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EXISTENCE AND MULTIPLICITY OF NONTRIVIAL SOLUTIONS FOR p-LAPLACIAN SYSTEM WITH NONLINEARITIES OF CONCAVE-CONVEX TYPE AND SIGN-CHANGING WEIGHT FUNCTIONS

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Abstract. This paper is concerned with the existence of multiple positive solutions for a quasilinear elliptic system involving concave-convex nonlinearities and sign-changing weight functions. With the help of the Nehari manifold and Palais-Smale condition, we prove that the system has at least two nontrivial positive solutions, when the pair of parameters (λ, μ) belongs to a certain subset of $\mathbb{R}^2$.
Keywords: Variational methods, Nehari manifold, Dirichlet boundary condition, sign-changing weight functions.
MSC(2010): Primary: 35J50; Secondary: 35J62, 35J92.

1. Introduction and notation

There is a wide literature that deals with the existence of multiple solutions to semilinear elliptic boundary value problems. Conditions that guarantee the existence of multiple solutions to differential equations are of interest, because physical processes described by differential equations can exhibit more than one solution. In recent years, many works have been carried out to discuss the existence and multiplicity of positive solutions for BVPs by variational methods, for example, see [1, 2, 5, 7, 9, 10, 12, 13, 18, 19].
In this paper, we are interested in the existence of two nontrivial positive solutions for the following nonlinear elliptic system:

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(1.1)
$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda\ f(x)|u|^{q-2}u + \dfrac{\alpha}{\alpha+\beta} b(x)|u|^{\alpha-2}u|v|^{\beta} & x \in \Omega, \\ -\Delta_p v + a(x)|v|^{p-2}v = \mu\ g(x)|v|^{q-2}v + \dfrac{\beta}{\alpha+\beta} b(x)|v|^{\beta-2}v|u|^{\alpha} & x \in \Omega, \\ u = v = 0 & x \in \partial\Omega, \end{cases}$$

where $0 \in \Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary and $\partial\Omega$, $\lambda, \mu > 0$, $1 \leq q < p < N$, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian. Also $\alpha > 1, \beta > 1$ satisfy $p < \alpha + \beta \leq p^*$, and $p^* = \dfrac{pN}{N-p}$ is the so-called critical Sobolev exponent. We make the following assumptions for the weight functions a, b, f, g :

(A) $a \in C(\overline{\Omega})$, $a(x) > 0$;

(B) $b \in C(\overline{\Omega})$, $b^+ = \max\{b, 0\} \not\equiv 0$ and $|b|_\infty = 1$

(C) $f, g \in C(\overline{\Omega})$, $f^+ = \max\{f, 0\} \not\equiv 0$, and $g^+ = \max\{g, 0\} \not\equiv 0$.

In many problems of mathematical physics and engineering it is not sufficient to deal with the classical solutions of differential equations. It is necessary to introduce variational methods involving Nehari manifold and Palais-Smale condition.

Here we give a variational method to prove the existence of at least two nontrivial nonnegative solutions of problem (1.1) in two cases.

Set $f(x) = g(x) = b(x) = 1$ and $\alpha + \beta = p^*$, $1 < q < p < N$, then (1.1) reduces to

(1.2)
$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda|u|^{q-2}u + \dfrac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} & x \in \Omega, \\ -\Delta_p v + a(x)|v|^{p-2}v = \mu|v|^{q-2}v + \dfrac{\beta}{\alpha+\beta}|v|^{\beta-2}v|u|^{\alpha} & x \in \Omega, \\ u = v = 0 & x \in \partial\Omega. \end{cases}$$

In our recent work ([15]), we proved that there exists $\Lambda_2 > 0$ such that if the parameters $\lambda, \mu > 0$, satisfy $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_2$ then problem (1.2) has at least two nontrivial positive solutions.

Wu in [18] has investigated the following semilinear elliptic system with subcritical nonlinearity:

$$(1.3)\qquad \begin{cases} -\Delta u = \lambda f(x)|u|^{q-2}u + \dfrac{\alpha}{\alpha+\beta}h(x)|u|^{\alpha-2}u|v|^{\beta} & x \in \Omega, \\ -\Delta v = \mu g(x)|v|^{q-2}v + \dfrac{\beta}{\alpha+\beta}h(x)|v|^{\beta-2}v|u|^{\alpha} & x \in \Omega, \\ u = v = 0 & x \in \partial\Omega, \end{cases}$$

where $1 < q < 2 < \alpha+\beta < 2^*$ with $\alpha > 1, \beta > 1$, and the weights f, g, h satisfy some suitable conditions. He proved that problem (1.3) has at least two nontrivial positive solutions when the pair of (λ, μ) belongs to a certain subset of $\mathbb{R}^2$.

Hsu in [13] also considered problem (1.3) in the case of the p-Laplacian operator. Motivated by the above paper, we consider the problem (1.1) and extend the results of the literature [13].

In this paper we use of the following notations.

$L^s(\Omega)$ where $1 \le s < \infty$, denote Lebesgue spaces and the norm in L^s is denoted by $|\cdot|_s$ for $1 \le s \le \infty$;

The dual space of a Banach space W will be denoted by W^{-1};
(u, v) is said to be nonnegative in Ω if $u \ge 0$ and $v \ge 0$ in Ω;
(u, v) is said to be positive in Ω if $u > 0$ and $v > 0$ in Ω;
$|\Omega|$ is the Lebesgue measure of Ω;
$O(\varepsilon^t)$ denotes $\dfrac{|O(\varepsilon^t)|}{\varepsilon^t} \le C$ as $\varepsilon \to 0$ for $t \ge 0$;
$o(1)$ denotes $o(1) \to 0$ as $n \to \infty$;
C, C_i denote various positive constants, the exact values of which are not important;
$p^* = \dfrac{pN}{N-p}(1 < p < N)$ is the critical Sobolev exponent;
S is the best Sobolev embedding constant defined by

$$(1.4)\qquad S = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega (|\nabla u|^p + a(x)|u|^p)dx}{\left(\displaystyle\int_\Omega |\, u\,|^{\alpha+\beta} dx\right)^{\frac{p}{\alpha+\beta}}};$$

By modifying the proof of Alves et al. [4, Theorem 5], we have

$$S_{\alpha,\beta} = \left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \right) S, \tag{1.5}$$

where S is the best Sobolev constant defined in (1.4) and

$$S_{\alpha,\beta} = \inf_{u,v \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega (|\nabla u|^p + a(x)|u|^p)dx + \int_\Omega (|\nabla v|^p + a(x)|v|^p)dx}{\left(\displaystyle\int_\Omega |u|^\alpha |v|^\beta dx\right)^{\frac{p}{\alpha+\beta}}}, \tag{1.6}$$

when $\alpha + \beta = p^*$. This is achieved if and only if $\Omega = \mathbb{R}^N$ by the function

$$U_\varepsilon(x) = C_N \left(\frac{\varepsilon^{\frac{1}{p}}}{\varepsilon + |x|^{\frac{p}{p-1}}} \right)^{(N-p)/p}, \quad \varepsilon > 0.$$

We organize this paper into four sections. In section 2, we give properties of Nehari manifold and set up the variational method. In section 3, we consider Palais-Smale condition and in the last section we give our main results.

2. **The Nehari manifold**

Problem (1.1) is posed in the framework of the Sobolev space $W = W_0^{1,p} \times W_0^{1,p}$ equipped with the norm

$$\| z \| = \left(\int_\Omega (|\nabla u|^p + a(x)|u|^p)dx + \int_\Omega (|\nabla v|^p + a(x)|v|^p)dx \right)^{1/p}, z = (u,v) \in W.$$

Moreover, z is said to be a weak solution of problem (1.1) if for all $(\varphi_1, \varphi_2) \in W$, there holds

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \nabla \varphi_1 + a(x) u \varphi_1)dx + \int_\Omega (|\nabla v|^{p-2}\nabla v \nabla \varphi_2 + a(x) v \varphi_2)dx$$
$$-\lambda \int_\Omega f(x)|u|^{q-2}u\varphi_1 - \mu \int_\Omega g(x)|v|^{q-2}v\varphi_2 dx$$
$$-\frac{\alpha}{\alpha+\beta}\int_\Omega b(x)|u|^{\alpha-2}u|v|^\beta \varphi_1 dx - \frac{\beta}{\alpha+\beta}\int_\Omega b(x)|v|^{\beta-2}v|u|^\alpha \varphi_2 dx = 0,$$

It is clear that problem (1.1) has a variational structure. Let $J_{\lambda,\mu} : W \to \mathbb{R}$ be the corresponding energy functional of problem (1.1), which is defined by

$$J_{\lambda,\mu}(z) = \frac{1}{p} \parallel z \parallel^p - \frac{1}{q} K_{\lambda,\mu}(z) - \frac{1}{\alpha+\beta} L(z), \ \forall z \in W$$

for which $K_{\lambda,\mu}, L : W \to \mathbb{R}$ are the functionals defined by

$$K_{\lambda,\mu}(z) = \int_\Omega (\lambda f(x)|u|^q + \mu g(x)|v|^q)dx, \qquad L(z) = \int_\Omega b(x)|u|^\alpha |v|^\beta dx.$$

It is well known that the weak solution of problem (1.1) is the critical point of the energy functional $J_{\lambda,\mu}$. Thus, to prove the existence of weak solutions for problem (1.1), it is sufficient to show that $J_{\lambda,\mu}$ admits a critical points. As the energy functional $J_{\lambda,\mu}$ is not bounded below on W, it is useful to consider the functional $J_{\lambda,\mu}$ on the Nehari manifold

$$N_{\lambda,\mu} = \{z \in W \setminus \{0\} | \langle J'_{\lambda,\mu}(z), z\rangle = 0\}.$$

Obviously $z \in N_{\lambda,\mu}$ if and only if

$$\langle J'_{\lambda,\mu}(z), z\rangle = \parallel z \parallel^p - K_{\lambda,\mu}(z) - L(z) = 0. \tag{2.1}$$

Note that $N_{\lambda,\mu}$ contains every nontrivial weak solution of problem (1.1). Define

$$\phi_{\lambda,\mu}(z) = \langle J'_{\lambda,\mu}(z), z\rangle.$$

Then, for any $z \in N_{\lambda,\mu}$,

$$\begin{aligned} \langle \phi'_{\lambda,\mu}(z), z\rangle &= p \parallel z \parallel^p - qK_{\lambda,\mu}(z) - (\alpha+\beta)L(z) && (2.2)\\ &= (p-q) \parallel z \parallel^p - (\alpha+\beta-q)L(z) && (2.3)\\ &= (\alpha+\beta-q)K_{\lambda,\mu}(z) - (\alpha+\beta-p) \parallel z \parallel^p . && (2.4) \end{aligned}$$

It is natural to split $N_{\lambda,\mu}$ into three disjoint parts:

$$\begin{aligned} N^+_{\lambda,\mu} &= \{z \in N_{\lambda,\mu} : \langle \phi'_{\lambda,\mu}(z), z\rangle > 0\},\\ N^0_{\lambda,\mu} &= \{z \in N_{\lambda,\mu} : \langle \phi'_{\lambda,\mu}(z), z\rangle = 0\},\\ N^-_{\lambda,\mu} &= \{z \in N_{\lambda,\mu} : \langle \phi'_{\lambda,\mu}(z), z\rangle < 0\}, \end{aligned}$$

similar to the method used in Tarantello ([16]). We now derive some important properties of $N^+_{\lambda,\mu}$, $N^0_{\lambda,\mu}$ and $N^-_{\lambda,\mu}$.

Lemma 2.1. *$J_{\lambda,\mu}$ is coercive and bounded from below on $N_{\lambda,\mu}$.*

Proof. If $z \in N_{\lambda,\mu}$, it follows from (2.1), (C), and the Hölder inequality and the Sobolev embedding theorem, that

$$\begin{aligned} J_{\lambda,\mu}(z) &= \frac{\alpha+\beta-p}{p(\alpha+\beta)} \parallel z \parallel^p - \frac{\alpha+\beta-q}{q(\alpha+\beta)} K_{\lambda,\mu}(z) \\ &\geq \frac{\alpha+\beta-p}{p(\alpha+\beta)} \parallel z \parallel^p - \frac{\alpha+\beta-q}{q(\alpha+\beta)} S^{-\frac{q}{p}} |\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}} \varrho_{\lambda,\mu}^{\frac{p-q}{p}} \parallel z \parallel^q, \end{aligned}$$

where S is the best Sobolev embedding constant defined in (1.4) and

$$\varrho_{\lambda,\mu} = (\lambda|f|_\infty)^{\frac{p}{p-q}} + (\mu|g|_\infty)^{\frac{p}{p-q}}. \tag{2.5}$$

Since $1 \leq q < p$, we get that $J_{\lambda,\mu}$ is coercive and bounded below on $N_{\lambda,\mu}$. □

Lemma 2.2. *Suppose that z_0 is a local minimizer for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$ and $z_0 \notin N^0_{\lambda,\mu}$. Then z_0 is a critical point of $J_{\lambda,\mu}$, that means, $J'_{\lambda,\mu}(z_0) = 0$ in W^{-1}.*

Proof. If z_0 is a local minimizer for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$, then z_0 is a solution of optimization problem. Since $\phi_{\lambda,\mu}(z) = \langle J'_{\lambda,\mu}(z), z\rangle$, then by the theory of Lagrange multipliers, there exists $\gamma \in \mathbb{R}$ such that

$$\langle J'_{\lambda,\mu}(z_0), z_0\rangle = \gamma \langle \phi'_{\lambda,\mu}(z_0), z_0\rangle.$$

Since $z_0 \in N_{\lambda,\mu}$ and $z_0 \notin N^0_{\lambda,\mu}$ we get $\langle \phi'_{\lambda,\mu}(z_0), z_0\rangle \neq 0$ and so $\gamma = 0$. This completes the proof. □

Lemma 2.3. *(i) if $z \in N^+_{\lambda,\mu}$, then $K_{\lambda,\mu}(z) > 0$;*

(ii) if $z \in N^0_{\lambda,\mu}$, then $K_{\lambda,\mu}(z) > 0$ and $L(z) > 0$;

(iii) if $z \in N^-_{\lambda,\mu}$, then $L(z) > 0$.

Proof. The proof is obtained from (2.2)-(2.4). □

Lemma 2.4. *Set*

(2.6)
$$\Lambda_0 = \left(\frac{p-q}{\alpha+\beta-q}\right)^{\frac{p}{\alpha+\beta-q}} \left(\frac{\alpha+\beta-q}{\alpha+\beta-p}|\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}}\right)^{-\frac{p}{p-q}} S^{\frac{\alpha+\beta}{\alpha+\beta-p}+\frac{q}{p-q}} > 0.$$

Then for (λ,μ) satisfying $0 < \varrho_{\lambda,\mu} < \Lambda_0$, we have $N^0_{\lambda,\mu} = \emptyset$.

Proof. Assume contrary, i.e., there exist $\lambda, \mu > 0$ with $0 < \varrho_{\lambda,\mu} < \Lambda_0$ such that $N^0_{\lambda,\mu} \neq \emptyset$. Then for $z \in N^0_{\lambda,\mu}$, by (2.3), (2.4) we have that

$$\| z \|^p = \frac{\alpha+\beta-q}{p-q} L(z), \qquad \| z \|^p = \frac{\alpha+\beta-q}{\alpha+\beta-p} K_{\lambda,\mu}(z).$$

Then (B) implies

$$\| z \| \geq \left(\frac{p-q}{\alpha+\beta-q} S^{\frac{\alpha+\beta}{p}} \right)^{\frac{1}{\alpha+\beta-p}}.$$

It follows from (C), the Hölder inequality and the Sobolev embedding theorem,

$$\| z \| \leq \left(\frac{\alpha+\beta-q}{\alpha+\beta-p} S^{-\frac{q}{p}} |\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}} \right)^{\frac{1}{p-q}} \varrho_{\lambda,\mu}.$$

This implies

$$\varrho_{\lambda,\mu} \geq \left(\frac{p-q}{\alpha+\beta-q} \right)^{\frac{p}{\alpha+\beta-p}} \left(\frac{\alpha+\beta-q}{\alpha+\beta-p} |\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}} \right)^{-\frac{p}{p-q}} S^{\frac{\alpha+\beta}{\alpha+\beta-p}+\frac{q}{p-q}} = \Lambda_0,$$

which is a contradiction. This proves the Lemma. □

Let $\Theta_\Lambda = \{(\lambda,\mu) \in \mathbb{R}^2 \backslash (0,0) : 0 < \varrho_{\lambda,\mu} < \Lambda\}$ and $\Lambda_1 = (\frac{q}{p})^{\frac{p}{p-q}} \Lambda_0 < \Lambda_0$. By lemma (2.4), for every $(\lambda,\mu) \in \Theta_{\Lambda_0}$, we have $N_{\lambda,\mu} = N^+_{\lambda,\mu} \cup N^-_{\lambda,\mu}$. So we define

$$\theta_{\lambda,\mu} = \inf_{z \in N_{\lambda,\mu}} J_{\lambda,\mu}(z),\ \theta^+_{\lambda,\mu} = \inf_{z \in N^+_{\lambda,\mu}} J_{\lambda,\mu}(z),\ \theta^-_{\lambda,\mu} = \inf_{z \in N^-_{\lambda,\mu}} J_{\lambda,\mu}(z).$$

Then we have the following result.

Theorem 2.5. *(i) If $(\lambda,\mu) \in \Theta_{\Lambda_0}$, then $\theta_{\lambda,\mu} \leq \theta^+_{\lambda,\mu} < 0$;*
(ii) If $(\lambda,\mu) \in \Theta_{\Lambda_1}$, then there exists $d_0 = d_0(\lambda,\mu,p,q,N,S,|\Omega|,|f|_\infty,$ $|g|_\infty) > 0$ such that $\theta^-_{\lambda,\mu} > d_0$.

Proof. (i) For any $z \in N^+_{\lambda,\mu}$, it follows from (2.4), that

$$K_{\lambda,\mu}(z) > \frac{\alpha+\beta-p}{\alpha+\beta-q} \| z \|^p$$

and

$$\begin{aligned} J_{\lambda,\mu}(z) &= \left(\frac{1}{p}-\frac{1}{\alpha+\beta}\right)\| z \|^p - \left(\frac{1}{q}-\frac{1}{\alpha+\beta}\right)K_{\lambda,\mu}(z) \\ &< \left(\frac{1}{p}-\frac{1}{\alpha+\beta}\right)\| z \|^p - \left(\frac{1}{q}-\frac{1}{\alpha+\beta}\right)\frac{\alpha+\beta-p}{\alpha+\beta-q}\| z \|^p \\ &< \frac{\alpha+\beta-p}{\alpha+\beta}\left(\frac{1}{p}-\frac{1}{q}\right)\| z \|^p < 0. \end{aligned}$$

This gives $\theta_{\lambda,\mu} \leq \theta^+_{\lambda,\mu} < 0$.

(ii) For any $z \in N^-_{\lambda,\mu}$, we obtain from (2.3) that

$$\frac{p-q}{\alpha+\beta-q}\| z \|^p < L(z). \tag{2.7}$$

Moreover, using (1.4) and (1.5) , since $S_{\alpha,\beta} > S$, we get

$$L(z) \leq S^{-\frac{(\alpha+\beta)}{p}} \| z \|^{\alpha+\beta}.$$

This implies that

$$\| z \| \geq \left(\frac{p-q}{\alpha+\beta-q}\right)^{\frac{1}{\alpha+\beta-p}} S^{\frac{\alpha+\beta}{p(\alpha+\beta-p)}}.$$

Using the main formula in the proof of Lemma (2.1), we have

$$\begin{aligned} J_{\lambda,\mu}(z) &\geq \| z \|^q \left[\frac{\alpha+\beta-p}{p(\alpha+\beta)}\| z \|^{p-q} - \frac{\alpha+\beta-q}{q(\alpha+\beta)}S^{-\frac{q}{p}}|\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}}\varrho_{\lambda,\mu}^{\frac{p-q}{p}}\right] \\ &> \left(\frac{p-q}{\alpha+\beta-q}\right)^{\frac{q}{\alpha+\beta-p}} S^{\frac{q(\alpha+\beta)}{p(\alpha+\beta-p)}} \\ &\quad \times \left[\frac{\alpha+\beta-p}{p(\alpha+\beta)}S^{\frac{(p-q)(\alpha+\beta)}{p(\alpha+\beta-p)}}\left(\frac{p-q}{\alpha+\beta-q}\right)^{\frac{p-q}{\alpha+\beta-p}} \right. \\ &\quad \left. -\frac{\alpha+\beta-q}{q(\alpha+\beta)}S^{-\frac{q}{p}}|\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}}\varrho_{\lambda,\mu}^{\frac{p-q}{p}}\right]. \end{aligned}$$

Thus, for any $(\lambda,\mu) \in \Theta_{\Lambda_1}$ and $z \in N^-_{\lambda,\mu}$, we have

$$J_{\lambda,\mu}(z) > d_0 = d_0(\lambda,\mu,p,q,N,S,|\Omega|,|f|_\infty,|g|_\infty) > 0.$$

□

For each $z \in W$ with $L(z) > 0$, we write

$$t_{\max} = \left(\frac{(p-q)\| z \|^p}{(\alpha+\beta-q)L(z)} \right)^{\frac{1}{\alpha+\beta-p}} > 0. \tag{2.8}$$

Then the following lemma holds.

Lemma 2.6. *Assume that $(\lambda, \mu) \in \Theta_{\Lambda_0}$, then for each $z \in W$ with $L(z) > 0$, we have:*

(i) if $K_{\lambda,\mu}(z) \leq 0$, then there is a unique $t^- = t^-(z) > t_{\max}$ such that $t^- z \in N^-_{\lambda,\mu}$ and

$$J_{\lambda,\mu}(t^- z) = \sup_{t \geq 0} J_{\lambda,\mu}(tz); \tag{2.9}$$

(ii) if $K_{\lambda,\mu}(z) > 0$, then there are unique $0 < t^+ = t^+(z) < t_{\max} < t^- = t^-(z)$, such that $t^\pm z \in N^\pm_{\lambda,\mu}$ and

$$J_{\lambda,\mu}(t^+ z) = \inf_{0 \leq t \leq t_{\max}} J_{\lambda,\mu}(tz); \qquad J_{\lambda,\mu}(t^- z) = \sup_{t \geq 0} J_{\lambda,\mu}(tz). \tag{2.10}$$

Proof. Fix $z \in W$ with $L(z) > 0$, we define

$$m(t) = t^{p-q}\| z \|^p - t^{\alpha+\beta-q} L(z),$$

for $t \geq 0$. Clearly $m(0) = 0$ and $m(t) \to -\infty$ as $t \to \infty$. Furthermore

$$m'(t) = (p-q)t^{p-q-1}\| z \|^p - (\alpha+\beta-q)t^{\alpha+\beta-q-1}L(z),$$

there is a unique $t_{\max} > 0$ such that $m(t)$ achieves its maximum at $t_{\max} > 0$, increasing for $t \in [0, t_{\max})$ and decreasing for $t \in (t_{\max}, \infty)$. Clearly, $tz \in N^+_{\lambda,\mu}$(or $N^-_{\lambda,\mu}$) if and only if $m'(t) > 0$(or < 0). Moreover,

$$m(t_{\max}) = \left(\frac{(p-q)\| z \|^p}{(\alpha+\beta-q)L(z)} \right)^{\frac{p-q}{\alpha+\beta-p}} \| z \|^p - \left(\frac{(p-q)\| z \|^p}{(\alpha+\beta-q)L(z)} \right)^{\frac{\alpha+\beta-q}{\alpha+\beta-p}} L(z)$$

$$= \| z \|^q \left[\left(\frac{p-q}{\alpha+\beta-q} \right)^{\frac{p-q}{\alpha+\beta-p}} - \left(\frac{p-q}{\alpha+\beta-q} \right)^{\frac{\alpha+\beta-q}{\alpha+\beta-p}} \right] \left(\frac{\| z \|^{\alpha+\beta}}{L(z)} \right)^{\frac{p-q}{\alpha+\beta-p}}$$

$$\geq \| z \|^q \left(\frac{p-q}{\alpha+\beta-q} \right)^{\frac{p-q}{\alpha+\beta-p}} \left(\frac{\alpha+\beta-q}{\alpha+\beta-q} \right) \left(S^{\frac{\alpha+\beta}{p}} \right)^{\frac{p-q}{\alpha+\beta-p}} \tag{2.11}$$

(i) If $K_{\lambda,\mu}(z) \leq 0$, then there exists a unique $t^- > t_{\max}$ such that $m(t^-) = K_{\lambda,\mu}(z)$ and $m'(t^-) < 0$. Now,

$$(p-q)(t^-)^p \| z \|^p - (\alpha+\beta-q)(t^-)^{\alpha+\beta} L(z) = (t^-)^{q+1} m(t^-) < 0,$$

and

$$\langle J'_{\lambda,\mu}(t^- z), (t^- z)\rangle = (t^-)^q [m(t^-) - K_{\lambda,\mu}(z)] = 0.$$

Thus, $t^- z \in N^-_{\lambda,\mu}$. Subsequently, $m'(t) < 0$ and $m''(t) < 0$ for $t > t_{\max}$. Then

$$J_{\lambda,\mu}(t^- z) = \sup_{t \geq 0} J_{\lambda,\mu}(tz).$$

(ii) Suppose that $K_{\lambda,\mu}(z) > 0$. Then for $(\lambda,\mu) \in \Theta_{\Lambda_1}$, we have

$$\begin{aligned} m(0) = 0 \quad &< \quad K_{\lambda,\mu}(z) \\ &\leq \quad S^{-\frac{q}{p}} |\Omega|^{\frac{\alpha+\beta-q}{\alpha+\beta}} \varrho_{\lambda,\mu}^{\frac{p-q}{p}} \| z \|^q \\ &\leq \quad \| z \|^q \left(\frac{p-q}{\alpha+\beta-q} \right)^{\frac{p-q}{\alpha+\beta-p}} \left(\frac{\alpha+\beta-q}{\alpha+\beta-q} \right) \left(S^{\frac{\alpha+\beta}{p}} \right)^{\frac{p-q}{\alpha+\beta-p}}, \end{aligned}$$

by (2.11), there are unique t^+ and t^- such that $0 < t^+ = t^+(z) < t_{\max} < t^- = t^-(z)$,

$$m(t^+) = K_{\lambda,\mu}(z) = m(t^-), \qquad m'(t^+) > 0 > m'(t^-).$$

Moreover, we have $t^{\pm} z \in N^{\pm}_{\lambda,\mu}$, and

$$J_{\lambda,\mu}(t^- z) \geq J_{\lambda,\mu}(tz) \geq J_{\lambda,\mu}(t^+ z), \qquad \forall t \in [t^+, t^-],$$
$$J_{\lambda,\mu}(t^+ z) \leq J_{\lambda,\mu}(tz), \qquad \forall t \in [0, t_{\max}].$$

Thus

$$J_{\lambda,\mu}\ (t^+ z) = \inf_{0 \leq t \leq t_{\max}} J_{\lambda,\mu}(tz); J_{\lambda,\mu}\ (t^- z) = \sup_{t \geq 0} J_{\lambda,\mu}(tz)$$

□

For each $z \in W$ with $L(z) > 0$, we write

$$\bar{t}_{\max} = \left(\frac{(\alpha+\beta-q) K_{\lambda,\mu}(z)}{(\alpha+\beta-q)) \| z \|^p} \right)^{\frac{1}{p-q}} > 0. \tag{2.12}$$

Then we have the following lemma.

Lemma 2.7. *Assume that* $(\lambda, \mu) \in \Theta_{\Lambda_0}$*, then for each* $z \in W$ *with* $K_{\lambda,\mu}(z) > 0$*, we have:*

(i) if $L(z) \leq 0$*, then there exists* $0 < t^+ = t^+(z) < \bar{t}_{\max}$ *such that* $t^+ z \in N^+_{\lambda,\mu}$ *and*

$$J_{\lambda,\mu}(t^+ z) = \inf_{t \geq 0} J_{\lambda,\mu}(tz); \tag{2.13}$$

(ii) if $L(z) > 0$*, then there exist* $0 < t^+ = t^+(z) < \bar{t}_{\max} < t^- = t^-(z)$*, such that* $t^{\pm} z \in N^{\pm}_{\lambda,\mu}$ *and*

$$J_{\lambda,\mu}(t^+ z) = \inf_{0 \leq t \leq \bar{t}_{\max}} J_{\lambda,\mu}(tz); \qquad J_{\lambda,\mu}(t^- z) = \sup_{t \geq 0} J_{\lambda,\mu}(tz). \tag{2.14}$$

Proof. Fix $z \in W$ with $K_{\lambda,\mu}(z) > 0$. Let

$$\bar{m}(t) = t^{p-\alpha-\beta} \| z \|^p - t^{q-\alpha-\beta} K_{\lambda,\mu}(z),$$

for $t \geq 0$. Clearly $\bar{m}(t) \to -\infty$ as $t \to 0^+$ and $\bar{m}(t) \to 0$ as $t \to \infty$. Since

$$\bar{m}'(t) = (p-\alpha-\beta) t^{p-\alpha-\beta-1} \| z \|^p - (q-\alpha-\beta) t^{q-\alpha-\beta-1} K_{\lambda,\mu}(z),$$

there is a unique $\bar{t}_{\max} > 0$ such that $m(t)$ achieves its maximum at $\bar{t}_{\max}$, increasing for $t \in [0, \bar{t}_{\max})$ and decreasing for $t \in (\bar{t}_{\max}, \infty)$. Similar to the argument in the proof of Lemma (2.6), we can derive the result of Lemma. □

3. Palais-Smale condition

At first, we give the following definitions about $(PS)_c$ -sequence and introduce the Brézis-Lieb lemma (see [8]) as a remark.

Definition 3.1. *Let* $c \in \mathbb{R}$*,* W *be a Banach space and* $J \in C^1(W, \mathbb{R})$*.*
(i) $\{z_n\}$ *is a* $(PS)_c$*- sequence in* W *for* J *if* $J(z_n) = c + o(1)$ *and* $J'(z_n) = o(1)$ *strongly in* W^{-1} *as* $n \to \infty$*.*

(ii) J *satisfies the* $(PS)_c$ *condition if any* $(PS)_c$*-sequence* $\{z_n\}$ *in* W *for* J *has a convergent subsequence.*

Remark 3.2. *Let $z_n \in W$ such that*
(i) $\| z_n \| \leq$ *a constant;*
(ii) $z_n \to z_0$ *almost every where in* Ω, *then*

$$\| \bar{z}_n \|^p = \| z_n \|^p - \| z_0 \|^p + o(1), \tag{3.1}$$

as $n \to \infty$ *where* $\bar{z}_n = z_n - z_0$.

Next, we will find the range of c where $(PS)_c$ condition holds for $J_{\lambda,\mu}$.

Lemma 3.3. *Assume that* $\{z_n\} \subset W$ *is a* $(PS)_c$*- sequence for* $J_{\lambda,\mu}$ *and* $z_n \rightharpoonup z$ *in W, then z is critical point of* $J_{\lambda,\mu}$, *and there exists a* $C_0 = C_0(p, q, N, S, |\Omega|) > 0$ *such that* $J_{\lambda,\mu} \geq -C_0 \varrho_{\lambda,\mu}$.

Proof. Let $z_n = (u_n, v_n)$ and assume that $\{z_n\}$ is a $(PS)_c$-sequence for $J_{\lambda,\mu}$ with $z_n \rightharpoonup z$ in W, it is easy to see that $J'_{\lambda,\mu}(z) = 0$, so $\langle J'_{\lambda,\mu}(z), z\rangle = 0$. It follows from (2.1) that

$$L(z) = \| z \|^p - K_{\lambda,\mu}(z).$$

Consequently,

$$J_{\lambda,\mu}(z) = \frac{\alpha+\beta-p}{p(\alpha+\beta)} \| z \|^p - \frac{\alpha+\beta-q}{q(\alpha+\beta)} K_{\lambda,\mu}(z).$$

By (C), the Hölder inequality and the Sobolev embedding theorem, we obtain

$$J_{\lambda,\mu}(z) \geq \frac{\alpha+\beta-p}{p(\alpha+\beta)} \| z \|^p - \frac{\alpha+\beta-q}{q(\alpha+\beta)} S^{-\frac{q}{p}} |\Omega|^{\frac{p^*-q}{p^*}}$$

$$\times \left[\lambda \left(\int_\Omega (|\nabla u|^p + a(x)|u|^p) dx \right) + \mu \left(\int_\Omega (|\nabla v|^p + a(x)|v|^p) dx \right) \right]^{\frac{q}{p}}.$$

It follows from the Young inequality, that

$$J_{\lambda,\mu}(z) \geq \frac{\alpha+\beta-p}{p(\alpha+\beta)} \| z \|^p - \frac{\alpha+\beta-p}{p(\alpha+\beta)} \| z \|^p - C_0 \varrho_{\lambda,\mu} = -C_0 \varrho_{\lambda,\mu},$$

in which $C_0 = C_0(p, q, N, S, |\Omega|) > 0$. □

Lemma 3.4. *Assume that* $\{z_n\} \subset W$ *is a* $(PS)_c$*-sequence for* $J_{\lambda,\mu}$, *then* $\{z_n\}$ *is bounded in* W.

Proof. Assume contrary, that $\| z_n \| \to \infty$. Let

$$z_n^* = (u_n^*, v_n^*) = \frac{z_n}{\| z_n \|} = \left(\frac{u_n}{\| z_n \|}, \frac{v_n}{\| z_n \|} \right), \tag{3.2}$$

$z_n^* \rightharpoonup z^* = (u^*, v^*)$ in W. This implies that $u_n^* \to u^*, v_n^* \to v^*$ strongly in $L^s(\Omega)$ for all $1 \le s < p^*$ and

$$K_{\lambda,\mu}(z_n^*) = K_{\lambda,\mu}(z^*) + o(1). \tag{3.3}$$

Now, since $\{z_n\} \subset W$ is a $(PS)_c$-sequence for $J_{\lambda,\mu}$ and $\| z_n \| \to \infty$, we have

$$\frac{\| z_n^* \|^p}{p} - \frac{\| z_n \|^{q-p}}{q} K_{\lambda,\mu}(z_n^*) - \frac{\| z_n \|^{\alpha+\beta-p}}{\alpha+\beta} L(z_n^*) = o(1) \tag{3.4}$$

and

$$\| z_n^* \|^p - \| z_n \|^{q-p} K_{\lambda,\mu}(z_n^*) - \| z_n \|^{\alpha+\beta-p} L(z_n^*) = o(1). \tag{3.5}$$

From (3.3)-(3.5), one can get

$$\| z_n^* \|^p = \frac{p(\alpha+\beta-q)}{q(\alpha+\beta-p)} \| z_n \|^{q-p} K_{\lambda,\mu}(z_n^*) + o(1). \tag{3.6}$$

Since $1 \le q < p$ and $\| z_n \| \to \infty$, (3.6) implies that $\| z_n^* \|^p \to 0$, as $n \to \infty$, which contradicts $\| z_n^* \|^p = 1$. □

Now, we need the following proposition.

Proposition 3.5. [16] *Suppose that $\psi(z) = b(x)|u|^\alpha |v|^\beta$ is positively homogeneous of degree p^*. Then there exists $M > 0$ such that $|\psi(z)| \le M(|u|^{p^*} + |v|^{p^*})$, where*

$$M = \max\{\psi(z) \mid |u|^{p^*} + |v|^{p^*} = 1\}.$$

Lemma 3.6. *Assume that $\psi(z)$ is positively homogeneous of degree p^*, then ψ_u, ψ_v are positively homogeneous of degree $p^* - 1$. Moreover, there exist $M_1, M_2 > 0$ such that*

$$|\psi_u| \le M_1(|u|^{p^*-1} + |v|^{p^*-1}), \qquad |\psi_v| \le M_2(|u|^{p^*-1} + |v|^{p^*-1}).$$

Proof. The proof is an immediate consequence of Proposition (3.5). □

Next, we need the following version of Brézis-Lieb lemma.

Lemma 3.7. *Suppose that $\{z_n\}$ is a bounded sequence in W, and $z_n \rightharpoonup z$ weakly in W. Let $\tilde{u}_n = u_n - u, \tilde{v}_n = v_n - v$, and $\tilde{z}_n = (\tilde{u}_n, \tilde{v}_n)$. Then one can get*

$$L(\tilde{z}_n) = L(z_n) - L(z) + o(1).$$

Proof. Let $\tilde{u}_n = u_n - u, \tilde{v}_n = v_n - v$, and $\tilde{z}_n = (\tilde{u}_n, \tilde{v}_n)$, then by the mean value theorem, for given $0 < |\theta| < 1$, it follows that

$$\begin{aligned}
|\psi(z_n) - \psi(\tilde{z}_n)| &= |\nabla\psi(\tilde{z}_n + \theta z).z| \\
&\leq M_1(|\tilde{u}_n + \theta u|^{p^*-1} + |\tilde{v}_n + \theta v|^{p^*-1})|u| \\
&\quad + M_2(|\tilde{u}_n + \theta u|^{p^*-1} + |\tilde{v}_n + \theta v|^{p^*-1})|v| \\
&\leq M_0\Big[(|\tilde{u}_n|^{p^*-1}|u| + |u|^{p^*} + |\tilde{v}_n|^{p^*-1}|u| + |v|^{p^*-1}|u|) \\
&\quad + (|\tilde{u}_n|^{p^*-1}|v| + |u|^{p^*-1}|v| + |\tilde{v}_n|^{p^*-1}|v| + |v|^{p^*})\Big] \\
&\leq M_0\Big[|\tilde{u}_n|^{p^*-1}|u| + |\tilde{v}_n|^{p^*-1}|v| + |\tilde{u}_n|^{p^*-1}|v| + |\tilde{v}_n|^{p^*-1}|u| \\
&\quad + |u|^{p^*} + |v|^{p^*} + |u|^{p^*-1}|v| + |v|^{p^*-1}|u|\Big],
\end{aligned}$$

where $M_0 = \max\{M_1, M_2\}$. Hence, for any $\varepsilon > 0$, applying the Young inequality to (3.7), there exists $M_\varepsilon > 0$ such that

$$|\psi(z_n) - \psi(\tilde{z}_n)| \leq \varepsilon(|\tilde{u}_n|^{p^*} + |\tilde{v}_n|^{p^*}) + M_\varepsilon(|u|^{p^*} + |v|^{p^*}).$$

Now, we define the functions

$$f_n = |\psi(z_n) - \psi(\tilde{z}_n) - \psi(z)|, \quad g_n = f_n - \varepsilon(|\tilde{u}_n|^{p^*} + |\tilde{v}_n|^{p^*}). \tag{3.7}$$

Then

$$f_n \leq \varepsilon(|\tilde{u}_n|^{p^*} + |\tilde{v}_n|^{p^*}) + M_\varepsilon(|u|^{p^*} + |v|^{p^*}) + \psi(z),$$

$$\begin{aligned}
g_n &\leq M_\varepsilon(|u|^{p^*} + |v|^{p^*}) + \psi(z) \\
&\leq M_\varepsilon(|u|^{p^*} + |v|^{p^*}) + M(|u|^{p^*} + |v|^{p^*}) \\
&= (M_\varepsilon + M)(|u|^{p^*} + |v|^{p^*}) \ \in L^1(\Omega).
\end{aligned}$$

Since $z_n \rightharpoonup z$ weakly in W, we can assume that $u_n \to u, v_n \to v$ a.e. in Ω. Thus we get $g_n \to 0$ a.e. in Ω as $n \to \infty$. The Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty} \int_\Omega g_n(x)dx = 0.$$

Therefore

$$\limsup_{n\to\infty} \int_\Omega f_n(x)dx \leq \limsup_{n\to\infty} \int_\Omega (g_n(x) + \varepsilon(|\tilde{u}_n|^{p^*} + |\tilde{v}_n|^{p^*}))dx$$

$$\begin{aligned}
&\leq \limsup_{n\to\infty}\int_\Omega g_n(x)d(x)+\varepsilon\limsup_{n\to\infty}\int_\Omega(|\tilde{u}_n|^{p^*}+|\tilde{v}_n|^{p^*}))d(x)\\
&\leq M_\varepsilon.
\end{aligned}$$

By the arbitrariness of $\varepsilon > 0$, one can get

$$\lim_{n\to\infty}\int_\Omega f_n(x)d(x)=0.$$

Thus,

$$L(\tilde{z}_n)=L(z_n)-L(z)+o(1).$$

□

Lemma 3.8. *Let* $C_{\lambda,\mu}=\dfrac{1}{N}S_{\alpha,\beta}^{\frac{N}{p}}-C_0\varrho_{\lambda,\mu}$, *where* C_0 *is the positive constant given in Lemma* (3.3), *then* $J_{\lambda,\mu}$ *satisfies the* $(PS)_c$ *condition with* $c\in(-\infty,C_{\lambda,\mu})$.

Proof. Let $\{z_n\}\subset W$ be a $(PS)_c$-sequence for $J_{\lambda,\mu}$ with $c\in(-\infty,C_{\lambda,\mu})$. By lemma (3.4) we have that $\{z_n\}$ is bounded in W. This implies that $z_n\rightharpoonup z$ up to a subsequence, when z is a critical point of $J_{\lambda,\mu}$. Furthermore we may assume

$$\left\{\begin{array}{ll} u_n\rightharpoonup u\ ,\ v_n\rightharpoonup v & \text{in } W_0^{1,p}(\Omega),\\ u_n\to u\ ,\ v_n\to v & \text{a.e on } \Omega,\\ u_n\to u\ ,\ v_n\to v & \text{in } L^s(\Omega)\ (1\leq s<p^*).\end{array}\right. \tag{3.8}$$

This implies that $J'_{\lambda,\mu}(z)=0$ and

$$K_{\lambda,\mu}(z_n)=K_{\lambda,\mu}(z)+o(1). \tag{3.9}$$

Let $\tilde{u}_n=u_n-u$, $\tilde{v}_n=v_n-v$, and $\tilde{z}_n=(\tilde{u}_n,\tilde{v}_n)$. Then by Remark (3.2), we obtain

$$\|\ \tilde{z}_n\ \|^p=\|\ z_n\ \|^p-\|\ z\ \|^p+o(1), \tag{3.10}$$

and from Lemma (3.7), deduce that

$$L(\tilde{z}_n)=L(z_n)-L(z)+o(1). \tag{3.11}$$

Since $J_{\lambda,\mu}(z_n)=c+o(1), J'_{\lambda,\mu}(z_n)=o(1)$, by (3.9)-(3.11), we get

$$\frac{1}{p}\|\ \tilde{z}_n\ \|^p-\frac{1}{\alpha+\beta}L(\tilde{z}_n)=c-J_{\lambda,\mu}(z)+o(1). \tag{3.12}$$

and

$$\|\ \tilde{z}_n\ \|^p-L(\tilde{z}_n)=o(1).$$

Thus, we may assume that

$$\|\ \tilde{z}_n\ \|^p\to h,\qquad L(\tilde{z}_n)\to h. \tag{3.13}$$

Assume that $h > 0$; by the definition of $S_{\alpha,\beta}$ and (B), (3.14), one can get

$$\begin{aligned} S_{\alpha,\beta} h^{\frac{p}{\alpha+\beta}} &= S_{\alpha,\beta} \lim_{n\to\infty} L(\tilde{z}_n)^{\frac{p}{\alpha+\beta}} \\ &\le |b|_\infty^{\frac{p}{\alpha+\beta}} \parallel \tilde{z}_n \parallel^p = h, \end{aligned}$$

which implies that $h \ge S_{\alpha,\beta}^{\frac{N}{p}}$. By (3.13) and (3.14), we have

$$c = \left(\frac{1}{p} - \frac{1}{\alpha+\beta}\right) h + J_{\lambda,\mu}(z),$$

then by Lemma (3.3), we get

$$c \ge \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{p}} - C_0 \varrho_{\lambda,\mu} = C_{\lambda,\mu}.$$

which is a contradiction. Hence $h = 0$; that is $z_n \to z$ strongly in W. $\square$

4. **Existence of solutions**

First, we state our main results.

Theorem 4.1. *Assume that conditions (A)-(C) hold. If* $\alpha+\beta \le p^*$, *and* λ, μ *satisfy* $0 < \varrho_{\lambda,\mu} < \Lambda_0$, *then* (1.1) *has at least one positive solution.*

Theorem 4.2. *Assume that conditions (A)-(C) hold. If* $\alpha+\beta < p^*$, *and* λ, μ *satisfy* $0 < \varrho_{\lambda,\mu} < \Lambda_1$, *then* (1.1) *has at least two positive solutions.*

Theorem 4.3. *Assume that conditions (A)-(C) hold. If* $\alpha+\beta = p^*$,*then there exists* $\Lambda_2 > 0$ *such that for* λ, μ *satisfying* $0 < \varrho_{\lambda,\mu} < \Lambda_2$, *problem* (1.1) *has at least two positive solutions.*

Note that, in Theorem 4.1 we claim the existence of one positive solution and in Theorem 4.2 and 4.3 we claim that the second positive solution exists in subcritical and critical case, respectively.

Proposition 4.4. [19] *(i) If* $(\lambda, \mu) \in \Theta_{\Lambda_0}$, *then there exists a* $(PS)_{\theta_{\lambda,\mu}}$-*sequence* $\{z_n\} \subset N_{\lambda,\mu}$ *in* W *for* $J_{\lambda,\mu}$;

(ii) If $(\lambda, \mu) \in \Theta_{\Lambda_1}$, *then there exists a* $(PS)_{\theta^-_{\lambda,\mu}}$-*sequence* $\{z_n\} \subset N^-_{\lambda,\mu}$ *in* W *for* $J_{\lambda,\mu}$,

where Λ_1 *is a positive constant given in* (2.6).

Now, we prove the existence of a local minimum for $J_{\lambda,\mu}$ on $N^+_{\lambda,\mu}$.

Theorem 4.5. *If* $(\lambda,\mu)\in\Theta_{\Lambda_0}$*, then* $J_{\lambda,\mu}$ *has a minimizer* z_0^+ *in* $N^+_{\lambda,\mu}$ *and it satisfies the following:*

(i) $J_{\lambda,\mu}(z_0^+)=\theta^+_{\lambda,\mu}=\theta_{\lambda,\mu}<0$*;*

(ii) z_0^+ *is a positive solution of (1.1).*

Proof. By proposition 4.4 (i), there exists a minimizing sequence $\{z_n\}$ for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$ such that

$$J_{\lambda,\mu}(z_n)=\theta_{\lambda,\mu}+o(1)\qquad\text{and}\qquad J'_{\lambda,\mu}(z_n)=o(1). \tag{4.1}$$

Since $J_{\lambda,\mu}$ is coercive on $N_{\lambda,\mu}$ (see Lemma (2.1)), there exists a subsequence $\{z_n\}=\{(u_n,v_n)\}$ and $z_0^+=(u_0^+,v_0^+)\in W$ such that

$$\begin{cases} u_n\rightharpoonup u_0^+\ ,\ v_n\rightharpoonup v_0^+ & \text{weakly in } W_0^{1,p}(\Omega),\\ u_n\to u_0^+\ ,\ v_n\to v_0^+ & \text{almost everywhere in } \Omega,\\ u_n\to u_0^+\ ,\ v_n\to v_0^+ & \text{strongly in } L^s(\Omega)\ (1\le s<p^*), \end{cases} \tag{4.2}$$

as $n\to\infty$. This implies

$$K_{\lambda,\mu}(z_n)=K_{\lambda,\mu}(z_0^+)+o(1)\qquad \text{as } n\to\infty. \tag{4.3}$$

First, we claim that z_0^+ is a nontrivial solution of (1.1). By (4.1) and (4.2), we can deduce that z_0^+ is a weak solution of (1.1). By (2.4) we have

$$\begin{aligned} J_{\lambda,\mu}(z_n) &= \frac{\alpha+\beta-p}{p(\alpha+\beta)}\parallel z_n\parallel^p-\frac{\alpha+\beta-q}{q(\alpha+\beta)}K_{\lambda,\mu}(z_n)\\ &\geq -\frac{\alpha+\beta-q}{q(\alpha+\beta)}K_{\lambda,\mu}(z_n). \end{aligned}$$

Let $n\to\infty$, we get

$$K_{\lambda,\mu}(z_0^+)\geq-\frac{q(\alpha+\beta)}{\alpha+\beta-q}\theta_{\lambda,\mu}>0. \tag{4.4}$$

Thus, $z_0^+ \in N_{\lambda,\mu}$ is a nontrivial solution of (1.1). Now, we prove that $z_n \to z_0^+$ strongly in W and $J_{\lambda,\mu}(z_0^+) = \theta_{\lambda,\mu}$. By applying Fatou's lemma and $z_0^+ \in N_{\lambda,\mu}$, we have

$$\begin{aligned}
\theta_{\lambda,\mu} &\leq J_{\lambda,\mu}(z_0^+) = \frac{\alpha+\beta-p}{p(\alpha+\beta)} \parallel z_0^+ \parallel^p - \frac{\alpha+\beta-q}{q(\alpha+\beta)} K_{\lambda,\mu}(z_0^+) \\
&\leq \liminf_{n\to\infty} \left(\frac{\alpha+\beta-p}{p(\alpha+\beta)} \parallel z_n \parallel^p - \frac{\alpha+\beta-q}{q(\alpha+\beta)} K_{\lambda,\mu}(z_n) \right) \\
&\leq \liminf_{n\to\infty} J_{\lambda,\mu}(z_n) = \theta_{\lambda,\mu}.
\end{aligned}$$

This implies that $J_{\lambda,\mu}(z_0^+) = \theta_{\lambda,\mu}$ and $\lim_{n\to\infty} \parallel z_n \parallel^p = \parallel z_0^+ \parallel^p$. Let $\bar{z}_n = z_n - z_0^+$,
then by Remark (3.2), we get

$$\parallel \bar{z}_n \parallel^p = \parallel z_n \parallel^p - \parallel z_0^+ \parallel^p .$$

Therefore, $z_n \to z_0^+$ strongly in W. Next, we show that $z_0^+ \in N_{\lambda,\mu}^+$. Suppose that $z_0^+ \in N_{\lambda,\mu}^-$, , then by (4.4) we have $K_{\lambda,\mu}(z_0^+) > 0$. Thus by Lemma (2.6), there are unique t_0^+ and t_0^- such that $t_0^{\pm} z_0^{\pm} \in N_{\lambda,\mu}^{\pm}$. In particular $t_0^+ < t_0^- = 1$. Since

$$\frac{\mathrm{d}}{\mathrm{dt}} J_{\lambda,\mu}(t_0^+ z_0^+) = 0 \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{dt}^2} J_\lambda(t_0^+ z_0^+) > 0,$$

there exists $t_0^+ < \bar{t} \leq t_0^-$ such that $J_{\lambda,\mu}(t_0^+ z_0^+) < J_{\lambda,\mu}(\bar{t} z_0^+)$. By Lemma (2.6), we have

$$J_{\lambda,\mu}(t_0^+ z_0^+) < J_{\lambda,\mu}(\bar{t} z_0^+) \leq J_{\lambda,\mu}(t_0^- z_0^+) = J_{\lambda,\mu}(z_0^+),$$

which contradicts $J_{\lambda,\mu}(z_0^+) = \theta_{\lambda,\mu}^+$. Thus $z_0^+ \in N_{\lambda,\mu}^+$. Since $J_{\lambda,\mu}(z_0^+) = J_{\lambda,\mu}(|z_0^+|)$ and $|z_0^+| \in N_{\lambda,\mu}^+$, by Lemma (2.2) we may assume that z_0^+ is a nontrivial nonnegative solution of (1.1). Moreover $u_0^+ > 0, v_0^+ > 0$ in Ω by the maximum principle.

Next, we prove the existence of a local minimizer for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}^-$ in the case $\alpha + \beta < p^*$. This implies that there exists the second positive solution in the subcritical case. $\square$

Theorem 4.6. *If $p < \alpha + \beta < p^*$ and $(\lambda, \mu) \in \Theta_{\Lambda_1}$, then $J_{\lambda,\mu}$ has a minimizer z_0^- in $N_{\lambda,\mu}^-$ and it satisfies the following:*

(i) $J_{\lambda,\mu}(z_0^-) = \theta_{\lambda,\mu}^-$;

(ii) z_0^- *is a positive solution of* (1.1).

Proof. Let $\{z_n\}$ be a minimizing sequence for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}^-$, i.e.,

$$\lim_{n\to\infty} J_{\lambda,\mu}(z_n) = \inf_{z\in N_{\lambda,\mu}^-} J_{\lambda,\mu}(z).$$

Then by coercivity of $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$ and the compact imbedding theorem, there exists a subsequence $\{z_n\}$ and $(z_0^-) = (u_0^-, v_0^-) \in W$ such that

$$\begin{cases} u_n \rightharpoonup u_0^- \ , \ v_n \rightharpoonup v_0^- & \text{weakly in } W_0^{1,p}(\Omega), \\ u_n \to u_0^- \ , \ v_n \to v_0^- & \text{strongly in } L^q(\Omega), L^{\alpha+\beta}(\Omega). \end{cases} \tag{4.5}$$

This implies

$$K_{\lambda,\mu}(z_n) = K_{\lambda,\mu}(z_0^-) + o(1), \quad L(z_n) = L(z_0^-) + o(1), \tag{4.6}$$

as $n \to \infty$. By Lemmas (2.3) and (2.7) we obtain that there exists $C_2 > 0$ such that $L(z_n) > C_2$. This implies

$$L(z_0^-) \geq C_2. \tag{4.7}$$

Now, we prove that $z_n \to z_0^-$ strongly in W. Assume contrary, then $\| z_0^- \| < \liminf_{n\to\infty} \| z_n \|$. By Lemma (2.6), there exists a unique t_0^- such that $t_0^- z_0^- \in N_{\lambda,\mu}^-$. Since $z_n \in N_{\lambda,\mu}^-, J_{\lambda,\mu}(z_n) \geq J_{\lambda,\mu}(tz_n)$ for all $t \geq 0$, we have

$$\theta_{\lambda,\mu}^- \leq J_{\lambda,\mu}(t_0^- z_0^-) < \lim_{n\to\infty} J_{\lambda,\mu}(t_0^- z_n) \leq \lim_{n\to\infty} J_{\lambda,\mu}(z_n) = \theta_{\lambda,\mu}^-,$$

and this is a contradiction. Hence $z_n \to z_0^-$ strongly in W. This implies that

$$J_{\lambda,\mu}(z_0^-) = \lim_{n\to\infty} J_{\lambda,\mu}(z_n) = \theta_{\lambda,\mu}^-.$$

Since $J_{\lambda,\mu}(z_0^-) = J_{\lambda,\mu}(|z_0^-|)$ and $|z_0^-| \in N_{\lambda,\mu}^-$, by Lemma (2.2) and (4.7) we deduce that z_0^- is a nontrivial nonnegative solution of (1.1). By the maximum principle, it follows that $u_0^- > 0, v_0^- > 0$ in Ω .

Now, we complete the proof of Theorem (4.1) and (4.2):

Proof of Theorem 4.1. By Theorem (4.5), we get that for all $\lambda, \mu > 0$ and $0 < \varrho_{\lambda,\mu} < \Lambda_0$ (or $(\lambda,\mu) \in \Theta_{\Lambda_0}$), (1.1) has a positive solution $z_0^+ \in N_{\lambda,\mu}^+$.

Proof of Theorem 4.2. By Theorems (4.5) and (4.6), we obtain that for all $\lambda, \mu > 0$, $\alpha + \beta < p^*$, and $0 < \varrho_{\lambda,\mu} < \Lambda_1 < \Lambda_0$ (or $(\lambda,\mu) \in \Theta_{\Lambda_1}$), (1.1) has two positive solutions z_0^+, z_0^- with $z_0^{\pm} \in N_{\lambda,\mu}^{\pm}$. Since $N_{\lambda,\mu}^+ \cap N_{\lambda,\mu}^- = \emptyset$, this implies that z_0^+ and z_0^- are distinct. This completes the proof of Theorem (4.2).

Now, we prove the existence of a local minimizer for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}^-$ in the case $\alpha + \beta = p^*$. This implies that there exists the second positive solution in the critical case. First, We point the following fact as a remark which will be used in the next lemma.

Remark 4.7. *Let $A, B > 0$, then using the auxiliary function $f(t) = \dfrac{t^p}{p}A - \dfrac{t^{\alpha+\beta}}{\alpha+\beta}B$, we have*

$$\sup_{t\geq 0}\left(\frac{t^p}{p}A - \frac{t^{\alpha+\beta}}{\alpha+\beta}B\right) = \frac{1}{N}A\left(\frac{A}{B}\right)^{\frac{N-p}{p}} = \frac{1}{N}A\left(\frac{A}{B^{\frac{p}{p^*}}}\right)^{\frac{N}{p}}.$$

Lemma 4.8. *There exists a nonnegative function $z \in W \setminus \{(0,0)\}$ and $\Lambda^* > 0$ such that for $(\lambda,\mu) \in \Theta_{\Lambda^*}$, we have*

$$\sup_{t\geq 0} J_{\lambda,\mu}(tz) < C_{\lambda,\mu}, \tag{4.8}$$

where $C_{\lambda,\mu}$ is the constant given in Lemma (3.8). In particular, $\theta_{\lambda,\mu}^- < c_{\lambda,\mu}$, for all $(\lambda,\mu) \in \Theta_{\Lambda^*}$.

Proof. Since $D_{f^+} \cap D_{g^+} \neq \emptyset$, there exists $x_0 \in \Omega, \rho_0, a_0, b_0 > 0$ such that $B(x_0, 2\rho_0) \subset \Omega$ and $f(x) \geq a_0$ and $g(x) \geq b_0$ for all $x \in B(x_0, 2\rho_0)$. In fact $a_0 = \min f(x)$ and $b_0 = \min g(x)$ on $B(x_0, 2\rho_0)$. Without loss of generality, we assume that $x_0 = 0$. Let $b(x) > 0$ for all $x_0 \in \Omega$, $|b|_\infty = b(0)$ and there exists $\delta_0 > \frac{N}{p-1}$ such that $b(x) = b(0) + o(|x|^{\delta_0})$ as $x \to 0$.

Now, we consider the functional $I : W \longrightarrow \mathbb{R}$ defined by

$$I(z) = \frac{1}{p} \parallel z \parallel^p - \frac{1}{\alpha+\beta} L(z),$$

for all $z \in W$, and define a cut-off function $\eta(x) \in C_0^\infty(\Omega)$ such that

$\eta(x) = \begin{cases} 1 & |x| < \rho_0, \\ 0 & |x| > 2\rho_0, \end{cases}$ where $0 \leq \eta \leq 1$ and $|\nabla \eta| \leq C$. For $\varepsilon > 0$, let

$$u_\varepsilon(x) = \frac{\eta(x)}{(\varepsilon + |x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}}. \tag{4.9}$$

Step 1. We show that $\sup_{t \geq 0} I_{\lambda,\mu}(tz_0) \leq \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{p}} + O(\varepsilon^{\frac{N-p}{p}})$.

From Hsu [14](Lemma 4.3), we have

$$\left(\int_\Omega |u_\varepsilon|^{p^*} dx \right)^{\frac{p}{p^*}} = \varepsilon^{-\frac{N-p}{p}} |U|^p_{L^{p^*}(\mathbb{R}^N)} + O(\varepsilon),$$

$$\int_\Omega |\nabla u_\varepsilon|^p dx = \varepsilon^{-\frac{N-p}{p}} |\nabla U|^p_{L^p(\mathbb{R}^N)} + O(1),$$

$$\frac{\displaystyle\int_\Omega |\nabla u_\varepsilon|^p dx}{\left(\displaystyle\int_\Omega |u_\varepsilon|^{p^*} dx \right)^{\frac{p}{p^*}}} = S + O\left(\varepsilon^{\frac{N-p}{p}} \right), \tag{4.10}$$

where $U(x) = (1 + |x|^{\frac{p}{p-1}})^{-\frac{N-p}{p}} \in W^{1,p}(\mathbb{R}^N)$.

Set $u_0 = \sqrt[p]{\alpha} u_\varepsilon, v_0 = \sqrt[p]{\beta} u_\varepsilon$ and $z_0 \in W$. Then from Remark (4.7), (1.4) and (4.10), we conclude that

$$
\begin{aligned}
\sup_{t\geq 0} I_{\lambda,\mu}(tz_0) &\leq \frac{1}{N}\left(\frac{(\alpha+\beta)\int_\Omega|\nabla u_\varepsilon|^p dx}{\left(\alpha^{\frac{\alpha}{p}}\beta^{\frac{\beta}{p}}\int_\Omega|u_\varepsilon|^{p^*}dx\right)^{\frac{p}{p^*}}}\right)^{\frac{N}{p}}\\
&\leq \frac{1}{N}\left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}+\left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right)^{\frac{N}{p}}\left(S+O(\varepsilon^{\frac{N-p}{p}})\right)^{\frac{N}{p}}\\
&= \frac{1}{N}\left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}+\left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right)^{\frac{N}{p}}\left(S^{\frac{N}{p}}+O(\varepsilon^{\frac{N-p}{p}})\right)\\
&= \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{p}}+O(\varepsilon^{\frac{N-p}{p}}).
\end{aligned}
$$

Step 2. We claim that if we set $\varepsilon=\varrho_{\lambda,\mu}^{\frac{p}{N-p}}$, then there exists $\Lambda^*>0$, such that for $(\lambda,\mu)\in\Theta_{\Lambda^*}$we have $\sup_{t\geq 0}J_{\lambda,\mu}(tz)<C_{\lambda,\mu}$.

Let C_0 be the positive constant given in Lemma (3.3). We can choose $\delta_1>0$ such that for all $(\lambda,\mu)\in\Theta_{\delta_1}$, we have

$$C_{\lambda,\mu}=\frac{1}{N}S_{\alpha,\beta}^{\frac{N}{p}}-C_0\varrho_{\lambda,\mu}>0.$$

Using the definition of $J_{\lambda,\mu}$ and z_0, we get

$$J_{\lambda,\mu}(tz_0)\leq\frac{t^p}{p}\|\, z_0\,\|^p=\frac{\alpha+\beta}{p}t^p|\nabla u_\varepsilon|^p_{L^p(\mathbb{R}^N)}\qquad\forall t\geq 0,\quad\lambda,\ \mu>0,$$

which implies that there exists $t_0\in(0,1)$ satisfying

$$\sup_{0\leq t\leq t_0}J_{\lambda,\mu}(tz_0)<C_{\lambda,\mu},\qquad\forall(\lambda,\mu)\in\Theta_{\delta_1}.$$

Using the definition of $J_{\lambda,\mu}$ and z_0 and by $\alpha,\beta>1$, (4.11), we have

$$\begin{aligned}\sup_{t\geq t_0} J_{\lambda,\mu}(tz_0) &= \sup_{t\geq t_0}\left(I(tz_0)-\frac{t^q}{q}K_{\lambda,\mu}(z_0)\right)\\ &\leq \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{p}}+O(\varepsilon^{\frac{N-p}{p}})-\frac{{t_0}^q}{q}\left(a_0\alpha^{\frac{q}{p}}\lambda+b_0\beta^{\frac{q}{p}}\mu\right)\int_{B(0,\rho_0)}|u_\varepsilon|^q dx\\ &\leq \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{p}}+O(\varepsilon^{\frac{N-p}{p}})-\frac{{t_0}^q}{q}(\lambda+\mu)\int_{B(0,\rho_0)}|u_\varepsilon|^q dx.\end{aligned}$$

Let $0<\varepsilon\leq {\rho_0}^{\frac{p}{p-1}}$, we have

$$\begin{aligned}\int_{B(0,\rho_0)}|u_\varepsilon|^q dx &= \int_{B(0,\rho_0)}\frac{1}{(\varepsilon+|x|^{\frac{p}{p-1}})^{q\frac{N-p}{p}}}dx\\ &\geq \int_{B(0,\rho_0)}\frac{1}{(2{\rho_0}^{\frac{p}{p-1}})^{q\frac{N-p}{p}}}dx\\ &= C_1=C_1(N,p,q,\rho_0).\end{aligned}$$

Then by (4.12) and (4.13), for all $\varepsilon\in(0,{\rho_0}^{\frac{p}{p-1}})$, one can get

$$\sup_{t\geq t_0} J_{\lambda,\mu}(tz_0)\leq \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{p}}+O\left((\lambda|f|_\infty)^{\frac{p}{p-q}}+(\mu|g|_\infty)^{\frac{p}{p-q}}\right)-\frac{{t_0}^q}{q}C_1(\lambda+\mu).$$

Hence, we can choose $\delta_2>0$ such that for all $(\lambda,\mu)\in\Theta_{\delta_2}$, we have

$$O(\varrho_{\lambda,\mu})-\frac{{t_0}^q}{q}C_1(\lambda+\mu)<C_0\varrho_{\lambda,\mu}.$$

If we set $\Lambda^*=\min\{\delta_1,{\rho_0}^{\frac{N-p}{p-1}},\delta_2\}$, then for $(\lambda,\mu)\in\Theta_{\Lambda^*}$, we have

$$\sup_{t\geq 0} J_{\lambda,\mu}(tz_0)<C_{\lambda,\mu}. \tag{4.11}$$

Step 3. We prove that $\theta^-_{\lambda,\mu}<C_{\lambda,\mu}$ for all $(\lambda,\mu)\in\Theta_{\Lambda^*}$.
By the definition of z_0 and u_ε, we have

$$L(z_0)>0,\qquad K_{\lambda,\mu}(z_0)>0.$$

Using this fact, Lemma (2.6)(ii), definition of $\theta^-_{\lambda,\mu}$ and (4.11) indicate that there exists $t_0 > 0$ such that $t_0 z_0 \in N^-_{\lambda,\mu}$ and

$$\theta^-_{\lambda,\mu} \leq J_{\lambda,\mu}(tz_0) \leq \sup_{t\geq 0} J_{\lambda,\mu}(tz_0) < C_{\lambda,\mu}$$

for all $(\lambda,\mu) \in \Theta_{\Lambda^*}$. □

Theorem 4.9. *If $(\lambda,\mu) \in \Theta_{\Lambda_2}$, then $J_{\lambda,\mu}$ has a minimizer $z_0{}^-$ in $N^-_{\lambda,\mu}$ and satisfies the following*

(i) $J_{\lambda,\mu}(z_0^-) = \theta^-_{\lambda,\mu}$;
(ii) z_0^- *is a positive solution of* (1.1),

where $\Lambda_2 = \min\{\Lambda^*, \Lambda_1\}$, Λ^* *is the same as in Lemma* (4.8).

Proof. If $(\lambda,\mu) \in \Theta_{\Lambda_1}$, then by Proposition (4.4), there exists a $(PS)_{\theta^-_{\lambda,\mu}}$-sequence $\{z_n\} \subset N^-_{\lambda,\mu}$ in W for $J_{\lambda,\mu}$. From Lemmas (3.8) and (4.8) and Theorem (2.5)(ii), for $(\lambda,\mu) \in \Theta_{\Lambda^*}$, $J_{\lambda,\mu}$ satisfies $(PS)_{\theta^-_{\lambda,\mu}}$ condition and $\theta^-_{\lambda,\mu} \in (0, C_{\lambda,\mu})$. By Lemma (2.1) and from coercivity of $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$, we get that $\{z_n\}$ is bounded in W. Therefore, there exists a subsequence still denoted by $\{z_n\}$ and a nontrivial solution $z_0^- \in N^-_{\lambda,\mu}$ such that $z_n \rightharpoonup z_0^-$ weakly in W. Finally by the same arguments as in the proof of Theorem (4.5), for all $(\lambda,\mu) \in \Theta_{\Lambda_2}$, we have that z_0^- is a positive solution of (1.1). □

Proof of Theorem 4.3. By Theorems (4.5) and (4.9), we obtain that for all $\lambda,\mu > 0$ and $0 < \varrho_{\lambda,\mu} < \Lambda_2 < \Lambda_0$ (or $(\lambda,\mu) \in \Theta_{\Lambda_2}$), (1.1) has two positive solutions z_0^+, z_0^- with $z_0^\pm \in N^\pm_{\lambda,\mu}$. Since $N^+_{\lambda,\mu} \cap N^-_{\lambda,\mu} = \emptyset$, this implies that z_0^+ and z_0^- are distinct. This completes the proof of Theorem (4.3). □

Conclusion. In this paper we investigate the existence and multiplicity of positive solutions for problem (1.1) in both cases, critical and subcritical growth terms. In the proof, we apply variational methods, via the extraction of Palais-Smale sequences in the Nehari manifold for subcritical Sobolev exponent. It consists of making precise comparisons between the critical and subcritical cases. In order to overcome the lack of compactness due to the critical growth, we use the ideas of Brezis and Nirenberg ([7]), besides the paper of Hsu ([13]), where it is proved that the existence of a certain range in $\mathbb{R}^2$, which plays an important

role when dealing with critical systems like (1.1). Actually, we use this certain range and adapt some calculations to localize the energy levels where Palais-Smale condition fails.
Finally, we would like to mention that, as a byproduct of our arguments, we can extend the existence results in Theorems (4.1), (4.2) and (4.3) for both critical and subcritical degrees of homogeneity of any perturbation term.

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