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NILPOTENT GROUPS WITH THREE CONJUGACY CLASSES OF NON-NORMAL SUBGROUPS

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Abstract. Let G be a finite group and $\nu(G)$ denote the number of conjugacy classes of non-normal subgroups of G. In this paper, all nilpotent groups G with $\nu(G) = 3$ are classified.
Keywords: Non-normal subgroup, conjugacy class, nilpotent group.
MSC(2010): Primary: 20D15; Secondary: 20E45.

1. Introduction

A group G is called a *Dedekind group* (*or Hamiltonian*) if any subgroup of G is normal. All such groups were determined by Dedekind in 1897; they are Abelian groups and direct products of a quaternion group of order 8 with a periodic Abelian group having no element of order 4.

If G is not a Dedekind group we denote the intersection and the number of conjugacy classes of all non-normal subgroups of G, by $R(G)$ and $\nu(G)$, respectively. Clearly $\nu(G) = 0$ if and only if G is Dedekind.

Brandl [2] and the present author [6] classified all finite groups with $\nu(G) = 1$ and $\nu(G) = 2$, respectively. Brandl conjectured that $1 + \nu(G)$ is an upper bound for $c(G)$, the nilpotency class of a nilpotent group G (with the exception of the Hamiltonian groups, of course). In [9] Poland and Rhemtulla proved this conjecture and determined when $c(G) = 1 + \nu(G)$. Also the present author [7] gave a simple proof for this conjecture and classified all finite non-Dedekind nilpotent groups with $c(G) = 1 + \nu(G)$.

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For another application of $\nu(G)$, La Haye [4] showed that for a non-Dedekind group G with a finite number of non-normal subgroups, $|G'| \leq \rho(G)^{\nu(G)+\varepsilon}$ and $|G/Z(G)| \leq \rho(G)^{\nu(G)+\varepsilon+1}$ where $\rho(G)$ denoted the largest prime p for which G has element of order p, and $\varepsilon = 1$ if G has element of order 2 and $\varepsilon = 0$ otherwise. This bound was improved by La Haye and Rhemtulla [5]. This goes to say that $\nu(G)$ can play an important role in the structures of finite groups.

The following theorem, which is the main result of this article, was announced without proof in the author's seminar article [8] at the 16th Seminar of Algebra in IASBS, Zanjan, Iran, 2004. Later in 2009, Chen [3] tries to characterize finite nilpotent groups with $\nu(G) = 3$; unfortunately the characterization is not complete, in the sense that one the groups are missing and also some of the presented groups are isomorphic.

Theorem 1.1. *Let G be a finite nilpotent group with $\nu(G) = 3$. Then G is isomorphic to one of the following groups.*

(1) $A \times \mathbb{Z}_{q^2}$*, where A is a p-group with $\nu(A) = 1$ and $q \neq p$ is prime.*
(2) $Q_8 \times \mathbb{Z}_4$*;*
(3) $\langle x, g \mid x^8, g^8, x^4g^4, g^x g\rangle$.
(4) $\langle x, g \mid x^8, g^4, [x^4, g^2], [g, x]x^4g^2\rangle$.
(5) $\langle x, y, z \mid x^2, y^2, z^{2^n}, [x, y]z^{2^{n-1}}, [x, z], [y, z]\rangle$*, where $n \geq 2$.*
(6) $\langle x, g \mid x^9, g^{3^n}, [x, g]g^{3^{n-1}}\rangle$*, where $n \geq 2$.*
(7) $\langle x, g \mid x^2, g^4, [x, g^2], [x, g]^2g^2\rangle$.

Our notation are standard and can be found in [10] or [11]. For example, the split extension of A by B is written as $A \rtimes B$ and $\mathbb{Z}_n$, D_{2n} and Q_{2^n} denote cyclic group of order n, the Dihedral group of order $2n$ and Quaternion group of order 2^n, respectively.

2. Preliminaries

In this section we let G be a finite non-Dedekind group. The following results of N. Blackburn [1] (1964) play an important role in the theory of finite non-Hamiltonian p-groups.

Theorem 2.1 (Blackburn)**.** *Let G be a finite p-group. Suppose that G is non-Dedekind and that $R(G) \neq 1$. Then $p = 2$ and one of the following holds:*

(1) *G is the direct product of a quaternion group of order 8, a cyclic group of order 4 and an elementary abelian group;*

(2) *G is the direct product of two quaternion group of order 8 and an elementary abelian group;*
(3) *G is a Q-group, that*

$$G =< x, A|x^4 = 1, x^2 \in A, a^x = a^{-1} \; \forall a \in A >,$$

where A is abelian with $\exp(A) \neq 2$.

In 1995, Brandl [2] proved the following result:

Theorem 2.2 ([2]). *Let G be a finite nilpotent group with $\nu(G) = 1$. Then*

$$G \cong \langle a, b | a^{p^n} = b^p = 1, \; a^b = a^{1+p^{n-1}} \rangle,$$

where p is a prime and $n \geq 2$ if $p \geq 3$, and $n \geq 3$ if $p = 2$.

In 1998 the present author [6] characterized finite groups G for which $\nu(G) = 2$, and proved the following theorem.

Theorem 2.3 ([6]). *Let G be a finite nilpotent group with $\nu(G) = 2$. Then G is isomorphic to one of the following groups:*
(1) $P \times \mathbb{Z}_q$*, where P is nilpotent with $\nu(P) = 1$ and q is prime;*
(2) $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$*;*
(3) Q_{16}*, the generalized quaternion group of order 16;*
(4) $\langle x, y | x^4 = y^{2^n} = 1, \; y^x = y^{1+2^{n-1}} \rangle$*, where $n \geq 3$;*
(5) D_8*, the dihedral group of order 8.*

Remark 2.4. *In the above theorem, groups $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$ and Q_{16} have two conjugacy classes of order 4 with non-trivial intersection and in groups (4), (5) non-normal subgroups have order 4 and 2 respectively with trivial intersection. All groups have cyclic non-normal subgroups which do not contain each other. In the group (4), $H = \langle x \rangle$ and $K = \langle xy^{2^{n-4}} \rangle$ are the non-normal and non-conjugate subgroups of G and also G contains two normal subgroups $\langle y^{2^{n-4}} \rangle$ and $L = \langle x^2 y^{2^{n-4}} \rangle$ of order 4 too. Let $Z = \langle x^2 \rangle$, then two subgroup KZ/Z and LZ/Z have a trivial intersection.*

La Haye and Rhemtulla in [5, Lemma 3] prove the following lemma stating that for non-Dedekindian p-group G with $\nu(G) = 3$, $p \leq 3$.

Lemma 2.5. *Let G be a finite p-group with $\nu(G) > 0$. Then either $\nu(G) = 1$ or $\nu(G) \geq p$.*

By above Lemma, for a p-group with $\nu(G) = 3$, we have $p \leq 3$. This fact is useful but does not help us, so we prove our results without using the above Lemma.

Lemma 2.6. *Let G be a finite p-group and $H \ntrianglelefteq G$ be of order p. If for some central element z of order p, $\langle H, z\rangle \trianglelefteq G$ then either $Z(G)$ is cyclic or for any central element y such that $z \notin \langle y\rangle$, $\langle H, y\rangle \ntrianglelefteq G$.*

Proof. Let $Z(G)$ be non-cyclic and $\langle H, y\rangle \trianglelefteq G$. Since for some $g \in G$, $\langle H, z\rangle = \langle H, H^g\rangle = \Omega_1(\langle H, y\rangle)$, then $z \in \Omega_1(\langle H, y\rangle)$. Therefore $H \leqslant \Omega_1(\langle H, y\rangle) = \langle z, y\rangle \leqslant Z(G)$, which is contradiction. □

For a finite group G we denote by $k(G)$ the number of conjugacy classes subgroups of G.

Lemma 2.7. [6, Lemma 4.1] *If A and B are groups, then*

$$\nu(A \times B) \geq k(A)\nu(B) + k(B)\nu(A) - \nu(A)\nu(B).$$

The equality holds if $(|A|, |B|) = 1$.

Since for any group G, $k(G) \geq \nu(G) + 2$, then

$$\nu(A \times B) \geq 2(\nu(A) + \nu(B)) + \nu(A)\nu(B).$$

3. The Structure Theorems

We first assume that G is not of prime power order or $R(G) \neq 1$, next in two subsections we consider the case G is of prime power order with $R(G) = 1$.

Theorem 3.1. *Let G be a finite nilpotent group which is not of prime power order. If $\nu(G) = 3$ then G is the group (1) presented in Theorem 1.1.*

Proof. We can write $G = A \times B$ where $(|A|, |B|) = 1$. Hence, $\nu(A)$ and $\nu(B)$ can not be both non-zero so we can assume that $\nu(B) = 0$; now $\nu(G) = k(B)\nu(A)$ implies that $k(B) = 3$ and $\nu(A) = 1$. Therefore, A is the group presented in Theorem 2.2 and B is a cyclic group of order q^2 for some prime $q \neq p$. So G is group (1) presented in Theorem 1.1. □

Theorem 3.2. *Let G be a finite p-group with $\nu(G) = 3$. If $R(G) \neq 1$ then $G \cong Q_8 \times \mathbb{Z}_4$.*

Proof. Obviously, $\nu(Q_8 \times Q_8) = 15$ and $\nu(Q_8 \times \mathbb{Z}_4) = 3$. Hence, by Lemma 2.7 and Theorem 2.1, we have either $G \cong Q_8 \times \mathbb{Z}_4$ or G is a Q-group.

Let G be a Q-group. If A is cyclic, then G is generalized Quaternion and $\nu(G)$ is even by [5, Proposition 2.5]. So, A is not cyclic. Since $\nu(G/R(G)) = 3$, we can consider $a \in A$ of order $\ell = \exp(A)$ such that

$a^2 \neq x^2$, also $\langle a \rangle$ has a complement A_a in A. Assume that $\exp(A_a) > 2$, $b \in A_a$ is of order $\exp(A_a)$ and A_b is a complement of $\langle b \rangle$ in A_a. Hence, $G/\langle A_b, x^2 \rangle$ has a quotient isomorphic to $\mathbb{Z}_2 \times D_{2\ell}$, where $\ell \geq 4$. Otherwise $\exp(A_a) = 2$ and $A_a \leqslant Z(G)$. Now as $G/\langle x^2 \rangle \cong A_a\langle x^2 \rangle/\langle x^2 \rangle \times D_{2\ell}$ and $\nu(G) = \nu(G/\langle x^2 \rangle)$, either $\ell > 4$ or $A_a\langle x^2 \rangle/\langle x^2 \rangle \neq 1$. So in each case either G has a quotient isomorphic to $\mathbb{Z}_2 \times D_8$ or D_{16}, so $\nu(G) \geq 4$. which is impossible. $\square$

Hence, in continuation we assume that G is a p-group with $\nu(G) = 3$ and $R(G) = 1$. Also we assume that H, K and L are representative of three non-normal non-conjugate subgroups of G.

3.1. **Non-normal subgroups do not contain each other.** In this subsection we see that any subgroup of every non-normal subgroups of G, is normal. So every non-normal subgroups must be cyclic.

Lemma 3.3. *Let a finite p-group G with $\nu(G) \geq 3$ have a non-normal subgroup of order p. If any two non-normal and non-conjugate subgroups of G have a trivial intersection, then all the non-normal subgroups of G have the same order p.*

Proof. By Lemma 2.6, $Z(G)$ must be cyclic. Since any non-normal subgroup of G is cyclic so Z, the central subgroup of order p, must be contained in any non-normal subgroup of order greater than p. Therefore, G has at most one conjugacy class of non-normal subgroups of order greater than p, say L. Suppose that H and K are non-normal and non-conjugate subgroups of order p; since $\nu(G/Z) = 1$, then $HZ/Z = KZ/Z$ is the central subgroup of order p of G/Z. Hence, $KZ = HZ$ is elementary abelian of order p^2 which contains both H and K. Therefore, H and K must be conjugate, which is a contradiction. $\square$

Theorem 3.4. *Let G be a finite p-group with $\nu(G) = 3$ and $R(G) = 1$ such that at least two subgroups of L, H and K have non-trivial intersection. Then G is isomorphic to one of the group (3) or (4), presented in Theorem 1.1.*

Proof. Without lose of generality, we can assume that $H \cap K \neq 1$. We set $\bar{G} = G/(H \cap K)$. Since all the non-normal subgroups of G are cyclic, $H \cap L = K \cap L = 1$, hence, $\nu(\bar{G}) = 2$ and $R(\bar{G}) = 1$. Furthermore, $|\bar{H}| = |\bar{K}| \leq 4$ (by Theorem 2.3). If $|\bar{H}| = 4$, then for a maximal subgroup M of H, we have $\nu(G/M) = 1$, hence, the center of G/M is cyclic but KM/M and LM/M are cyclic normal subgroup with trivial

intersection (Remark 2.4), which is a contradiction. Therefore, $\bar{G} \cong D_8$ and $G = \langle H, K\rangle$ so $H \cap K \leqslant Z(G)$. Furthermore for $\bar{L} \trianglelefteq \bar{G}$ we have $|L| = |\bar{L}| \leq 4$.
Now, set $H = \langle x\rangle$ and $K = \langle y\rangle$ so we have $G = \langle x, y\rangle$ and $(xy)^4 \in H \cap K$, also $\bar{L} \leqslant \langle \bar{x}\bar{y}\rangle$, then $L \leqslant \langle xy\rangle(H \cap K)$. As $x^2, y^2 \in Z(G)$ we have $(xy)^2 = x^2y^2[x, y]$ and $(xy^{-1})^2 = x^2y^{-2}[x, y]$, so either $x^2 = y^2$ and $(xy^{-1})^2 = [x, y]$ or $x^2 = y^{-2}$ and $(xy)^2 = [x, y]$. Therefore, in either case $\langle [x, y]\rangle \trianglelefteq G$ and $G' = \langle [x, y]\rangle$.
Let z be an involution of $H \cap K$, since $\nu(G/\langle z\rangle) = 2$ and $(H \cap K)/\langle z\rangle = R(G/\langle z\rangle)$ then $|H \cap K| \leq 4$. If $|H \cap K| = 2$, then $x^2 = y^2$ and $(xy)^2 = [x, y]$ thus $\langle xy\rangle \trianglelefteq G$. Assume that $L = \langle t\rangle$, since $t \in \langle xy\rangle(H \cap K)$ we can write $t = (xy)^i x^2$. Now, we have

$$t^x = (xy^x)^i x^2 = (xy[y, x])^i x^2 = (xy)^{-i} x^2 = t^{-1},$$

similarly $t^y = (xy)^{3i} x^2 = t^3$. So $L \trianglelefteq G$, which is impossible. Hence, $|H \cap K| = 4$, and $|G| = 32$ also $o(x) = o(y) = 8$ and $x^4 = y^4$.

Now we distinguish two cases.

Case $|L| = 2$. By Lemma 2.6, the center of G is cyclic so $|G'| = 4$ and xy has order 8. If $Z(G) \neq H \cap K$ then $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $|G'| = 2$. Therefore, $Z(G) = H \cap K$. We set $g = xy$, since $x^2 \in Z(G)$ and $\langle g\rangle \trianglelefteq G$, (otherwise $(xy)^2 \in H \cap K$ and $\bar{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$) thus $[g, x] \in \langle g\rangle$, since $[g, x] = [y, x]$ has order 4, $[g, x] = g^{-1}$ or g^3. Suppose that $g^x = g^{-1}$, set $g_1 = x^2 g$, then $G = \langle x, g_1\rangle$ and $g_1^8 = 1$, $g_1^4 = x^4$ also $g_1^x = x^2 g^x = g_1^4 x^{-2} g^{-1} = g_1^3$. Hence, we can consider $g^x = g^{-1}$. Therefore, G presented the group (3) of Theorem 1.1.

Case $|L| = 4$. Since $L \cap Z(G) \neq 1$, then $|Z(G)| = 8$ and so $G' \leqslant Z(G)$ hence, $|G'| = 2$ and $o(xy) = 4$. If $\langle xy\rangle \trianglelefteq G$, then $\widetilde{G} = G/\langle xy\rangle$ must be cyclic, thus, $G' \leqslant \langle xy\rangle$, which implies that $\langle xy\rangle \cap L = \langle xy\rangle \cap K = 1$. Thus, $\widetilde{L} \leqslant \widetilde{K}$ and we must have $t = y^2(xy)^2$ where t is a generator of L. Hence, $t^2 = y^4(xy)^4 = 1$ which not happen. Therefore $\langle xy\rangle \ntrianglelefteq G$ and we can assume that $L = \langle xy\rangle$. Now we set $g = xy$, so $G' \leqslant \langle g^2, x^4\rangle$ hence, $[g, x] = g^2 x^4$. Therefore, G is the group (4) presented in Theorem 1.1. $\square$

The hypothesis of the next theorem implies that, all the non-normal subgroups of G have the same order. These groups are known, but the search for the target group of these groups do not appropriately, so we create these groups.

Theorem 3.5. *Let G be a finite p-group with $\nu(G) = 3$ and $R(G) = 1$ such that any two non-normal and non-conjugate subgroups of G have trivial intersection. Then G is isomorphic to one of the group (5) or (6), presented in Theorem 1.1.*

Proof. Let $H = \langle x \rangle$ and $K = \langle y \rangle$. First we let $|H| = p$; then all the non-normal subgroups of G have the same order p, by Lemma 3.1. Now by Lemma 2.6, the center of G is cyclic. Assume that Z is the central subgroup of order p. Since $\nu(G/Z) = 0$, we have $[x, y] \in Z$. If p is odd then for all $1 \leq i \leq p-1$, $(xy^i)^p = 1$ and $\langle xy^i \rangle \ntrianglelefteq G$, because xy^i is not central. Since $\langle H, Z \rangle \trianglelefteq G$ and contains all the conjugates of H, the subgroups $\langle x \rangle$, $\langle y \rangle$ and $\langle xy^i \rangle$, $(1 \leq i \leq p-1)$ are non-normal and non-conjugate, so we have a contradiction $\nu(G) \geq 4$. Therefore, $p = 2$.

Now, set $C_x = \mathcal{C}_G(x)$, $C_y = \mathcal{C}_G(y)$ and $C_{xy} = \mathcal{C}_G(xy)$. Since $xy \notin Z(G)$, $C_{xy} \lneqq G$. If $g \in G$ and $g \notin C_x \cup C_y$, then $[xy, g] = 1$ which implies that $g \in C_{xy}$ so $G = C_x \cup C_y \cup C_{xy}$ and hence, $G/Z(G)$ is the four group and $G = \langle x, y, z \rangle$ where $Z(G) = \langle z \rangle$. By Theorem 2.3, $|Z(G)| \geq 4$ and G is the group (5) of Theorem 1.1.

Next, we assume that $|H| \geq p^2$. We set $\bar{G} = G/\langle z \rangle$, where $z \in H$ is of prime order p; since $\nu(\bar{G}) = 1$ then $|H| = p^2$ by Theorem 2.2, so all the non-normal subgroups of G have same order p^2, also $G = \langle x, g \rangle$ for some $g \in G$. In the proof of [6, Theorem 4.4], step 1 and 2, $\langle x \rangle \cap \langle g \rangle = 1$, $|G'| = p$ and $\langle g \rangle \trianglelefteq G$ (otherwise we have a contradiction $|G| = p^4$ and $|Z(G)| \geq p^3$), therefore, $G \cong \langle g \rangle \rtimes \langle x \rangle$. If we consider $G' = \langle g^{p^{n-1}} \rangle$ for some n, then the subgroups $\langle x^\ell g^{p^{n-2}} \rangle$ for $1 \leq \ell \leq p-1$ are non-normal and non-conjugate. Since $\nu(G) = 3$, we have $p = 3$ and G is the group (6) presented in Theorem 1.1. □

3.2. **At least one non-normal subgroup contains the other.** In this subsection, without lose of generality, we can assume that $H \leqslant K$.

Theorem 3.6. *Let G be a finite p-group with $\nu(G) = 3$ and $R(G) = 1$ such that $H \leqslant K$. Then G is isomorphic to the group (7) presented in Theorem 1.1.*

Proof. Let z be a central element of order p. We prove the theorem in several steps.

Step 1: H does not contains any conjugate of L and also K does not contained in any conjugate of L.
Let H contains a conjugate of L, without loss of generality we can assume

that $L \leqslant H$ then $|L| = p$, because $R(G) = 1$. If $\langle L, z\rangle \trianglelefteq G$, then H must be cyclic (otherwise L has p conjugates that are contained in H, hence, $H \trianglelefteq G$ which is a contradiction) also $\langle H, z\rangle \ntrianglelefteq G$ because $L = \Phi(\langle H, z\rangle)$, thus, $K = \langle H, z\rangle$. Now $\nu(G/\langle z\rangle) = 1$ but $|K/\langle z\rangle| = |H| = p^2$, which is a contradiction. So $H = \langle L, z\rangle$. In this case $\nu(G/\langle z\rangle) = 2$ but $H/\langle z\rangle \leqslant K/\langle z\rangle$; again we have a contradiction by Remark 2.4. Therefore, $L \nleqslant H$ and similarly $K \nleqslant L$.

Step 2: H does not contained in any conjugate of L so $H \cap L = R(G) = 1$ and $|H| = p$.
Without loss of generality, we assume that $H \leqslant L$, then L and also any conjugate of L can not be contained in K (similar to the previous step) hence, H must be maximal in K and L so $H = K \cap L$ and $|H| = p$, because $R(G) = 1$. Now if one of K or L is cyclic, say K, then $\langle K, z\rangle \trianglelefteq G$; thus H must be normal, a contradiction. So K and L are non-cyclic, hence, there exists central elements z_1 and z_2 of prime order p such that $K = \langle H, z_1\rangle$ and $L = \langle H, z_2\rangle$. Since $\langle H, z_1z_2\rangle \trianglelefteq G$, H has p conjugates which are contained in both K and L, therefore $K = L$, a contradiction. Hence $H \nleqslant L$ and $H \cap L = R(G) = 1$. Now we must have $|H| = p$ by Remark 2.4.

Step 3: K does not contain any conjugate of L so $K = \langle H, z\rangle$ is of order p^2 and $Z(G)$ is cyclic.
Again we can assume that $L \leqslant K$; then both H and L must be maximal in K also we have $|H| = |L| = p$, thus, $|K| = p^2$ and H is conjugate to L, which is impossible. Therefore, H is maximal in K and so $|K| = p^2$. Since K can not be cyclic, $K = \langle H, z\rangle$ for some central element z of order p, which implies that $Z(G)$ is cyclic.

Step 4: $K \cap L = \langle z\rangle$, $p = 2$ and L is cyclic of order 4.
We show that $K \cap L \neq 1$. If not then $|L| = p$ and $\langle H, L\rangle \trianglelefteq G$ hence, $z \in \langle H, L\rangle$. So $K \trianglelefteq \langle H, L\rangle$, thus, $\nu(\langle H, L\rangle) = 2$ and we must have $\langle H, L\rangle \cong D_8$. Therefore, H has only two conjugates in G, which implies that $K \trianglelefteq G$, a contradiction. So $K \cap L = \langle z\rangle$ and L is cyclic. Since $\nu(G/\langle z\rangle) = 2$ we conclude that $G/\langle z\rangle \cong D_8$, thus, $|L| = 4$.

Now set $H = \langle x\rangle$ and $L = \langle y\rangle$; then $G = \langle x, y\rangle$. Since $[x, y] \neq z$, G' is cyclic of order 4. Also $y^2 = z$ and $[x, y^2] = 1$ as desired. □

Now we show the converse.

Theorem 3.7. *If G is one of the groups presented in Theorem 1.1, then* $\nu(G) = 3$.

Proof. By Lemma 2.7 we see that the group (1) has three conjugacy classes of non-normal subgroups. Also other groups except groups (5) and (6) have small order, so we can easily check by GAP [12] software that this groups, have three conjugacy classes of non-normal subgroups too.

In groups (5) and (6), $Z(G) = \Phi(G)$ and G' is central of prime order. In group (5), $Z(G)$ is cyclic so all the subgroups of order greater than 2 contain G'; so is normal. Therefore, non-normal subgroups have just prime order 2. Hence, if $H \ntrianglelefteq G$ then $H = \langle x \rangle$, $\langle y \rangle$ or $\langle xyz^{2^{n-2}} \rangle$ which are non-normal and non-conjugate. In the group (6), $\Omega_1(G) \leqslant Z(G)$ and any cyclic subgroup of order greater than 9 contains G', so all the non-normal subgroups are of order 9, and must be cyclic. Let t be an element of order 9; then $t \notin Z(G)$, $\langle g \rangle$, also we have $\langle t \rangle^g = \langle tg^{3^{n-1}} \rangle$, so $t = x^i g^{j3^{n-2}}$ where $3 \nmid i$, $i < 9$ and $0 \leq j \leq 2$. We set $t_1 = xg^{3^{n-2}}$ and $t_2 = xg^{2\cdot 3^{n-2}}$; then subgroups $\langle x \rangle$, $\langle t_1 \rangle$ and $\langle t_2 \rangle$ are non-normal and non-conjugate. We show that for all i and j, $\langle t \rangle$ is conjugate to one of these three subgroups. For $j = 0$ obviously $t \in \langle x \rangle$; for $j \neq 0$ we show that in the following table, t is equal to the entry (i, j) for all i and j, as desired. □

$j \backslash i$	1	2	4	5	7	8
1	t_1	$(t_2^g)^2$	$(t_1^{g^2})^4$	t_2^5	$(t_1^g)^7$	$(t_2^{g^2})^8$
2	t_2	t_1^2	$(t_2^g)^4$	$(t_1^g)^5$	$(t_2^{g^2})^7$	$(t_1^{g^2})^8$

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