there exists a Hall p'-subgroup M_1 of G such that $y \in M_1$. Noticing that M_1 and M are conjugate, we conclude that M_1 is also abelian and thus $|y^G| = \frac{|G|}{|M_1|} = |G'|$. If $G'K$ is 3-decomposable in G, then $|G'||K| =$

$|G'K| = 2|G'|$ and $|K| = 2$. If $G'K$ is 4-decomposable in G, then $|G'||K| = |G'K| = 3|G'|$ and $|K| = 3$. Therefore, M is a cyclic group of order 6. On the other hand, M acts transitively and fixed-point freely on $G'\backslash\{1\}$, so $|G'| - 1 = p^n - 1 = 6$. It follows that $|G'| = 7$ and there exists $i \in \{2,3,4,5,6\}$ such that $G = \langle a, b \mid a^7 = b^6 = 1, b^{-1}ab = a^i\rangle$. It is easy to see that $i = 5$ and G is the first group described in this theorem.

Now, suppose that there exists $1 \neq x \in M$ such that $C_G(x) = G$. Then $Z(G) \neq 1$. If $G' \leq Z(G)$, then $|G'| = 2$ and $|G : M| = |G'| = 2$. So $G' \leq M$, which is a contradiction. Therefore, $G' \nleq Z(G)$. The minimality of G' implies that $G' \cap Z(G) = 1$. Let $H = G' \times Z(G)$. Then H is abelian and thus $H < G$. In this case, $Z(G)$ must be 2-decomposable and so H is 4-decomposable in G. Therefore, there exists a prime q such that $|G/H| = q$. Since $|H| = 2p^n$, we have $|G| = 2p^nq$. If $p = 2$, then $q \neq 2$ as G is not a 2-group. As G' is 2-decomposable and $H \leq C_G(G')$, we have that $|G'| - 1 = 2^n - 1 = q$. Let Q be a Sylow q-subgroup of G and $K = G'Q$. Then K is normal in G and $|K| = 2^nq$. If K is 3-decomposable in G, then $|K| - |G'| = 2^n(q-1)$ divides $|G| = 2^{n+1}q$. It follows that $q = 3, n = 2$ and $|G| = 24$. Since $Q \ntrianglelefteq G$ and $Z(G) \neq 1$, $G/Z(G) \cong A_4$ by [11, Theorem 4.3.4], which is a contradiction. If K is 4-decomposable in G, then K is a Frobenius group by [13, Theorem 2]. So all elements of order q in K form two G-conjugacy classes. Let $y \in K$ be an element of order q. We can see that $|C_G(y)| = 2q$, and thus $|y^G| = 2^n$. Now we have $2^nq = |K| = 2^n + 2^n + 2^n$ and thus $q = 3, n = 2$ and $|K| = 12$. It is easy to see that $K \cong A_4$ and $K \cap Z(G) = 1$. Therefore, $G \cong A_4 \times Z_2$. However, G is $\{1,2,4\}$-decomposable by Theorem C, which is a contradiction. If $p \neq 2$, then there exist elements in H of order 2, p and $2p$. So all elements of order p in H form one G-conjugacy class. Noticing that H is abelian, we conclude that $p^n - 1 = q$ and thus $p^n = 3$ and $q = 2$. In this case, $|G| = 12$ and G is an extension of a cyclic group H of order 6 by a cyclic group of order 2. Suppose that $H = \langle a\rangle$ and let $1 \neq b \in G\backslash H$. Then $b^{-1}ab = a^{-1}$ since $b^{-1}ab \neq a$. On the other hand, $b^2 \in H$ since $|G/H| = 2$. If $b^2 = a^2$ or $b^2 = a^4$, then b is of order 6. It is easy to see that $|\langle a\rangle \cap \langle b\rangle| = 3$, and thus $|Z(G)| \geq 3$, which contradicts to that $|Z(G)| = 2$. If $b^2 = a^3$, then $G = \langle a, b \mid a^6 = 1, b^2 = a^3, b^{-1}ab = a^{-1}\rangle \cong Q_{12}$. However, G is $\{1,2,4\}$-decomposable by Theorem C. Therefore, $b^2 = 1$ and $G = \langle a, b \mid a^6 = b^2 = 1, b^{-1}ab = a^{-1}\rangle \cong D_{12}$, and G is X-decomposable by Example 2.1.

□

Now, from the above three theorems, we come to our main theorem.

Theorem 3.4 (**Main theorem).** *Let G be a finite non-perfect X-decomposable group. Then G is one of the following groups:*

(1) $|G| = 216$ *and* $G = \langle a, b, c, d, e, f \mid a^3 = d^2 = e^3 = f^3 = 1, b^2 = c^2 = d, b^a = cd, c^a = bc, c^b = cd, e^a = f^2, e^b = e^2 f, e^c = f^2, e^d = e^2, f^a = ef^2, f^b = ef, f^c = e, f^d = f^2\rangle$.

(2) $|G| = 600$ *and* $G = \langle a, b, c, d, e, f \mid a^3 = d^2 = e^5 = f^5 = 1, b^2 = c^2 = d, b^a = bc, c^a = b, c^b = cd, e^a = ef^3, e^b = e^3 f^3, e^c = e^3, e^d = e^4, f^a = e^4 f^3, f^b = f^2, f^c = e^4 f^2, f^d = f^4\rangle$.

(3) $|G| = 42$ *and* $G = \langle a, b \mid a^7 = b^6 = 1, b^{-1}ab = a^5\rangle$.

(4) $G = D_{12}$.

Appendix

Program 1 : A Magma Program

```
SetLogFile("nnn..txt");
   P:=SmallGroupProcess(600);
   repeat
   G:=Current(P);
   "................................................................group ";
   CurrentLabel(P);
   M:=NormalSubgroups(G); m:=0;
   for j in [1..♯M] do
   N:=M[j]‘subgroup;
   S:=[n:n in N—Order(n) ge 1];
   while ♯S gt 1 do
   h:=1;X:=S[1];Remove( S,1);
            for k in [♯S..1 by -1] do
                     if IsConjugate(G, X, S[k]) then
                            h:=h+1;Remove( S,k);
                     end if;
            end for;
   "c,",h;
   end while; "c,",♯S; N;m:=m+1; "........................................" ,m;
   if ♯N eq 1 then "1"; end if;
   end for;
   "..............................................................................";
   Advance( P);
   until IsEmpty(P);
```

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