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# BAER'S LOWER NILRADICAL AND CLASSICAL PRIME SUBMODULES

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Abstract. Let $N$ be a submodule of a module $M$ and a minimal primary decomposition of $N$ is known. A formula to compute Baer's lower nilradical of $N$ is given. The relations between classical prime submodules and their nilradicals are investigated. Some situations in which semiprime submodules can be written as finite intersection of classical prime submodule are stated.
**Keywords:** Envelopes, nilradical, classical prime submodules, semiprime submodules.
**MSC(2010):** Primary: 13E05; Secondary: 13E15, 13C99, 13P99.

## 1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary.

Let $R$ be a ring and $M$ be an $R$-module. A proper submodule $P$ of $M$ is said to be *primary* if whenever $rm \in P$ where $r \in R$ and $m \in M$ then $m \in P$ or $r^k M \subseteq N$ for some positive integer $k$.

Recall that $(P : M) = \{r \in R \mid rM \subseteq P\}$. If $P$ is a primary submodule of $M$ and $p = \sqrt{P : M}$, then $P$ is called $p$-primary submodule (see [10]).

A *primary decomposition* of a submodule $N$ of $M$ is a representation of $N$ as an intersection of finitely many primary submodules of $M$. Such a primary decomposition $N = \cap_{i=1}^{n} Q_i$ with $p_i$-primary submodules $Q_i$

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is called minimal if $p_i$'s are pairwise distinct and $Q_j \not\supseteq \cap_{i \neq j} Q_i$ for all $j = 1, \ldots, n$.

If $R$ is a Noetherian ring and $M$ is a finitely generated module, then every proper submodule $N$ has a minimal primary decomposition. The first uniqueness theorem states that for such a minimal primary decomposition the set of primes $\{p_1, \ldots, p_m\}$ is uniquely defined. These primes are called the associated primes of $N$. We denote this set by $\mathrm{Ass}(M/N)$. It is clear that for any $p \in \mathrm{Ass}(M/N)$, $(N : M) \subseteq p$.

The prime ideals in $\mathrm{Ass}(M/N)$ that are minimal with respect to inclusion are called the isolated primes of $N$, the remaining associated prime ideals are the embedded primes of $N$.

The second uniqueness theorem states that not only the primes but also the primary components corresponding to isolated primes, the isolated components of $N$ in $M$, are uniquely defined. The other primary components, the embedded components of $N$ in $M$, need not be defined uniquely. The concepts and theorems about the primary decomposition of modules can be found in chapter 9 of [12].

The radical $\sqrt{I}$ of an ideal $I \subset R$ is characterized as the the set of elements $a \in R$ such that $a^n \in I$ for some positive integer $n$. The concept of envelope of a submodule is the generalization of this characterization to the modules. If $N$ is a submodule of an $R$-module $M$, then the envelope of $N$ in $M$ is defined to be the set

$$E_M(N) = \{rm : r \in R, m \in M \, and \, r^k m \in N \, for \, some \, k \in \mathbb{Z}^+\}.$$

The submodule generated by the envelope is called (Baer's) lower nilradical and denoted by $\sqrt[nil]{N}$. Although some methods to compute radical of a submodule, which defined to be intersection of prime submodules containing $N$, are given in [9] and [11]. It seems there is no description for the computation of the lower nilradical of a submodule in the literature. In Section 1, we give a formula for the computation of $\sqrt[nil]{N}$ if a minimal primary decomposition of $N$ is known. In this section, we use extensively the concepts and results from [8].

When $M$ is a module, a proper submodule $N$ of an $R$-module $M$ is called *a classical prime submodule* if for each $m \in M$ and $a, b \in R$; $abm \in N$ implies that $am \in N$ or $bm \in N$. A proper submodule $N$ of an $R$-module $M$ is called *a classical primary submodule* if $abm \in N$ where $a, b \in R$ and $m \in M$, then either $bm \in N$ or $a^k m \in N$ for some $k \geq 1$. We remark that these two concept are sometimes referred to in the literature as weakly prime submodules and weakly primary submodules, respectively. This notion of classical (weakly) prime submodule was first introduced and studied in [4] and recently has received a good deal of attention from several authors; see for example [1, 2, 5]. Also, this notion of classical primary submodule was first introduced and studied in [3]. In Section 2, we investigated relations between classical prime submodules and their lower nilradicals. We also give an example to show a conjecture given in [3] is false.

## 2. Baer's lower nilradicals of submodules

**Lemma 2.1.** *Let $N$ be a submodule of a module $M$ over a ring $R$. If $N = Q_1 \cap Q_2 \cap \cdots \cap Q_k$ is a minimal primary decomposition of $N$ where $\sqrt{Q_i : M} = p_i$ for all $i = 1, 2, \ldots, k$. If $S = \{1, 2, \ldots, k\}$ and $\emptyset \neq T \subsetneq S$, then*

$$(\bigcap_{i \in T} p_i)(\bigcap_{i \in S \setminus T} Q_i) \subseteq \sqrt[nil]{N}$$

*Proof.* Let $n \in (\bigcap_{i \in T} p_i)(\bigcap_{i \in S \setminus T} Q_i)$. Then there exist $r_j \in \bigcap_{i \in T} p_i$ and $m_j \in \bigcap_{i \in S \setminus T} Q_i$ such that

$$n = r_1 m_1 + r_2 m_2 + \cdots + r_s m_s$$

for some $s \in \mathbb{Z}^+$.

Since $r_j \in \bigcap_{i \in T} p_i$, we have ${r_j}^{k_j} M \subseteq \bigcap_{i \in T} Q_i$ for some $k_j \in \mathbb{Z}^+$. In particular, ${r_j}^{k_j} m_j \in \bigcap_{i \in T} Q_i$ for all $j = 1, 2, \cdots, s$.

Since $m_j \in \bigcap_{i \in S-T} Q_i$, we have ${r_j}^{k_j} m_j \in \bigcap_{i \in S-T} Q_i$ for all $j$. Thus, we have ${r_j}^{k_j} m_j \in \bigcap_{i=1}^{k} Q_i = N$ which means that $r_j m_j \in E_M(N)$ for all $j$. Therefore, $n \in \sqrt[nil]{N}$. □

Before giving a formula for the nilradical of a submodule in terms of its associated primes and primary submodules in its primary decomposition, we need some technical prerequisites.

**Definition 2.2.** *Let $N$ be a submodule of a module $M$ over a ring $R$. If $I$ is an ideal of $R$, then the set*

$$N : I^\infty = \{m \in M : I^k m \subseteq N \text{ for some positive integer } k\}$$

*is called the stable quotient of $N$ by $I$ in $M$.*

**Lemma 2.3.** [8, Lemma 1] *Let $P \subset M$ be a primary submodule of $M$ and $f \in R$.*

$$(i)\ P : \langle f\rangle^{\infty} = M \text{ if } f \in \sqrt{P : M},$$

$$(ii)\ P : \langle f\rangle^{\infty} = P \text{ if } f \notin \sqrt{P : M}.$$

*More generally, for arbitrary submodule $N$ of $M$ and its primary decomposition $N = \bigcap P_i$ into $p_i$-primary submodules $P_i$ we get*

$$(iii)\ N : \langle f\rangle^{\infty} = \bigcap_{f \notin p_i} P_i$$

*and for arbitrary ideal $I$ of $R$*

$$(iv)\ N : I^{\infty} = \bigcap_{I \not\subset p_i} P_i.$$

We can easily show that.

**Lemma 2.4.** *Let $N$ be $p$-primary submodule of an $R$-module $M$. Then*

$$(i)\ N : h = N, \text{ if } h \notin p,$$

$$(ii)\ N : h = M, \text{ if } h \in (N : M).$$

The following theorem is the main result of this section.

**Theorem 2.5.** *With the notation in Lemma 2.1,*

$$\sqrt[nil]{N} = N + (\bigcap_{i=1}^{k} p_i)M + \sum_{\emptyset \neq T \subsetneq S} (\bigcap_{i \in T} p_i)(\bigcap_{i \in S \setminus T} Q_i).$$

*Proof.* Let $m \in \sqrt[nil]{N}$. Then there exist $m_j \in M$, and $r_j \in R$ such that

$$m = r_1 m_1 + r_2 m_2 + \cdots + r_t m_t.$$

By the definition of $\sqrt[nil]{N}$, we have $m_j \in N : \langle r_j\rangle^{\infty}$ for each $j = 1, 2, \ldots, t$.

For each $r_j$, $r_j \in R \setminus \bigcup_{i=1}^{k} p_i$, there is a maximal proper subset $T$ of $S$ such that $r_j \in \bigcap_{i \in T} p_i$ or $r_j \in p_i$ for all $i$.

If $r_j \in R \setminus \bigcup_{i=1}^{k} p_i$, then $N : \langle r_j \rangle^\infty = N$ by Lemma 2.3. Hence $m_j \in N$ and so $r_j m_j \in N$.

If $r_j \in \bigcap_{i \in T} p_i$, then

$$N : \langle r_j \rangle^\infty = \bigcap_{i=1}^{k} (Q_i : \langle r_j \rangle^\infty) = \bigcap_{i \in S \setminus T} Q_i$$

by Lemma 2.3. Hence,

$$r_j m_j \in (\bigcap_{i \in T} p_i)(\bigcap_{i \in S \setminus T} Q_i).$$

If $r_j \in \bigcap_{i=1}^{k} p_i = \sqrt{N : M}$, then $r_j m_j \in \sqrt{N : M}M$.

Thus, we can conclude that

$$\sqrt[nil]{N} \subseteq N + (\bigcap_{i=1}^{k} p_i)M + \sum_{\emptyset \neq T \subsetneq S} (\bigcap_{i \in T} p_i)(\bigcap_{i \in S \setminus T} Q_i).$$

For the other side of the inclusion, Lemma 2.1 implies that

$$\sum_{\emptyset \neq T \subsetneq S} (\bigcap_{i \in T} p_i)(\bigcap_{i \in S \setminus T} Q_i) \subseteq \sqrt[nil]{N}.$$

Moreover $N$ and $(\bigcap_{i=1}^{k} p_i)M = \sqrt{N : M}M$ are clearly in $\sqrt[nil]{N}$. $\square$

**Corollary 2.6.** *If $N$ is a $p$-primary submodule, then*

$$\sqrt[nil]{N} = N + pM.$$

Now, we will give an application of Theorem 2.5. The computer algebra system Singular was used for the computations (see [7]).

**Example 2.7.** *Let $R = \mathbb{Q}[x, y, z]$ and let $M = R \oplus R \oplus R$. Consider the submodule $N = \langle xz\mathbf{e}_3 - z\mathbf{e}_1, x^2\mathbf{e}_3, x^2y^3\mathbf{e}_1 + x^2y^2z\mathbf{e}_2 \rangle$.*

*Primary decomposition of $N$ is $N = Q_1 \cap Q_2 \cap Q_3$ where*

$$Q_1 = \langle \mathbf{e}_3, z\mathbf{e}_1, y\mathbf{e}_1 + z\mathbf{e}_2, z^2\mathbf{e}_2 \rangle \text{ is } \langle z \rangle - \text{primary},$$
$$Q_2 = \langle \mathbf{e}_1, \mathbf{e}_3, y^2\mathbf{e}_2 \rangle \text{ is } \langle y \rangle - \text{primary and}$$
$$Q_3 = \langle x\mathbf{e}_1, x\mathbf{e}_3 - \mathbf{e}_1, x^2\mathbf{e}_2 \rangle \text{ is } \langle x \rangle - \text{primary.}$$

*By Theorem 2.5,*

$$\sqrt[nil]{N} = N + (p_1 \cap p_2 \cap p_3)M + p_1(Q_2 \cap Q_3) + p_2(Q_1 \cap Q_3) \\ + p_3(Q_1 \cap Q_2) + (p_1 \cap p_2)Q_3 + (p_1 \cap p_3)Q_2 + (p_2 \cap p_3)Q_1.$$

*It is clear that* $(p_1 \cap p_2 \cap p_3)M = \langle xyz\mathbf{e}_1, xyz\mathbf{e}_2, xyz\mathbf{e}_3\rangle$. *We also get*

$$\begin{aligned} p_1(Q_2 \cap Q_3) &= \langle xz\mathbf{e}_1, xz\mathbf{e}_3 - z\mathbf{e}_1, x^2y^2z\mathbf{e}_2\rangle \\ p_2(Q_1 \cap Q_3) &= \langle xyz\mathbf{e}_3 - yz\mathbf{e}_1, x^2y\mathbf{e}_3, x^2y^2\mathbf{e}_1 + x^2yz\mathbf{e}_2\rangle \\ p_3(Q_1 \cap Q_2) &= \langle x\mathbf{e}_3, xz\mathbf{e}_1, xy^3\mathbf{e}_1 + xy^2z\mathbf{e}_2\rangle \\ (p_1 \cap p_2)Q_3 &= \langle xyz\mathbf{e}_1, xyz\mathbf{e}_3 - yz\mathbf{e}_1, x^2yz\mathbf{e}_2\rangle \\ (p_1 \cap p_3)Q_2 &= \langle xz\mathbf{e}_1, xz\mathbf{e}_3, xy^2z\mathbf{e}_2\rangle \\ (p_2 \cap p_3)Q_1 &= \langle xy\mathbf{e}_3, xyz\mathbf{e}_1, xy^2\mathbf{e}_1 + xyz\mathbf{e}_2, xyz^2\mathbf{e}_2\rangle \end{aligned}$$

*Thus*

$$\sqrt[nil]{N} = \langle z\mathbf{e}_1, x\mathbf{e}_3, xyz\mathbf{e}_2, xy^2\mathbf{e}_1\rangle.$$

**Corollary 2.8.** *If* $\sqrt[nil]{N} = N$, *then each isolated component of primary decomposition of* $N$ *must be prime.*

*Proof.* Let $N = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ with $Q_i$'s are $p_i$-primary submodules. Let $Q_k$ be one of the isolated components of $N$. If $Q_k$'s were not a prime submodule, then there would exist $x \in p_k \backslash (Q_k : M)$. Hence, there exists $m \in M$ such that $xm \notin Q_k$. Since $p_k$ is an isolated prime, we can find an element $y \in (\bigcap_{j \neq k} p_j) \setminus p_k$. Then

$$xym \in (\bigcap_{j=1}^{n} p_j)M \subseteq \sqrt[nil]{N} = N \subseteq Q_k.$$

Since $Q_k$ is $p_k$-primary and $xm \notin Q_k$, $y \in p_k$ which is a contradiction. $\square$

## 3. **Classical prime submodules**

In this section we investigate the relations between classical prime submodules and their nilradicals.

**Lemma 3.1.** *If* $N$ *is a classical prime submodule, then* $\sqrt[nil]{N} = N$.

*Proof.* Let $x \in \sqrt[nil]{N}$. Then there exist elements $r_i \in R$ and $m_i \in M$ $(1 \leq i \leq k)$ such that

$$x = r_1 m_1 + \cdots + r_k m_k \qquad with \quad r_i^{t_i} m_i \in N$$

for some $t_i \in \mathbb{Z}^+$. Since $N$ is classical prime, $r_i^{t_i} m_i \in N$ implies that $r_i m_i \in N$ or $r_i^{t_i - 1} m_i \in N$. If $r_i m_i \in N$, then $x = r_1 m_1 + \cdots + r_k m_k \in N$. If $r_i^{t_i-1} m_i \in N$, then $r_i m_i \in N$ or $r_i^{t_i-2} m_i \in N$. By the same process, $r_i m_i \in N$ for all cases. Hence, $x \in N$, which means that $\sqrt[nil]{N} \subseteq N$. Other side of the inclusion is obvious. □

The classical quasi-primary submodules are introduced in [6]. We will take one of the equivalence definitions in Noetherian modules.

**Definition 3.2.** *A proper submodule $N$ of a Noetherian module $M$ is called classical quasi-primary if $abm \in N$ where $a, b \in R$ and $m \in M$ implies that either $a^k m \in N$ or $b^k m \in N$ for some $k \in \mathbb{N}$.*

**Proposition 3.3.** [6, Proposition 3.4] *Let $M$ be a Noetherian $R$-module and $N$ is a proper submodule of $M$. Suppose that $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$ where each $Q_i$ is $p_i$-primary submodule. Then $N$ is classical quasi-primary if and only if $\{p_1, p_2, \ldots, p_s\}$ is a chain of prime ideals.*

**Theorem 3.4.** *If $N$ is a classical quasi-primary submodule and $\sqrt[nil]{N} = N$, then $N$ is classical prime submodule.*

*Proof.* Suppose that $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$ where each $Q_i$ is $p_i$-primary submodule with $p_1 \subset p_2 \subset \cdots \subset p_s$. Since $p_1 \subset p_2 \subset \cdots \subset p_s$, by the Theorem 2.5

$$N = \sqrt[nil]{N} = N + p_1 M + \sum_{i=2}^{s} p_i \Big(\bigcap_{j=1}^{i-1} Q_j\Big).$$

Let $abm \in N$ with $a, b \in R$ and $m \in M$. Let $i$ be the first index for which $m \notin Q_i$. Since $Q_i$ is $p_i$-primary, $ab \in p_i$ and so either $a \in p_i$ or $b \in p_i$. If $i = 1$, then since $p_1 M \subset \sqrt[nil]{N} = N$, either $am \in N$ or $bm \in N$. Let $i > 1$. Since $p_i(\bigcap_{j=1}^{i-1} Q_j) \subset \sqrt[nil]{N} = N$, either $am \in N$ or $bm \in N$. Hence $N$ is a classical prime submodule. □

The following conjecture is stated in [3]: Let $R$ be a ring and $M$ be an $R$-module. Then for every classical primary submodule $Q$ of $M$, $\sqrt[nil]{Q}$ is a classical prime submodule.

The next example shows that the conjecture is false.

**Example 3.5.** *Let $R = \mathbb{Q}[x, y]$ and let $M = R \oplus R$. Consider the submodule $N = \langle x\mathbf{e}_1 + y^3\mathbf{e}_2, x^2\mathbf{e}_1, x\mathbf{e}_2\rangle$. One can easily see that $(N : M) = \langle x^2\rangle$ and $N$ is $\langle x\rangle$-primary submodule. Hence*

$$\sqrt[nil]{N} = N + \langle x\rangle M = \langle x\mathbf{e}_1, x\mathbf{e}_2, y^3\mathbf{e}_2\rangle$$

*Then $\sqrt[nil]{N}$ is not classical prime submodule since $y^2(0, y) = (0, y^3) \in \sqrt[nil]{N}$ but $y(0, y) = (0, y^2) \notin \sqrt[nil]{N}$.*

If we weaken the conditions of the conjecture as follows, then we can obtain the desired result as a consequence of Theorem 3.4.

**Corollary 3.6.** *Let $R$ be a Noetherian ring and $M$ be a finitely generated $R$-module. Then for every classical primary submodule $Q$ of $M$; if $\sqrt[nil]{Q} = Q$, then $Q$ is classical prime.*

We also have the following.

**Corollary 3.7.** *Let $N = Q_1 \cap Q_2$ be a submodule of $M$ where $Q_i$ is $p_i$-primary. If $\sqrt[nil]{N} = N$, then either $Q_1$ and $Q_2$ are both prime or $N$ is classical prime.*

*Proof.* We have two cases: $p_1 \nsubseteq p_2$ or $p_1 \subseteq p_2$. If $p_1 \nsubseteq p_2$, then both $p_1$ and $p_2$ are isolated primes. From Corollary 2.8, $Q_1$ and $Q_2$ are prime submodules. If $p_1 \subseteq p_2$, then Theorem 3.4 implies that $N$ is classical prime. □

**Definition 3.8.** *A proper submodule $N$ of an $R$-module $M$ is called semiprime if whenever $r^k m \in N$ for some $r \in R, m \in M$ and natural number $k$, then $rm \in N$.*

The question when a semiprime submodule can be expressed as a finite intersection of classical prime submodules was discussed in [5]. We try to make some contributions to this discussion. First of all, the following lemma shows that semiprime submodules can be defined in terms of their lower nilradicals.

**Lemma 3.9.** *A proper submodule $N$ is semiprime if and only if $\sqrt[nil]{N} = N$.*

*Proof.* Suppose that $N$ is semiprime. Let $x \in \sqrt[nil]{N}$. Then there exist elements $r_i \in R$, $m_i \in M$ $(1 \leq i \leq k)$ such that

$$x = r_1m_1 + \cdots + r_km_k \qquad \textit{with} \quad r_i^{t_i}m_i \in N$$

for some $t_i \in \mathbb{Z}^+$. Since $N$ is semiprime, $r_i m_i \in N$ for all $i$. Hence $x \in N$ and $\sqrt[nil]{N} = N$.

Conversely, suppose that $\sqrt[nil]{N} = N$. Let $r^k m \in N$ for some $r \in R, m \in M$ and natural number $k$. By definition of the envelope, $rm \in E_M(N) \subseteq \sqrt[nil]{N} = N$. Hence, $N$ is semiprime. □

**Proposition 3.10.** *A finite intersection semiprime submodules is also semiprime.*

*Proof.* Let $N = N_1 \cap N_2 \cap \cdots \cap N_s$ where each $N_i$ is semiprime. If $x \in \sqrt[nil]{N}$, then $x = r_1 m_1 + r_2 m_2 + \cdots r_t m_t$ where $r_i^{k_i} m_i \in N$ for some $k_i \in \mathbb{N}$. Therefore for each $i$ and $j$, $r_i^{k_i} m_i \in N_j$. Since each $N_j$ is semiprime, $r_i m_i \in N_j$ for $j = 1, \ldots, s$. Hence $x \in N$. By Lemma 3.9 , $N$ is semiprime. □

**Definition 3.11.** *A submodule $N$ is called a quasi-p-primary submodule in $M$, if $N$ has a unique isolated prime $p$ (and possibly embedded primes).*

**Definition 3.12.** *A quasi-p-primary submodule $N$ is called simple quasi-p-primary if for any distinct associated primes $p_i, p_j$ and $p_k$ of $N$, $p_i \subset p_k$ and $p_j \subset p_k$ implies either $p_i \subset p_j$ or $p_j \subset p_i$.*

In the language of graph theory, we can say $N$ is a simple quasi-primary submodule, if Hasse diagram of associated primes of $N$ with respect to set inclusion form a rooted tree.

**Lemma 3.13.** *Let $N$ be a simple quasi-$p_1$-primary submodule for a prime ideal $p_1$. If $N$ is also semiprime, then $N$ can be expressed as an intersection of finitely many classical prime submodules containing $N$.*

*Proof.* Let $\text{Ass}(M/N) = \{p_1, \ldots, p_s\}$ and $S = \{1, \ldots, s\}$. If $N$ contains only one maximal associated prime with respect to inclusion, then its associated primes form a chain $p_1 \subset \cdots \subset p_s$. Hence, $N$ is classical prime by Theorem 3.4.

Suppose that $N$ has more than one maximal element. For each maximal $p_j$, we have a unique chain of associated primes $p_1 = p_{j_1} \subset p_{j_2} \subset \cdots \subset p_{j_t} = p_j$. Let $N_j = Q_{j_1} \cap Q_{j_2} \cdots \cap Q_{j_t}$ where $Q_{j_1} = Q_1$ and $Q_{j_t} = Q_j$. From Theorem 2.5,

$$\sqrt[nil]{N} = N + p_1 M + \sum_{T \subset S} (\bigcap_{i \in T} p_i)(\bigcap_{i \in S \setminus T} Q_i)$$

and

$$\sqrt[nil]{N_j} = N_j + p_1M + \sum_{i=2}^{t} p_{j_i}(\bigcap_{k=1}^{i-1} Q_{j_k}).$$

Our aim is to show that $\sqrt[nil]{N_j} = N_j$. Clearly $p_1M \subset \sqrt[nil]{N} = N \subset N_j$. Let $B = \mathrm{Ass}(M/N) \setminus \mathrm{Ass}(M/N_j)$. Take $x \in p_{j_i}$ and $m \in \bigcap_{k=1}^{i-1} Q_{j_k}$. Since $p_j$ is a maximal prime and $N$ is simple quasi-primary, there exists $y \in (\bigcap_{p\in B} p) \setminus p_j$. Hence,

$$yxm \in (p_{j_i} \cap (\bigcap_{p\in B} p))(\bigcap_{k=1}^{i-1} Q_{j_k}) \subset \sqrt[nil]{N} = N \subset N_j \subset Q_{j_k}.$$

Since each $Q_{j_k}$ is $p_{j_k}$-primary and $y \notin p_{j_k}$, $xm \in Q_{j_k}$. Hence, $xm \in N_j$. This implies $\sqrt[nil]{N_j} = N_j$ and $N_j$ is classical prime by Theorem 3.4. Since $N = \cap N_j$, $N$ is intersection of finitely many classical prime submodules. $\square$

The following proposition is crucial for computing primary decompositions and is quite useful for our purpose.

**Proposition 3.14.** [8, Proposition 1] *Assume that $L = \{p_1, \ldots, p_k\}$ are the isolated primes of $N$. For $i, j = 1, \ldots, k$ take $f_i \in R$ such that $f_i \in p_j$ if $i \neq j$, but $f_i \notin p_i$, $N_i = N : f_i^{\infty}$ and take integers $e_i$ such that $f_i^{e_i} N_i \subset N$.*

*Then:*

*(i) $N_i$ is a quasi-$p_i$-primary module in $M$.*

*(ii) The sets $A_i = \mathrm{Ass}(M/N_i) = \{p \in \mathrm{Ass}(M/N) : f_i \notin p\}$ are pairwise disjoint.*

*(iii) For $J := \langle e_1, e_2, \ldots, e_k\rangle$ we have*

$$N = (\bigcap N_i) \cap (N + JM)$$

*This is a decomposition of $N$ into quasi-primary components $N_i$ and a component $N' := N + JM \subset M$ of lower (relative) dimension.*

**Theorem 3.15.** *Assume that $L = \{p_1, \ldots, p_k\}$ are the isolated primes of a semiprime submodule $N$ and define $N_i$'s as in the previous proposition. If $N = N_1 \cap N_2 \cap \cdots N_k$ and each $N_i$ is simple quasi-$p_i$-primary, then $\sqrt[nil]{N_i} = N_i$ for $i = 1, \ldots, k$. Hence $N$ can be written as a finite intersection of classical prime submodules.*

*Proof.* For a fixed $i$, let $\operatorname{Ass}(M/N_i) = \{p_{i_1} = p_i, p_{i_2}, \ldots, p_{i_{s_i}}\}$ and $p_i \subseteq p_{i_k}$ for every $k$ and let $N_i = Q_{i_1} \cap \cdots \cap Q_{i_{s_i}}$ where each $Q_{i_k}$ is $p_{i_k}$-primary. By Theorem 2.5,

$$\sqrt[nil]{N} = N + (\bigcap_{i=1}^{k} p_i)M + \sum_{\emptyset \neq T \subsetneq S} (\bigcap_{j \in T} p_{i_j})(\bigcap_{j \in S \setminus T} Q_{i_j})$$

and

$$\sqrt[nil]{N_i} = N_i + p_i M + \sum_{\emptyset \neq T \subsetneq S_i} (\bigcap_{r \in T} p_{i_r})(\bigcap_{r \in S_i \setminus T} Q_{i_r}\rangle$$

where $S_i = \{i_1, i_2, \ldots, i_{s_i}\}$ and $S = \bigcup_{i=1}^{k} S_i$.

Let $x \in p_i$ and $m \in M$. Take $y \in \big(\bigcap_{j \neq i} p_j\big) \setminus \big(\bigcup_{t=2}^{s_i} p_{i_t}\big)$. This is possible since associated primes of $N_i$'s are pairwise disjoint. Then

$$yxm \in (\bigcap_{j=1}^{k} p_j)M \subseteq \sqrt[nil]{N} \subseteq Q_{i_t}$$

for $t = 1, \ldots, s_i$. Since $Q_{i_t}$ is primary and $y \notin p_{i_t}$, $xm \in Q_{i_t}$. Hence $xm \in N_i$.

Now, let $x \in \bigcap_{r \in T} p_{i_r}, m \in \bigcap_{S_i \setminus T} Q_{i_r}$ for some $T \subsetneq S_i$. Take

$$y \in (\bigcap_{j \neq i} p_j) \setminus (\bigcup_{t=2}^{s_i} p_{i_t}).$$

Then

$$yxm \in \Big[(\bigcap_{j \neq i} p_j) \cap (\bigcap_{r \in T} p_{i_t})\Big](\bigcap_{r \in S_i \setminus T} Q_{i_r}).$$

Since

$$\bigcap_{j \neq i} p_j = \bigcap_{j \neq i} \bigcap_{t=1}^{s_j} p_{j_t},$$

we have

$$\Big[(\bigcap_{j \neq i} p_j) \cap (\bigcap_{r \in T} p_{i_t})\Big](\bigcap_{r \in S_i \setminus T} Q_{i_r}) \subseteq \sqrt[nil]{N} \subseteq N_i.$$

Thus $yxm \in Q_{i_t}$ for $t = 1, \ldots, s_i$. Since $Q_{i_t}$ is primary and $y \notin p_{i_t}$, $xm \in Q_{i_t}$ and hence $xm \in N_i$. Therefore $\sqrt[nil]{N_i} = N_i$ and the conclusion easily follows. $\square$

We also have the following result.

**Proposition 3.16.** *Let $N$ be a classical primary submodule of $M$. Then $N$ is semiprime if and only if $N$ is classical prime.*

*Proof.* Suppose $N$ is semiprime. Then by Lemma 3.9, $\sqrt[nil]{N} = N$. Since $N$ is classical primary, $N$ is classical prime by Corollary 3.6.

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