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The finite $\mathcal{S}-$determinacy of singularities in positive characterisitc, $\mathcal{S} = \mathcal{R}_{\mathcal{G}}, \mathcal{R}_{\mathcal{A}}, \mathcal{K}_{\mathcal{G}}, \mathcal{K}_{\mathcal{A}}$

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THE FINITE $\mathcal{S}-$DETERMINACY OF SINGULARITIES IN POSITIVE CHARACTERISITC, $\mathcal{S}=\mathcal{R}_{\mathcal{G}},\mathcal{R}_{\mathcal{A}},\mathcal{K}_{\mathcal{G}},\mathcal{K}_{\mathcal{A}}$

L. HENGXING AND L. JINGWEN*

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Abstract. For singularities $f\in K[[x_1,\ldots,x_n]]$ over an algebraically closed field K of arbitrary characteristic, we introduce the finite $\mathcal{S}-$determinacy under $\mathcal{S}-$equivalence, where $\mathcal{S}=\mathcal{R}_{\mathcal{G}}$, $\mathcal{R}_{\mathcal{A}}$, $\mathcal{K}_{\mathcal{G}}$, $\mathcal{K}_{\mathcal{A}}$. It is proved that the finite $\mathcal{R}_{\mathcal{G}}(\mathcal{K}_{\mathcal{G}})-$determinacy is equivalent to the finiteness of the relative $\mathcal{G}-$Milnor ($\mathcal{G}-$Tjurina) number and the finite $\mathcal{R}_{\mathcal{A}}(\mathcal{K}_{\mathcal{A}})-$determinacy is equivalent to the finiteness of the relative $\mathcal{A}-$Milnor ($\mathcal{A}-$Tjurina) number. Moreover, some estimates are provided on the degree of the $\mathcal{S}-$determinacy in positive characteristic.
Keywords: Finite $\mathcal{R}_{\mathcal{G}}$ ($\mathcal{R}_{\mathcal{A}}$)$-$determinacy, finite $\mathcal{K}_{\mathcal{G}}$ ($\mathcal{K}_{\mathcal{A}}$)$-$ determinacy, the relative $\mathcal{G}(\mathcal{A})-$Milnor number, relative $\mathcal{G}(\mathcal{A})-$ Tjurina number.
MSC(2010): Primary: 14B05; Secondary: 32S10, 32S25, 58K40.

1. **Introduction**

In this paper, we assume that K is an algebraically closed field of arbitrary characteristic unless otherwise stated explicitly. Let

$$K[[\mathbf{x}]]=K[[x_1,\ldots,x_n]]=\left\{\sum_{\alpha\in N^n}a_\alpha\mathbf{x}^\alpha|a_\alpha\in K\right\}$$

be the formal power series ring over K. We use the usual multi-index notation $\mathbf{x}^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ for $\alpha=(\alpha_1,\ldots,\alpha_n)\in N^n$. We denote $\mathcal{M}=$

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$\langle x_1, \ldots, x_n \rangle$ the unique maximal ideal of $K[[\mathbf{x}]]$, so that the set of units in $K[[\mathbf{x}]]$ is $K[[\mathbf{x}]]^* = K[[\mathbf{x}]] \backslash \mathcal{M}$,

Let $\mathcal{S}$ be a subgroup of $\mathrm{Aut}(K[[\mathbf{x}]])$. Then an equivalence relation can be introduced on $K[[\mathbf{x}]]$ via $\mathcal{S}$. For the given equivalence relation, a fundamental question is: when is a function $f \in K[[\mathbf{x}]]$ equivalent to a finite number of terms of its power series. This question is concerned with the finite determinacy theory and the classification theory for map-germs.

If K is the field of complex numbers and $K[[x]]$ is the ring of formal power series defined by the convergent ones, this question is well studied by John Mather and some authors (see, e.g. [1,2,4–6,11–15,17]). In the complex case, let $\mathcal{O}_{n+1,0}$ be the local ring of analytic function germs on analytic space $(\mathbb{C}^{n+1}, 0)$. Let $\{y_1, \ldots, y_{n+1}\}$ be a coordinate system in $\mathbb{C}^{n+1}$ and $\mathcal{M}$ be the maximal ideal of $\mathcal{O}_{n+1,0}$. Let $\mathcal{R}$ be the group of all the holomorphic automorphisms of the germ $(\mathbb{C}^{n+1}, 0)$. Take L as the y_1-axis in $(\mathbb{C}^{n+1}, 0)$, then the defining ideal of L is $\mathcal{G} = \langle y_2, \ldots, y_{n+1} \rangle$. Let

$$\mathcal{R}_L \doteq \{\phi \in \mathcal{R} \mid \phi(L) = L\},$$

be the subgroup of the holomorphic automorphisms $\phi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ such that $\phi(L) = L$ for all $\phi \in \mathcal{R}$. $\mathcal{R}_L$ can act on $\mathcal{M} \cdot \mathcal{G}$ from right and this defines an equivalence relation on $\mathcal{M} \cdot \mathcal{G}$. Two germs $f, \ g \in \mathcal{M} \cdot \mathcal{G}$ are called $\mathcal{R}_L-$*equivalent* if there exists a $\phi \in \mathcal{R}_L$ such that $f = g \circ \phi$. A germ $f \in \mathcal{M} \cdot \mathcal{G}$ is called $k - \mathcal{R}_L-$*determined* in $\mathcal{M} \cdot \mathcal{G}$ if for each $g \in \mathcal{M} \cdot \mathcal{G}$ such that $f - g \in \mathcal{M}^{k+1} \cap \mathcal{G} = \mathcal{M}^k \cdot \mathcal{G}, \ g$ is $\mathcal{R}_L-$equivalent to f.

Siersma studied the problem of finite $\mathcal{R}_L-$determinacy in [16]. He gave the list of $\mathcal{R}_L-$simple singularities and studied the Milnor fiber of a generic deformation of a certain class of such singularities.

Jiang and Siersma proved the following theorem (see Theorem 2.2. of [9]):

If $\mathcal{M}^k \cdot \mathcal{G} \subset \mathcal{M} \cdot \tau_{\mathcal{G}}(f) + \mathcal{M}^{k+1} \cdot \mathcal{G}$, then f is $k - \mathcal{R}_L-$determined, where

$$\tau_{\mathcal{G}}(f) \doteq \mathcal{M} \cdot \langle \frac{\partial f}{\partial y_1} \rangle + \mathcal{G} \cdot \langle \frac{\partial f}{\partial y_2}, \ldots, \frac{\partial f}{\partial y_{n+1}} \rangle$$

is the tangent space at f of the $\mathcal{R}_L-$orbit $\mathcal{R}_L(f)$.

In [4], When $(X, 0)$ is the germ of an analytic subvariety of $(\mathbb{C}^n, 0)$, let $\mathcal{R}_X$ be the group of all analytic automorphisms of $(\mathbb{C}^n, 0)$ which preserve X. $\mathcal{R}_X$ can act on $\mathcal{O}_{n,0}$ and induce an equivalence relation. If

f is again a function germ on $\mathbb{C}^n$ at 0, Bruce and Roberts generalized the definition of Milnor number $\mu(f)$ as follows. Let $\Theta_{X,0}$ denote the $\mathcal{O}_{n,0}$ module of germs of vector fields on $\mathbb{C}^n$ at 0 which are tangent to X, or equivalently, the submodule of germs of derivations of $\mathcal{O}_{n,0}$ which preserve the ideal defining X. For an $f \in \mathcal{O}_{n,0}$ define $j_X(f)$ the ideal in $\mathcal{O}_{n,0}$ given by the image of the homomorphism

$$\Theta_{X,0} \to \mathcal{O}_{n,0}, \delta \mapsto \delta f,$$

and define the Milnor number $\mu_X(f)$ of f on X to be $\dim_{\mathbb{C}}\mathcal{O}_{n,0}/j_X(f)$. Bruce and Roberts stated Damon's result as (see Theorem 2.2. of [4]): *A germ f in $\mathcal{O}_{n,0}$ is finitely determined with respect to the $\mathcal{R}_X$ action if $\mu_X(f) < \infty$.*

In [3], Yousra Boubakri, Gert-Martin Greuel, and Thomas Markwig studied the finite determinacy of singularities $f \in K[[\mathbf{x}]]$ over an algebraically closed field K of arbitrary characteristic under the equivalence relation on the power series ring $K[[\mathbf{x}]]$ induced by the action of either $\mathcal{R} = \mathrm{Aut}(K[[\mathbf{x}]])$ or the semidirect product $\mathcal{K} = K[[\mathbf{x}]]^* \ltimes \mathcal{R}$. For an $f \in K[[\mathbf{x}]]$, they established that the finiteness of the Milnor number and the Tjurina number is equivalent to the finite $\mathcal{R}-$determinacy of f and the finite $\mathcal{K}-$determinacy of f respectively. The Milnor number $\mu(f)$ is defined as $\dim_K K[[\mathbf{x}]]/j(f)$ where $j(f)$ is the Jacobian ideal of f, generated by the partial derivatives f_{x_i} of f, $(i = 1, \ldots, n)$. The Tjurina number $\tau(f)$ is defined as $\dim_K K[[\mathbf{x}]]/\langle f\rangle + j(f)$ where $\langle f\rangle$ is the ideal generated by f. Their results are as follows (see Theorem 5 of [3]):

Let $0 \neq f \in \mathcal{M} \subset K[[\mathbf{x}]]$ be a power series.

1. *$\mu(f) < \infty$ if and only if f is finitely $\mathcal{R}-$determined.*
2. *$\tau(f) < \infty$ if and only if f is finitely $\mathcal{K}-$determined.*

Since the proofs of Jiang's theorem and Damon's result need to use the solution of a differential equation, it seems that their methods do not work in the case of positive characteristic. Motivated by Jiang's theorem and Damon's result, following the ideas of [3], we discuss the finite determinacy of singularities $f \in K[[x_1, \ldots, x_n]]$ under the equivalence relation on the power series ring $K[[\mathbf{x}]]$ induced by the action of the subgroup of automorphisms preserving the line $x_2 = \cdots = x_n = 0$ or the subgroup of automorphisms preserving a given hypersurface. We try to obtain some results which are similar to Jiang's theorem, respectively to Damon's result in case of X is a smooth hypersurface.

In this paper, We have two main results :

(1) *For a singularity* $f \in \mathcal{M}^2 \subset K[[\mathbf{x}]]$ *over an algebraically closed field* K *of arbitrary characteristic, the finite* $\mathcal{R}_{\mathcal{G}}$ *(or* $\mathcal{K}_{\mathcal{G}}-$*)determinacy of* f *is equivalent to the relative* $\mathcal{G}-$*isolatedness of the singularity* f *(or* R_f*), when* $\mathcal{R}_{\mathcal{G}}$ *is the subgroup of automorphisms preserving the line* $x_2 = \cdots = x_n = 0$ *and* $\mathcal{K}_{\mathcal{G}} \doteq K[[\mathbf{x}]]^* \ltimes \mathcal{R}_{\mathcal{G}}$. *(see Theorem 3.7)*

(2) *Let* $0 \neq f \in \mathcal{M}^2 \subseteq K[[\mathbf{x}]]$. *The finite* $\mathcal{R}_{\mathcal{A}}$ *(or* $\mathcal{K}_{\mathcal{A}}-$*)determinacy of* f *is equivalent to the relative* $\mathcal{G}-$*isolatedness of the singularity* f *(or* R_f*), when* $\mathcal{R}_{\mathcal{A}}$ *is the subgroup of automorphisms preserving a given hypersurface and* $\mathcal{K}_{\mathcal{A}} \doteq K[[\mathbf{x}]]^* \ltimes \mathcal{R}_{\mathcal{A}}$. *(see Theorem 4.7)*

The above results also provide some estimates on the degree of determinacy in positive characteristic (for details, see section 3 and 4).

Moreover, the results we obtain can be applied to classify the $f \in K[[\mathbf{x}]]$ which are finitely $\mathcal{S}-$determined.

2. **Preliminaries**

Lemma 2.1. (see [7] p. 210) *Let* R *be any ring and let* $f_1, \ldots, f_n \in \langle x_1, \ldots, x_n \rangle \cdot R[[x_1, \ldots, x_n]]$ *be power series. If* φ *is the endomorphism*

$$\varphi : R[[x_1, \ldots, x_n]] \to R[[x_1, \ldots, x_n]],\ x_i \mapsto f_i, i = 1, \ldots, n$$

and the Jacobian matrix $J(\varphi)$ *of* φ *is the matrix* $\left((\varphi_i)_{x_j}\right)$, *then* φ *is an isomorphism if and only if* $DetJ(\varphi)(0)$ *is a unit in* K.

Lemma 2.2. (see [3]) *Let* K *be an algebraically closed field of arbitrary characteristic and* $K[[\mathbf{x}]] = K[[x_1, \ldots, x_n]]$. *Let* $Q \geq 1$ *be an integer and let* $b_{p,0} \in \mathcal{M}^{Q+p-1}$ *and* $b_{p,i} \in \mathcal{M}^{Q+p}$ *for* $i = 1, \ldots, n$ *and* $p \geq 1$. *Consider the units* $v_p = 1 + b_{p,0} \in K[[\mathbf{x}]]^*$ *and the automorphisms* $\phi_p \in Aut(K[[\mathbf{x}]])$ *given by* $\phi_p : x_i \mapsto x_i + b_{p,i}$ *for* $i = 1, \ldots, n$. *We denote by*

$$\varphi_p = \phi_p \circ \phi_{p-1} \circ \cdots \circ \phi_1 \in Aut(K[[\mathbf{x}]])$$

the composition of the first p *automorphisms, and we define inductively* $u_p = v_p \cdot \phi_p(u_{p-1})$, *where* $u_0 = 1$. *Then the following hold true:*

(a) The sequences $\left(\varphi_p(x_i)\right)_{p\geq 1}$ *converge in the* $\mathcal{M}-$*adic topology of* $K[[\mathbf{x}]]$ *to power series* $x_i + b_i$ *with* $b_i \in \mathcal{M}^{Q+1}$ *for* $i = 1, \ldots, n$. *In particular, the map*

$$\varphi :\ K[[\mathbf{x}]] \to K[[\mathbf{x}]] :\ x_i \mapsto x_i + b_i$$

is a local K-algebra automorphism of $K[[\mathbf{x}]]$.

(b) The sequence $(u_p)_{p\geq 1}$ *converges in the* $\mathcal{M}-$*adic topology to a unit* $u = 1 + b_0 \in K[[\mathbf{x}]]^*$ *with* $b_0 \in \mathcal{M}^Q$.

(*c*) *For any power series* $f_0 \in K[[\mathbf{x}]]$ *the sequence* $\left(\varphi_p(f_0)\right)_{p\geq 1}$ *converges in the* $\mathcal{M}-$*adic topology to* $\varphi(f_0)$.

(*d*) *For any power series* $f_0 \in K[[\mathbf{x}]]$ *the sequence* $\left(u_p \cdot \varphi_p(f_0)\right)_{p\geq 1}$ *converges in the* $\mathcal{M}-$*adic topology to* $u \cdot \varphi(f_0)$.

3. Finite $\mathcal{S}-$determinacy of singularities in positive characteristic, $\mathcal{S} = \mathcal{R}_{\mathcal{G}},\ \mathcal{K}_{\mathcal{G}}$

Definition 3.1. *Let* $\mathcal{G}$ *be the ideal* $\langle x_2, \ldots, x_n\rangle$ *of* $K[[\mathbf{x}]]$ *and* $\mathcal{R} = Aut(K[[\mathbf{x}]])$. *Define* $\mathcal{R}_{\mathcal{G}} \doteq \{\varphi \in \mathcal{R} |\ \varphi(\mathcal{G}) = \mathcal{G}\}$. *We say that two power series* $f, g \in K[[\mathbf{x}]]$ *are right line equivalent or* $\mathcal{R}_{\mathcal{G}}-$*equivalent if there is an automorphism* $\varphi \in \mathcal{R}_{\mathcal{G}}$ *such that* $f = \varphi(g)$. *We denote this relation by* $f \sim_{r_{\mathcal{G}}} g$. *A power series* $f \in K[[\mathbf{x}]]$ *is called* $k-\mathcal{R}_{\mathcal{G}}-$*determined if for each* $g \in K[[\mathbf{x}]]$ *such that the same* $k-$*jet as* f, g *is right line equivalent to* f.

Let $\mathcal{K}_{\mathcal{G}} \doteq K[[\mathbf{x}]]^* \ltimes \mathcal{R}_{\mathcal{G}}$. *Two power series* $f, g \in K[[\mathbf{x}]]$ *are contact line equivalent or* $\mathcal{K}_{\mathcal{G}}-$*equivalent if there is an automorphism* $\varphi \in \mathcal{R}_{\mathcal{G}}$ *and a unit* $u \in K[[\mathbf{x}]]^*$ *such that* $f = u \cdot \varphi(g)$, *we denote this relation by* $f \sim_{c_{\mathcal{G}}} g$. *A power series* $f \in K[[\mathbf{x}]]$ *is* $k - \mathcal{K}_{\mathcal{G}}-$*determined if for each* $g \in K[[\mathbf{x}]]$ *such that the same* $k-$*jet as* f, g *is contact line equivalent to* f .

We say that f *is finitely* $\mathcal{R}_{\mathcal{G}}(\mathcal{K}_{\mathcal{G}})-$*determined if it is* $k - \mathcal{R}_{\mathcal{G}}(\mathcal{K}_{\mathcal{G}})-$*determined for some positive integer* k.

For an $f \in K[[\mathbf{x}]]$, we call the K-algebra $R_f = K[[\mathbf{x}]]/\langle f\rangle$ the induced *hypersurface singularities.*

We denote by $j_{\mathcal{G}}(f) = \mathcal{M} \cdot \langle f_{x_1}\rangle + \mathcal{G} \cdot \langle f_{x_2}, \ldots, f_{x_n}\rangle$ the *relative* $\mathcal{G}-$*Jacobian ideal of of* f, where f_{x_i} is the formal partial derivative of f with respect to x_i. We call the associated algebra $M_{\mathcal{G}}(f) = \frac{K[[\mathbf{x}]]}{j_{\mathcal{G}}(f)}$ the *relative* $\mathcal{G}-$*Milnor algebra* and its dimension $\mu_{\mathcal{G}}(f) = \dim_K\left(M_{\mathcal{G}}(f)\right)$ the *relative* $\mathcal{G}-$*Milnor number of* f. We then call f a *relative* $\mathcal{G}-$*isolated singularity* if $\mu_{\mathcal{G}}(f) < \infty$ or, equivalently, if there is a positive integer such that $\mathcal{M}^k \subseteq j_{\mathcal{G}}(f)$.

The *relative* $\mathcal{G}-$ *Tjurina ideal of* f is defined by $tj_{\mathcal{G}}(f) = \langle f\rangle + j_{\mathcal{G}}(f)$. The associated algebra $T_{\mathcal{G}}(f) = \frac{K[[\mathbf{x}]]}{tj_{\mathcal{G}}(f)}$ is called the *relative* $\mathcal{G}-$ *Tjurina algebra of* f. The dimension $\tau_{\mathcal{G}}(f) = \dim_K\left(T_{\mathcal{G}}(f)\right)$ of $T_{\mathcal{G}}(f)$ is called the *relative* $\mathcal{G}-$ *Tjurina number of* f. We then call R_f a *relative* $\mathcal{G}-$*isolated hypersurface singularity* if $\tau_{\mathcal{G}}(f) < \infty$, which is equivalent to the existence of a positive integer k such that $\mathcal{M}^k \subseteq tj_{\mathcal{G}}(f)$.

Note that the ideal $j_{\mathcal{G}}(f)$ is basically the tangent space to the orbit of f under the action of $\mathcal{R}_{\mathcal{G}}$, and similarly that $tj_{\mathcal{G}}(f)$ is basically the tangent space to the orbit of f under the action of $\mathcal{K}_{\mathcal{G}}$. The precise statement and its proof will be given in Proposition 3.6.

Let $f \in K[[\mathbf{x}]]$ be a non-zero power series, we denote by $\operatorname{ord}(f)$ the largest integer k such that $f \in \mathcal{M}^k$. We set $\operatorname{ord}(0) = \infty$.

Theorem 3.2. *Let $0 \neq f \in \mathcal{M}^2$ and $k \in \mathbf{N}$.*

(a) If

$$\mathcal{M}^{k+2} \subseteq \mathcal{M}^2 \cdot \langle f_{x_1} \rangle + \mathcal{M} \cdot \mathcal{G} \cdot \langle f_{x_2}, \dots, f_{x_n} \rangle,$$

then f is $(2k - ord(f) + 2) - \mathcal{R}_{\mathcal{G}}-$determined.

(b) If

$$\mathcal{M}^{k+2} \subseteq \mathcal{M} \cdot \langle f \rangle + \mathcal{M}^2 \cdot \langle f_{x_1} \rangle + \mathcal{M} \cdot \mathcal{G} \cdot \langle f_{x_2}, \dots, f_{x_n} \rangle,$$

then f is $(2k - ord(f) + 2) - \mathcal{K}_{\mathcal{G}}-$determined.

Proof. We first prove (b). Let $o = \operatorname{ord}(f)$. It follows that

$$\operatorname{ord}(f_{x_i}) \geq o - 1 \text{ for all } (i = 1, \dots, n)$$

and by assumption we have

$$\mathcal{M}^{k+2} \subseteq \mathcal{M} \cdot \langle f \rangle + \mathcal{M}^2 \cdot \langle f_{x_1} \rangle + \mathcal{M} \cdot \mathcal{G} \cdot \langle f_{x_2}, \dots, f_{x_n} \rangle \subseteq \mathcal{M}^{o+1}.$$

This implies $k \geq o - 1$.

Set $N = 2k - o + 2 \geq k + 1$, and take $g \in K[[\mathbf{x}]]$ such that $g - f \in \mathcal{M}^{N+1}$, i.e., f and g have the same N-jet. We shall show that f and g are $\mathcal{K}_{\mathcal{G}}-$equivalent, i.e., there exists an automorphism $\varphi \in \mathcal{R}_{\mathcal{G}}$ and a unit $u \in K[[\mathbf{x}]]^*$ such that

$$g = u \cdot \varphi(f).$$

We construct φ and u inductively, i.e., we construct inductively sequences of automorphisms $(\varphi_p)_{p\geq 1}$ and units $(u_p)_{p\geq 1}$ such that $u_p \cdot \varphi_p(f)$ converges in the $\mathcal{M}-$adic topology to $u \cdot \varphi(f)$ for some automorphism $\varphi \in \mathcal{R}_{\mathcal{G}}$ and some unit $u \in K[[\mathbf{x}]]^*$ and at the same time

$$g - u_p \cdot \varphi_p(f) \in \mathcal{M}^{N+1+p},$$

for all $p \geq 1$. The latter implies that $u_p \cdot \varphi_p(f)$ converges to g as well, and thus

$$g = u \cdot \varphi(f).$$

By Lemma 2.2 and its terminology with $Q = N - k \geq 1$ it suffices to construct certain series $b_{p,0} \in \mathcal{M}^{Q+p-1}$, $b_{p,1} \in \mathcal{M}^{Q+p}$, and $b_{p,i} \in \mathcal{M}^{Q+p-1} \cdot \mathcal{G} \subset \mathcal{M}^{Q+p}$ for $i = 2, \ldots, n$ and $p \geq 1$.

In fact, note that by assumption

$$g - f \in \mathcal{M}^{N+1} = \mathcal{M}^{Q-1} \cdot \mathcal{M}^{k+2} \subset \mathcal{M}^{Q} \cdot tj_{\mathcal{G}}(f)$$

there exist $b_{1,0} \in \mathcal{M}^{Q}$, $b_{1,1} \in \mathcal{M}^{Q+1}$, and $b_{1,i} \in \mathcal{M}^{Q}\mathcal{G} \subset \mathcal{M}^{Q+1}$ for $i = 2, \ldots, n$ such that

$$g - f = b_{1,0} f + b_{1,1} f_{x_1} + \sum_{i=2}^{n} b_{1,i} f_{x_i}. \tag{3.1}$$

Let $v_1 = 1 + b_{1,0} \in K[[\mathbf{x}]]^*$ and $\phi_1 : K[[\mathbf{x}]] \to K[[\mathbf{x}]] :\ x_i \mapsto x_i + b_{1,i},\ i = 1, \ldots, n$, where $b_{1,1} \in \mathcal{M}^{Q+1}$, $b_{1,i} \in \mathcal{M}^{Q} \cdot \mathcal{G} \subset \mathcal{M}^{Q+1}$ for $i = 2, \ldots, n$.

Now We prove $\phi_1 \in \mathcal{R}_{\mathcal{G}}$.

In fact, by Lemma 2.1 ϕ_1 is an automorphism. For any g in $\mathcal{G} = \langle x_2, \ldots, x_n \rangle$, there exist power series $g_2, \ldots, g_n \in K[[\mathbf{x}]]$ such that $g = g_2 \cdot x_2 + \cdots + g_n \cdot x_n$. We have

$$\begin{aligned} \phi_1(g) &= \phi_1(g_2) \cdot (x_2 + b_{1,2}) + \cdots + \phi_1(g_n)(x_n + b_{1,n}) \\ &= \sum_{i=2}^{n} \phi_1(g_i) \cdot x_i + \sum_{i=2}^{n} \phi_1(g_i) b_{1,i}. \end{aligned}$$

Since $b_{1,i} \in \mathcal{M}^{Q} \cdot \mathcal{G} \subseteq \mathcal{G},\ i = 2, \ldots, n$, we have $\phi_1(g) \in \mathcal{G}$.

Next, we want to show that

$$g - v_1 \cdot \phi_1(f) \in \mathcal{M}^{N+2}.$$

If the above formula is true, we can replace f in the above argument by $v_1 \cdot \phi_1(f)$ and go on inductively. Note first that

$$(x_1 + z_1)^{\beta_1} \cdots (x_n + z_n)^{\beta_n} = \sum_{\gamma_1=0}^{\beta_1} \cdot \sum_{\gamma_2=0}^{\beta_2} \cdots \sum_{\gamma_n=0}^{\beta_n} c_{\beta,\gamma} \mathbf{x}^{\beta-\gamma} \cdot \mathbf{z}^{\gamma}$$

where $c_{\beta,\gamma} = \begin{pmatrix} \beta_1 \\ \gamma_1 \end{pmatrix} \begin{pmatrix} \beta_2 \\ \gamma_2 \end{pmatrix} \cdots \begin{pmatrix} \beta_n \\ \gamma_n \end{pmatrix} \in \mathbb{Z}$. For $f = \sum_{|\beta| \geq 0} k_{\beta} \cdot \mathbf{x}^{\beta}$, consider

$$\begin{aligned} & f\left((x_1 + z_1), \ldots, (x_n + z_n)\right) \\ &= \textstyle\sum_{|\beta| \geq \mathrm{ord}(f)} k_{\beta} \cdot \sum_{\gamma_1=0}^{\beta_1} \sum_{\gamma_2=0}^{\beta_2} \cdots \sum_{\gamma_n=0}^{\beta_n} c_{\beta,\gamma} \mathbf{x}^{\beta-\gamma} \cdot \mathbf{z}^{\gamma} \\ &= \textstyle\sum_{\alpha \in \mathrm{N}^n} w_{\alpha} \cdot \mathbf{z}^{\alpha}, \end{aligned} \tag{3.2}$$

where

$$w_\alpha = \sum_{|\beta|\geq \text{ord}(f),\beta\geq\alpha} k_\beta \cdot c_{\beta,\alpha}\cdot \mathbf{x}^{\beta-\alpha}$$

if we define $\beta \geq \alpha$ by $\beta_i \geq \alpha_i$ for all $i = 1, 2, \ldots, n$. It follows that

$$\text{ord}(w_\alpha) = \min\left\{|\beta| - |\alpha|\,\middle|\, |\beta| \geq \text{ord}(f),\ |\beta| \geq |\alpha|\right\} \geq o - |\alpha|.$$

We notice that $w_\alpha = \frac{D^\alpha f(\mathbf{x})}{\alpha_1!\alpha_2!\cdots\alpha_n!}$ whenever $\alpha_i <$char(K) for all $i = 1, 2, \ldots, n$. In particular, the constant term is $w_0 = f$. For every unit vector e_i $(1 \leq i \leq n)$ $w_{e_i} = f_{x_i}$.

Applying ϕ_1 to f amounts to substituting z_1 by $b_{1,1}$ and z_i by $b_{1,i}$ in (3.2) we thus find $\phi_1(f) = f + f_{x_1}\cdot b_{1,1} + \sum_{i=2}^n f_{x_i}\cdot b_{1,i} + w$, where $w = \sum_{|\alpha|\geq 2} w_\alpha \cdot b_{1,1}^{\alpha_1}\cdots b_{1,n}^{\alpha_n}$. Since

$$\begin{aligned}\text{ord}\left(w_\alpha\cdot b_{1,1}^{\alpha_1}\cdot b_{1,2}^{\alpha_2}\cdots b_{1,n}^{\alpha_n}\right) &\geq \text{ord}(w_\alpha) + \text{ord}(b_{1,1})\cdot\alpha_1 + \sum_{i=2}^n \text{ord}(b_{1,i})\cdot\alpha_i\\ &\geq o- \mid\alpha\mid +(Q+1)\cdot\mid\alpha\mid\\ &\geq o + 2\cdot Q = N+2,\end{aligned}$$

we have $w \in \mathcal{M}^{N+2}$. Multiplying $\phi_1(f)$ by $v_1 = 1 + b_{1,0}$ and using (3.1) we get

$$\begin{aligned}g - v_1\cdot\phi_1(f) &= g - (1 + b_{1,0})\cdot(f + f_{x_1}\cdot b_{1,1} + \sum_{i=2}^n f_{x_i}\cdot b_{1,i} + w)\\ &= -f_{x_1}\cdot b_{1,1}\cdot b_{1,0} - \sum_{i=2}^n f_{x_i}\cdot b_{1,i}\cdot b_{1,0} - (1 + b_{1,0})\cdot w.\end{aligned}$$

Since

$$\text{ord}(b_{1,0}\cdot b_{1,i}\cdot f_{x_i}) \geq Q + (Q+1) + (o-1) = N+2,\ i = 1, 2, \ldots, n,$$

we have

$$g - v_1\cdot\phi_1(f) \in \mathcal{M}^{N+2}. \tag{3.3}$$

Thus we can proceed inductively to construct sequences $\{b_{p,i}\}_{p\geq 1}$ for $i = 0, \ldots, n$ with $b_{p,0} \in \mathcal{M}^{Q+p-1}$, $b_{p,1} \in \mathcal{M}^{Q+p}$ and $b_{p,i} \in \mathcal{M}^{Q+p-1}\cdot\mathcal{G} \subseteq \mathcal{M}^{Q+p}$ for $i = 2, \ldots, n$. The generalization of (3.3) holds by induction. Using Lemma 2.2 we have

$$g - u_p\cdot\varphi_p(f) \in \mathcal{M}^{N+1+p}.$$

Again using Lemma 2.2, we obtain an automorphism $(u,\varphi) \in \mathcal{K}_{\mathcal{G}}$ such that $g = u \cdot \varphi(f)$.

The proof for right equivalence can be done in the same lines. The condition $\mathcal{M}^{k+2} \subseteq \mathcal{M}^1 \cdot j_{\mathcal{G}}(f) \subseteq \mathcal{M}^{o+1}$ implies also that $k \geq o-1$. For any g with

$$g - f \in \mathcal{M}^{N+1} = \mathcal{M}^{Q-1} \cdot \mathcal{M}^{k+2} \subseteq \mathcal{M}^{Q} \cdot j_{\mathcal{G}}(f)$$

where $N = 2k-o+2 \geq k+1$ and $Q = N-k \geq 1$, there exist $b_{1,1} \in \mathcal{M}^{Q+1}$ and $b_{1,i} \in \mathcal{M}^{Q} \cdot \mathcal{G} \subseteq \mathcal{M}^{Q+1},\ i = 2, \ldots, n$ with

$$g - f = b_{1,1} \cdot f_{x_1} + b_{1,2} \cdot f_{x_2} + \cdots + b_{1,n} \cdot f_{x_n}.$$

We can then define ϕ_1 as above. It is easy to show

$$g - \phi_1(f) = h \in \mathcal{M}^{N+2}.$$

Going on by induction and applying Lemma 2.2, we get an automorphism $\varphi \in \mathcal{R}_{\mathcal{G}}$ such that $g = \varphi(f)$. □

Corollary 3.3. *Let* $0 \neq f \in \mathcal{M}^2 \subseteq K[[\mathbf{x}]]$.

(a) If $\mu_{\mathcal{G}}(f) < \infty$, *then* f *is* $\big(2\mu_{\mathcal{G}}(f) - ord(f)\big) - \mathcal{R}_{\mathcal{G}}-$*determined.*

(b) If $\tau_{\mathcal{G}}(f) < \infty$, *then* f *is* $\big(2\tau_{\mathcal{G}}(f) - ord(f)\big) - \mathcal{K}_{\mathcal{G}}-$*determined.*

The converse also holds in arbitrary characteristic.

Theorem 3.4. *Let* $0 \neq f \in \mathcal{M} \subseteq K[[\mathbf{x}]]$.

(a) If f *is* $\mathcal{R}_{\mathcal{G}} - k-$*determined, then* $\mathcal{M}^{k+1} \subseteq j_{\mathcal{G}}(f)$.

(b) If f *is* $\mathcal{K}_{\mathcal{G}} - k-$*determined, then* $\mathcal{M}^{k+1} \subseteq \langle f \rangle + j_{\mathcal{G}}(f)$.

The proof of Theorem 3.4 is analogous to the result established in [3]. Before we begin the proof, we need some notations.

Denote $J_l = K[[\mathbf{x}]]/\mathcal{M}^{l+1}$ the space of $l-$jets of power series in $K[[\mathbf{x}]]$. Each $K-$algebra automorphism φ of $K[[\mathbf{x}]]$ is a tuple $(\varphi_1, \varphi_2, \ldots, \varphi_n) \in K[[\mathbf{x}]]^n$ of power series such that $\varphi_i(0) = 0$, for all $i = 1, 2, \ldots, n$ and Det $\left(\frac{\partial \varphi_i}{\partial x_j}(0)\right)_{i,j=1,2,\ldots,n}$ is invertible. The $l-$jet of the automorphism φ is $\mathrm{jet}_l(\varphi) = \big(\ \mathrm{jet}_l(\varphi_1),\ \ldots,\ \mathrm{jet}_l(\varphi_n)\ \big)$. The $l-$jet of the right line equivalence group is $\mathcal{R}_{\mathcal{G},l} = \{\mathrm{jet}_l(\varphi) | \varphi \in \mathcal{R}_{\mathcal{G}}\}$ and the $l-$jet of the contact line equivalence group is $\mathcal{K}_{\mathcal{G},l} = jet_l(K[[\mathbf{x}]]^*) \ltimes \mathcal{R}_{\mathcal{G},l}$. $\mathcal{K}_{\mathcal{G},l}$ acts on J_l via

$$\phi_l : \mathcal{K}_{\mathcal{G},l} \times J_l \to J_l : (\mathrm{jet}_l(u),\ \mathrm{jet}_l(\varphi),\ \mathrm{jet}_l(f)) \mapsto \mathrm{jet}_l\left(u \cdot \varphi(f)\right).$$

Similarly, we define the action of the $l-$jet $\mathcal{R}_{\mathcal{G},l}$ on J_l.

Remark 3.5. (a) From [3], we know that J_l is an affine space and $\mathcal{K}_l$ and $\mathcal{R}_l$ are affine algebraic groups acting on J_l via a regular separable algebraic action.

(b) $\mathcal{K}_{\mathcal{G},l}$ and $\mathcal{R}_{\mathcal{G},l}$ are affine algebraic groups acting on J_l via a regular separable algebraic action.

In fact, given $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \in \mathcal{R}_{\mathcal{G}}$, we have $\varphi_i = \varphi(x_i) \in \mathcal{G}$ for $i = 2, \ldots, n$. It implies that $\frac{\partial \varphi_i}{\partial x_1}(x_1, 0, \ldots, 0) = 0$ for $i = 2, \ldots, n$. Let $\text{jet}_l(f) = \sum_{|\alpha|=0}^{l} a_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, $\text{jet}_l(\varphi_i) = \sum_{|\beta|=1}^{l} b_{i,\beta} x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$ and $\text{jet}_l(u) = \sum_{|\gamma|=0}^{l} c_\gamma x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n}$.

We can choose coordinate variables $(a_\alpha, b_{i,\beta}, c_\gamma)_{\alpha,i,\beta,\gamma}$ on $\mathcal{K}_l \times J_l$ with $c_0 \neq 0$ and $\text{Det}(B) \neq 0$ where $B = (B_{ij})$ with $B_{ij} = \frac{\partial \varphi_i}{\partial x_j}(0) = b_{i,e_j}$ and e_j the $j-$th canonical basic vectors in $\mathbb{Z}^n$.

We note that $\mathcal{K}_{\mathcal{G},l} \times J_l$ is a subvariety of $\mathcal{K}_l \times J_l$. This is because $\mathcal{K}_{\mathcal{G},l} \times J_l$ is defined by a system of equations $b_{i,k\cdot e_1} = 0$, for all $i = 2, \ldots, n$ and $k = 1, \ldots, l$. Again by Remark 2 of [3], the extension $K(\mathcal{K}_l \times J_l)$ of the field K is a purely transcendental extension of $K(J_l)$ and it is thus a separably generated extension. Since $\mathcal{K}_{\mathcal{G},l} \times J_l \subseteq \mathcal{K}_l \times J_l$, we have $K(\mathcal{K}_{\mathcal{G},l} \times J_l)$ is a separably generated extension of $K(J_l)$.

Now we can obtain the tangent space to the orbits also in positive characteristic.

Proposition 3.6. *Let $f \in K[[\mathbf{x}]]$. Then the tangent space to the orbit of $jet_l(f)$ under the action of $\mathcal{R}_{\mathcal{G},l}$ and $\mathcal{K}_{\mathcal{G},l}$ considered as a subspace of J_l are*

$$T_{jet_l(f)}\left(\mathcal{R}_{\mathcal{G},l} \cdot jet_l(f)\right) = \left(j_{\mathcal{G}}(f) + \mathcal{M}^{l+1}\right)/\mathcal{M}^{l+1}$$
$$T_{jet_l(f)}\left(\mathcal{K}_{\mathcal{G},l} \cdot jet_l(f)\right) = \left(\langle f\rangle + j_{\mathcal{G}}(f) + \mathcal{M}^{l+1}\right)/\mathcal{M}^{l+1}$$

Proof. Let G be one of the two above groups, then the action of G on J_l induces a surjective separable morphism $G \to G \cdot \text{jet}_l(f)$ of smooth varieties. As $K(\mathcal{K}_{\mathcal{G},l} \times J_l)$ is a separably generated extension of $K(J_l)$, the induced differential map on the tangent spaces is generically surjective (see e.g. the proof of [8], Ch.3. Lemma 10.5.]).

Because each point in G can be translated to the identity element of G and this translation is an isomorphism, it thus suffices to understand the image of the tangent space to G at the identity element of G and its image under the differential map. We restrict here to the case $G = \mathcal{K}_{\mathcal{G},l}$ since the proof for $\mathcal{R}_{\mathcal{G},l}$ is analogous to $\mathcal{K}_{\mathcal{G},l}$.

We now describe the tangent space to $\mathcal{K}_{\mathcal{G},l}$ at $(1,\ id)$, through the local $K-$algebra homomorphisms from the local ring of $\mathcal{K}_{\mathcal{G},l}$ to $K[[t]]$ with $t^2 = 0$. In this sense, a tangent vector of $\mathcal{K}_{\mathcal{G},l}$ at $(1,\ id)$ can be represented by the residue class modulo $\mathcal{M}^{l+1}$ of a tuple $(1+t\cdot a,\ id+t\cdot\phi)$ in $\mathcal{K}_{\mathcal{G},l}$ with $a \in K[[\mathbf{x}]]$ and $\phi = (\phi_1, \phi_2 \ldots, \phi_n)$, where $\phi_1 \in \mathcal{M}$ and $\phi_i \in \mathcal{G},\ i = 2, \ldots, n$.

The tangent space to $\mathcal{K}_{\mathcal{G},l}\cdot\mathrm{jet}_l(f)$ at $\mathrm{jet}_l(f)$ can be described as follows. We apply the differential map by acting with the above tuple on f modulo $\mathcal{M}^{l+1}$. Expanding the power series as in (3.2), we have

$$(1 + t \cdot a) \cdot f\left((\mathbf{x}) + t\phi\right) = f + t \cdot \left(a \cdot f + f_{x_1}\phi_1 + \sum_{i=2}^{n} f_{x_i}\phi_i\right) + t^2 h(\mathbf{x}, t).$$

Hence, in $K[[\mathbf{x}]][[t]]/\langle t^2\rangle$,

$$(1 + t \cdot a) \cdot f\left((\mathbf{x}) + t \cdot \phi\right) = f + t \cdot \left(a \cdot f + f_{x_1} \cdot \phi_1 + \sum_{i=2}^{n} f_{x_i} \cdot \phi_i\right).$$

In J_l this tangent vector is just the $l-$jet of

$$a \cdot f + f_{x_1} \cdot \phi_1 + \sum_{i=2}^{n} f_{x_i} \cdot \phi_i.$$

This implies that

$$T_{\mathrm{jet}_l(f)}\left(\mathcal{K}_{\mathcal{G},l} \cdot \mathrm{jet}_l(f)\right) = \left(\langle f\rangle + j_{\mathcal{G}}(f) + \mathcal{M}^{l+1}\right)/\mathcal{M}^{l+1}.$$

□

Now we prove Theorem 3.4.

Proof. We give only the proof of $\mathcal{K}_{\mathcal{G},l}-$determinacy since the proof of the other case is analogous. If f is $k-\mathcal{K}_{\mathcal{G},l}-$determined and $g \in \mathcal{M}^{k+1}$, then for any $t \in K$ the $(k+1)-$jet $\mathrm{jet}_{k+1}(f) + t\cdot\mathrm{jet}_{k+1}(g)$ is in the orbit of $\mathrm{jet}_{k+1}(f)$ under $\mathcal{K}_{\mathcal{G},k+1}$. Hence

$$\mathrm{jet}_{k+1}(g) \in T_{\mathrm{jet}_l(f)}\left(\mathcal{K}_{\mathcal{G},k+1} \cdot \mathrm{jet}_{k+1}(f)\right) = \left(\langle f\rangle + j_{\mathcal{G}}(f) + \mathcal{M}^{k+2}\right)/\mathcal{M}^{k+2}.$$

This implies that

$$g \in \langle f\rangle + j_{\mathcal{G}}(f) + \mathcal{M}^{k+2},$$

and hence

$$\mathcal{M}^{k+1} \subseteq \langle f\rangle + j_{\mathcal{G}}(f) + \mathcal{M}^{k+2}.$$

By Nakayama's Lemma we get $\mathcal{M}^{k+1} \subseteq \langle f\rangle + j_{\mathcal{G}}(f)$. $\square$

From the formulas in Proposition 3.6, the geometrical meaning of the ideals $j_{\mathcal{G}}(f)$ and $tj_{\mathcal{G}}(f)$ are the tangent space to the orbit of f under the action of $\mathcal{R}_{\mathcal{G}}$ and $\mathcal{K}_{\mathcal{G}}$ respectively.

Combining Corollary 3.3 and Theorem 3.4, we obtain:

Theorem 3.7. *Let $0 \neq f \in \mathcal{M}^2 \subset K[[\mathbf{x}]]$ be a power series.*

1. f is a relative $\mathcal{G}-$isolated singularity if and only if f is finitely $\mathcal{R}_{\mathcal{G}}-$determined.

2. $\mathcal{R}_f$ is a relative $\mathcal{G}-$isolated hypersurface singularity if and only if f is finitely $\mathcal{K}_{\mathcal{G}}-$determined.

4. finite $\mathcal{S}-$determinacy of singularities in positive characteristic, $\mathcal{S} = \mathcal{R}_{\mathcal{G}},\ \mathcal{K}_{\mathcal{G}}$

Definition 4.1. *Let $h \in K[[\mathbf{x}]]$ with $h(0) = 0$ and $\frac{\partial h}{\partial x_n}(0) \neq 0$. For a hypersurface ideal $\mathcal{A} = \langle h\rangle$ of $K[[\mathbf{x}]]$, $\mathcal{R}_{\mathcal{A}} \doteq \{\varphi \in \mathcal{R} |\ \varphi(\mathcal{A}) = \mathcal{A}\}$.*

Two power series $f, g \in K[[\mathbf{x}]]$ are right hypersurface equivalent or $\mathcal{R}_{\mathcal{A}}-$ equivalent if there is an automorphism $\varphi \in \mathcal{R}_{\mathcal{A}}$ such that $f = \varphi(g)$. We denote this relation by $f \sim_{r_{\mathcal{A}}} g$.

A power series $f \in K[[\mathbf{x}]]$ is $k-\mathcal{R}_{\mathcal{A}}-$determined if for each $g \in K[[\mathbf{x}]]$ such that the same $k-$jet as f, g is right hypersurface equivalent to f.

We define $\mathcal{K}_{\mathcal{A}} \doteq K[[\mathbf{x}]]^* \ltimes \mathcal{R}_{\mathcal{A}}$. Two power series $f, g \in K[[\mathbf{x}]]$ are contact hypersurface equivalent or $\mathcal{K}_{\mathcal{A}}-$ equivalent if there is an automorphism $\varphi \in \mathcal{R}_{\mathcal{A}}$ and a unit $u \in K[[\mathbf{x}]]^*$ such that $f = u \cdot \varphi(g)$, where $(u, \varphi) \in \mathcal{K}$. We denote this relation by $f \sim_{c_{\mathcal{A}}} g$.

A power series $f \in K[[\mathbf{x}]]$ is $k-\mathcal{K}_{\mathcal{A}}-$determined if for each $g \in K[[\mathbf{x}]]$ such that the same $k-$jet as f, g is contact hypersurface equivalent to f.

We say that f is finitely $\mathcal{R}_{\mathcal{A}}(\mathcal{K}_{\mathcal{A}})-$determined if it is $k - \mathcal{R}_{\mathcal{A}}(\mathcal{K}_{\mathcal{A}})-$determined for some positive integer k.

For a power series $f \in K[[\mathbf{x}]]$, Let

$$j_{\mathcal{A}}(f) = \mathcal{M} \cdot \left\langle h_{x_n} \cdot f_{x_i} - h_{x_i} \cdot f_{x_n} \middle|\ i = 1, \ldots, n-1 \right\rangle + \mathcal{A} \cdot \langle f_{x_n}\rangle$$

be the relative $\mathcal{A}-$Jacobian ideal of of f.

The *relative $\mathcal{A}-$Milnor algebra $M_{\mathcal{A}}(f)$ of f* is defined as $M_{\mathcal{A}}(f) = \frac{K[[\mathbf{x}]]}{j_{\mathcal{A}}(f)}$. We call its dimension $\mu_{\mathcal{A}}(f) = \dim_K (M_{\mathcal{A}}(f))$ the *relative $\mathcal{A}-$Milnor number of f*. We call f a *relative $\mathcal{A}-$isolated singularity* if $\mu_{\mathcal{A}}(f) < \infty$ or, equivalently, if there is a positive integer such that $\mathcal{M}^k \subseteq j_{\mathcal{A}}(f)$.

The *relative* $\mathcal{A}-$*Tjurina ideal of* f is defined as $tj_{\mathcal{A}}(f) = \langle f \rangle + j_{\mathcal{A}}(f)$ and the associated *relative* $\mathcal{A}-$*Tjurina algebra of* f is $T_{\mathcal{A}}(f) = \frac{K[[\mathbf{x}]]}{tj_{\mathcal{A}}(f)}$. The dimension $\tau_{\mathcal{A}}(f) = \dim_K(T_{\mathcal{A}}(f))$ of $T_{\mathcal{A}}(f)$ is called the *relative* $\mathcal{A}-$*Tjurina number of* f. We then call R_f a *relative* $\mathcal{A}-$*isolated hypersurface singularity* if $\tau_{\mathcal{A}}(f) < \infty$, which is equivalent to the existence of a positive integer k such that $\mathcal{M}^k \subseteq tj_{\mathcal{A}}(f)$.

Note that the ideal $j_{\mathcal{A}}(f)$ is basically the tangent space to the orbit of f under the action of $\mathcal{R}_{\mathcal{A}}$, and similarly that $tj_{\mathcal{A}}(f)$ is basically the tangent space to the orbit of f under the action of $\mathcal{K}_{\mathcal{A}}$. The precise statement and its proof will be given in Proposition 4.4.

Remark 4.2. *In the complex case, when* $(X, 0)$ *is the germ of an analytic subvariety of* $(\mathbb{C}^n, 0)$ *and* f *again a function germ on* $\mathbb{C}^n$ *at* 0, *J.W.Bruce defined the Milnor number of* f *on* X *by*

$$\mu_X(f) = dim_{\mathbb{C}}\mathcal{O}_{n,0}/j_X(f)$$

(*see [4]*)*. If* X *is a hypersurface defined by* $h : \mathbb{C}^n \to \mathbb{C}$ *in analytic space* $(\mathbb{C}^n, 0)$, *where* $h(0) = 0$ *and* $h_{x_n}(0) \neq 0$, *then*

$$\Theta_{X,0} = \left\langle h_{x_n} \cdot \frac{\partial}{\partial x_i} - h_{x_i} \cdot \frac{\partial}{\partial x_n} | \ i = 1, \dots, n-1 \right\rangle + \left\langle h \cdot \frac{\partial}{\partial x_n} \right\rangle$$

and

$$j_X(f) = \langle h_{x_n} \cdot f_{x_i} - h_{x_i} \cdot f_{x_n} \mid \ i = 1, \dots, n-1 \rangle + \langle h \cdot f_{x_n} \rangle .$$

However, the number $\mu_X(f)$ *does not coincide with the number* $\mu_{\mathcal{A}}(f)$. *The number* $\mu_X(f)$ *coincides with the usual Milnor number* $\mu(f)$ *in the case that* $X = \emptyset$. *On the other hand, it is not the codimension of the orbit of* f *under the group action of* $\mathcal{R}_X$, *while this is the case for the number* $\mu_{\mathcal{A}}(f)$ *under the group action of* $\mathcal{R}_{\mathcal{A}}$.

Theorem 4.3. *Let* $0 \neq f \in \mathcal{M} \subseteq K[[\mathbf{x}]]$.
(a) If f *is* $\mathcal{R}_{\mathcal{A}} - k-$*determined, then* $\mathcal{M}^{k+1} \subseteq j_{\mathcal{A}}(f)$.
(b) If f *is* $\mathcal{K}_{\mathcal{A}} - k-$*determined, then* $\mathcal{M}^{k+1} \subseteq tj_{\mathcal{A}}(f)$.

In order to prove Theorem 4.3, we need some facts and propositions. Consider the map $\psi : \ K[[\mathbf{x}]] \to K[[\mathbf{x}]], \ x_i \mapsto x_i, (1 \leq i \leq n-1), x_n \mapsto h$. By Lemma 2.1, ψ is an isomorphism. Let φ be an element of $\mathcal{R}_{\mathcal{A}}$.

Set $\overline{\varphi} \doteq \psi^{-1} \circ \varphi \circ \psi$. Then $\varphi = \psi \circ \overline{\varphi} \circ \psi^{-1}$. We have

$$\varphi\left(\langle h \rangle\right) = \langle h \rangle \Leftrightarrow \overline{\varphi}\left(\langle x_n \rangle\right) = \langle x_n \rangle.$$

So $\mathcal{R}_{\mathcal{A}} = \{ \ \varphi \in \mathcal{R} | \ \varphi(\mathcal{A}) = \mathcal{A}\}$ is isomorphic to $\mathcal{R}_{\overline{\mathcal{A}}} \doteq \{ \ \overline{\varphi} \in \mathcal{R} | \ \overline{\varphi}(\langle x_n \rangle) = \langle x_n \rangle\}$.

The $l-$jet of $\mathcal{R}_{\overline{\mathcal{A}}}$ is $\mathcal{R}_{\overline{\mathcal{A}},l} = \left\{ \text{jet}_l(\overline{\varphi}) | \ \overline{\varphi} \in \mathcal{R}_{\overline{\mathcal{A}}} \right\}$ and the $l-$jet of $\mathcal{K}_{\overline{\mathcal{A}}}$ is $\mathcal{K}_{\overline{\mathcal{A}},l} = \text{jet}_l \left(K[[x_1, \ldots, x_n]]^* \right) \ltimes \mathcal{R}_{\overline{\mathcal{A}},l}$.

Now we show that $\mathcal{K}_{\overline{\mathcal{A}},l}$ and $\mathcal{R}_{\overline{\mathcal{A}},l}$ are affine algebraic groups acting on J_l via a regular *separable* algebraic action.

For $u \in K[[\mathbf{x}]]^*, f \in K[[\mathbf{x}]]$, let $\text{jet}_l(u) = \sum_{|\gamma|=0}^{l} c_\gamma \mathbf{x}^\gamma$, $\text{jet}(f) = \sum_{|\alpha|=0}^{l} a_\alpha \mathbf{x}^\alpha$. If $\overline{\varphi} \in \mathcal{R}_{\overline{\mathcal{A}}}$, then $\overline{\varphi} = (\overline{\varphi}_1, \ldots, \overline{\varphi}_n)$ and there exists a $g \in K[[\mathbf{x}]]$ such that $\overline{\varphi}(x_n) = \overline{\varphi}_n = x_n \cdot g$. Let $\text{jet}_l(\overline{\varphi}_i) = \sum_{|\beta|=1}^{l} b_{i,\beta} \mathbf{x}^\beta$, and $\text{jet}_l(g) = \sum_{|\lambda|=0}^{l} g_\lambda \mathbf{x}^\lambda$. Then $\text{jet}_l(\overline{\varphi_n}) = \text{jet}_l(x_n \cdot g)$. We can obtain a system of equations by comparing the coefficients of the monomials x^β on both sides of the equation $\text{jet}_l(\overline{\varphi}_n) = \text{jet}(x_n \cdot g)$. So the coordinates $b_{n,\beta}$ are given by polynomial maps $b_{n,\beta} = W_\beta(g_\lambda)$, where $0 \leq |\lambda| \leq l-1, \ 1 \leq |\beta| \leq l$, and $g_0 \neq 0$. In fact, if $g_0 = 0$, then the first term of $\overline{\varphi}_n$ is $b_{n,\beta} x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$, where $|\beta| = 2$, so that $(\overline{\varphi}_n)_{x_i}(0) = 0, \ i = 1, \ldots, n$. It is a contradiction to the fact that Det $\text{J}(\overline{\varphi})(0)$ is a unit in K.

So we can take coordinates

$$\begin{aligned} &(a_\alpha, \ b_{i,\beta}, \ g_\lambda, \ c_\gamma)_{\alpha, \ i, \ \beta, \ \gamma, \ \lambda}, \quad 1 \leq i < n, \\ &0 \leq |\beta| \leq l, \ 1 \leq |\alpha|, \ |\gamma| \leq l, \ 0 \leq |\lambda| \leq l-1 \end{aligned}$$

on $\mathcal{K}_{\overline{\mathcal{A}},l} \times J_l$, it satisfies the following conditions: (1) $c_0 \neq 0$; (2) $\text{Det}(B) \neq 0$ where $B = (B_{ij})$ with $B_{ij} = (\overline{\varphi}_i)_{x_j}(0) = b_{i,e_j}$ where e_j is the $j-$th canonical basis vector in $\mathbb{Z}^n$ and the coordinates $b_{n,e_j} = W_{e_j}(g_\lambda), \ 0 \leq |\lambda| \leq l-1, \ 1 \leq j \leq n$; (3) $g_0 \neq 0$. Using in the same manner the coordinates $\left(a_\delta^{'} \right)_{|\delta|=0,\cdots,l}$ on the target space, we define the action by polynomial maps

$$a_\delta^{'} = F_\delta \left(a_\alpha, \ b_{i,\beta}, \ g_\lambda, \ c_\gamma \right).$$

It is important to note that the inverse of this action is given by the rational maps

$$a_\alpha = \frac{G_\delta \left(a_\delta^{'}, \ b_{i,\beta}, \ g_\lambda, \ c_\gamma \right)}{H_\delta \left(a_\delta^{'}, \ b_{i,\beta}, \ g_\lambda, \ c_\gamma \right)}.$$

The reason for this is that we can solve the a_α step by step starting with Cramer's rule. This property ensures the extension of the field of rational functions induced by the action of Φ_l. We have

$$K(J_l) = K(a_\delta^{'}) \subset K(\mathcal{K}_{\overline{\mathcal{A}},l} \times J_l) = K(a_\alpha,\ b_{i,\beta},\ g_\lambda,\ c_\gamma)$$
$$= K(a_\delta^{'},\ b_{i,\beta},\ g_\lambda,\ c_\gamma) = K(J_l)(\ b_{i,\beta},\ g_\lambda,\ c_\gamma).$$

The $b_{i,\beta},\ g_\lambda$ and c_γ are algebraically independent over $K(a_\alpha)$. Comparing transcendence degrees they must be also algebraically independent over $K(J_l)$. Thus $K(\mathcal{K}_{\overline{\mathcal{A}},l} \times J_l)$ is a purely transcendental extension of $K(J_l)$, and it is a separably generated extension in the sense of [8, p.27]. Hence $\mathcal{K}_{\overline{\mathcal{A}},l}$ operates separably on J_l.

Let $F : \mathcal{R}_{\mathcal{A}} \to \{\overline{\varphi} |\ \overline{\varphi} \in \mathcal{R} \text{ and } \overline{\varphi}(\langle x_n \rangle) = \langle x_n \rangle\},\ \varphi \mapsto \psi^{-1} \circ \varphi \circ \psi$. Then F from $\mathcal{R}_{\mathcal{A}}$ to $\mathcal{R}_{\overline{\mathcal{A}}}$ is one-to-one and onto. So $K(\mathcal{K}_{\mathcal{A},l} \times J_l)$ is a separably generated extension of $K(J_l)$.

Now we can prove the following proposition.

Proposition 4.4. *Let $f \in K[[\mathbf{x}]]$. The tangent space to the orbit of $jet_l(f)$ under the actions of $\mathcal{R}_{\mathcal{A},l}$ and $\mathcal{K}_{\mathcal{A},l}$ considered as subspaces of J_l are, respectively,*

$$T_{jet_l(f)}\left(\mathcal{R}_{\mathcal{A},l} \cdot jet_l(f)\right) = \left(j_A(f) + \mathcal{M}^{l+1}\right) / \mathcal{M}^{l+1}$$

and

$$T_{jet_l(f)}\left(\mathcal{K}_{\mathcal{A},l} \cdot jet_l(f)\right) = \left(tj_A(f) + \mathcal{M}^{l+1}\right) / \mathcal{M}^{l+1}.$$

Proof. We note that the action of $G = \mathcal{R}_{\mathcal{A},l}$ or $G = \mathcal{K}_{\mathcal{A},l}$ on J_l induces a surjective separable morphism $G \to G \cdot \mathrm{jet}_l(f)$ of smooth varieties. The proof is similar to the first part of the proof of Proposition 3.6.

We give only the proof in the case $G = \mathcal{K}_{\mathcal{A},l}$ since the proof of $\mathcal{R}_{\mathcal{A},l}$ is completely similar to the case of $\mathcal{K}_{\mathcal{A},l}$.

Now we compute the tangent space $T_{\mathrm{jet}_l(f)}\left(\mathcal{K}_{\mathcal{A},l} \cdot \mathrm{jet}_l(f)\right)$.

Let ψ be the map $\psi:\ K[[\mathbf{x}]] \to K[[\mathbf{x}]],\ x_i \mapsto x_i, (1 \le i \le n-1), x_n \mapsto h$. By Lemma 2.1, ψ is an isomorphism. The tangent space to $\mathcal{K}_{\mathcal{A},l}$ at $(1,\ id)$ can be described via the local K-algebra homomorphisms from the local ring of $\mathcal{K}_{\mathcal{A},l}$ at $(1,\ id)$ to $K[[t]]/\langle t^2 \rangle$. A tangent vector of $\mathcal{K}_{\mathcal{A},l}$ at $(1,\ id)$ can be represented by the residue class modulo $\mathcal{M}^{l+1}$ of a tuple $(1 + t \cdot a,\ id + t \cdot \varphi^*)$ with $a \in K[[\mathbf{x}]]$ and $\varphi^* = (\varphi_1^*, \varphi_2^*. \cdots, \varphi_n^*)$ where $\varphi_i^* \in \mathcal{M},\ i = 1, \ldots, n$. This means in particular that $t \in \mathcal{K}[[t]]/\langle t^2 \rangle$, i.e., $t^2 = 0$

If $(1+t\cdot a,\ id+t\cdot\varphi^*)$ is a tangent vector of $\mathcal{K}_{\mathcal{A},l}$ at $(1,\ id)$, then

$$\delta=\sum_{i=1}^{n}\varphi_i^*\frac{\partial}{\partial x_i}$$

is a derivation that satisfies $\delta(h)\subseteq\langle h\rangle$. Thus there exists a power series $g\in K[[x_1,\dots,x_n]]$ such that

$$g\cdot h=\delta(h)=\sum_{i=1}^{n}\varphi_i^*\frac{\partial h}{\partial x_i}.$$

This implies that

$$\varphi_n^*=\frac{1}{h_{x_n}}\cdot\left(g\cdot h-\sum_{i=1}^{n-1}\varphi_i^*\cdot h_{x_i}\right).$$

Plugging this into the definition of δ we get

$$\delta=\frac{1}{h_{x_n}}\cdot\left(\sum_{i=1}^{n-1}\varphi_i^*\cdot(h_{x_n}\cdot\frac{\partial}{\partial x_i}-h_{x_i}\cdot\frac{\partial}{\partial x_n})+h\cdot g\cdot\frac{\partial}{\partial x_n}\right). \tag{4.1}$$

Applying this to f we find that

$$\delta(f)\in j_{\mathcal{A}}(f),$$

since $\varphi_i^*\in\mathcal{M}$ for $i=1,\dots n-1$ and $g\cdot h\in\mathcal{A}$. Then we have

$$(1+ta)\cdot f(\mathbf{x}+t\varphi^*)=f+t\cdot(af+\delta(f))$$

and

$$af+\delta(f)\in tj_{\mathcal{A}}(f).$$

Thus (4.1) implies that:

$$T_{\mathrm{jet}_l(f)}\left(\mathcal{K}_{\mathcal{A},l}\cdot\mathrm{jet}_l(f)\right)=\left(tj_A(f)+\mathcal{M}^{l+1}\right)/\mathcal{M}^{l+1}.$$

□

Now we prove Theorem 4.3.

Proof. We only prove the $K_{\mathcal{A},k+1}-$determinacy since the other case is completely analogous. If f is k-$K_{\mathcal{A},k+1}-$determined and $g\in\mathcal{M}^{k+1}$, then for any $t\in K$ the $(k+1)-$jet $\mathrm{jet}_{k+1}(f)+t\cdot\mathrm{jet}_{k+1}(g)$ is in the orbit of $\mathrm{jet}_{k+1}(f)$ under $K_{\mathcal{A},k+1}$. So

$$\mathrm{jet}_{k+1}(g)\in T_{\mathrm{jet}_{k+1}(f)}\left(\mathcal{K}_{\mathcal{A},k+1}\cdot\mathrm{jet}_{k+1}(f)\right)=\left(tj_{\mathcal{A}}(f)+\mathcal{M}^{k+2}\right)/\mathcal{M}^{k+2}.$$

This implies that $g \in tj_{\mathcal{A}}(f)+\mathcal{M}^{k+2}$, and hence $\mathcal{M}^{k+1} \subseteq tj_{\mathcal{A}}(f)+\mathcal{M}^{k+2}$. By Nakayama's Lemma we get $\mathcal{M}^{k+1} \subseteq tj_{\mathcal{A}}(f)$. □

Theorem 4.5. *Let* $0 \neq f \in \mathcal{M}^2$ *and* $k \in \mathbf{N}$

(a) If $\mathcal{M}^{k+2} \subseteq \mathcal{M}\cdot j_{\mathcal{A}}(f)$, *then* f *is* $(2k-ord(f)+2)-\mathcal{R}_{\mathcal{A}}-$*determined.*

(b) If $\mathcal{M}^{k+2} \subseteq \mathcal{M}\cdot tj_{\mathcal{A}}(f)$, *then* f *is* $(2k-ord(f)+2)-\mathcal{K}_{\mathcal{A}}-$*determined.*

Proof. We first prove (b). Let $o=$ord(f). By assumption and the fact that ord$(f_{x_i}) \geq o-1$ for $i=1,\dots,n$, we have $\mathcal{M}^{k+2} \subseteq \mathcal{M}\cdot tj_{\mathcal{A}}(f) \subseteq \mathcal{M}^{o+1}$. This implies that $k \geq o-1$.

Set $N=2k-o+2 \geq k+1$, and take a $g \in K[[\mathbf{x}]]$ such that $g-f \in \mathcal{M}^{N+1}$, i.e., f and g have the same N-jet. The key point of the proof is to show that f and g are contact hypersurface equivalent, i.e., there are an automorphism $\varphi \in \mathcal{R}_{\mathcal{A}}$ and a unit $u \in K[[\mathbf{x}]]^*$ such that $g=u\cdot\varphi(f)$.

In order to construct φ and u, we must use Lemma 2.2 and consider the following three cases:

(1): $h \in x_nK[[x_1,\dots,x_{n-1}]]$;

(2): $h=x_n+h_1(x_1,\dots,x_{n-1})$;

(3): $h=H_1(x_1,\dots,x_n)\cdot x_n+h_1(x_1,\dots,x_{n-1})$, where $H_1 \in K[[\mathbf{x}]]$.

Case (1): Let $h \in x_nK[[x_1,\dots,x_{n-1}]]$. Then there exits $H \in K[[\mathbf{x}]]$ such that $h=H(\mathbf{x})\cdot x_n$.

Set $Q=N-k \geq 1$, by assumption

$$\begin{aligned} g-f \in \mathcal{M}^{N+1} =& \mathcal{M}^{Q-1}\cdot\mathcal{M}^{k+2} \subseteq \mathcal{M}^{Q}\cdot\langle f\rangle+\mathcal{M}^{Q}\cdot j_{\mathcal{A}}(f) \\ =& \mathcal{M}^{Q}\cdot\langle f\rangle+\mathcal{M}^{Q}\cdot\mathcal{A}\cdot\langle f_{x_n}\rangle+ \\ &+\mathcal{M}^{Q+1}\cdot\left\langle\left\{h_{x_n}\cdot f_{x_j}-h_{x_j}\cdot f_{x_n};\ 1\leq j<n\right\}\right\rangle. \end{aligned}$$

Thus there exist $a_{1,0} \in \mathcal{M}^{Q}$, $a_{1,j} \in \mathcal{M}^{Q+1}$, $1 \leq j < n$ and $a_{1,n} \in \mathcal{M}^{Q}\cdot\mathcal{A} \subset \mathcal{M}^{Q+1}$ such that

$$\begin{aligned} g-f &= a_{1,0}f+\sum_{1\leq j<n}a_{1,j}\left(h_{x_n}\cdot f_{x_j}-h_{x_j}\cdot f_{x_n}\right)+a_{1,n}f_{x_n} \\ &= a_{1,0}f+\sum_{j=1}^{n-1}\left(a_{1,j}h_{x_n}\right)f_{x_j}-\sum_{j=1}^{n-1}\left(a_{1,j}h_{x_j}\right)f_{x_n}+a_{1,n}f_{x_n}. \end{aligned} \tag{4.2}$$

Let $b_{1,0} \doteq a_{1,0}$, $b_{1,j} \doteq a_{1,j}h_{x_n}$, $j=1,\dots,n-1$, $b_{1,n} \doteq -\sum_{j=1}^{n-1}\left(a_{1,j}h_{x_j}\right)+a_{1,n}$, then

$$g-f=b_{1,0}\cdot f+\sum_{j=1}^{n}b_{1,j}\cdot f_{x_j}.$$

Now define $v_1 = 1 + b_{1,0} \in K[[\mathbf{x}]]^*$ and $\phi_1 : K[[\mathbf{x}]] \to K[[\mathbf{x}]] : x_j \mapsto x_j + b_{1,j} = x_j + a_{1,j}h_{x_n},\ (j = 1,\dots,n-1), x_n \mapsto x_n + b_{1,n} = x_n - \sum_{j=1}^{n-1}\left(a_{1,j}h_{x_j}\right) + a_{1,n}$. We want to show that

$$g - v_1 \cdot \phi_1(f) \in \mathcal{M}^{N+2}. \tag{4.3}$$

If the formula (4.3) is true, we can replace f in the above argument by $v_1 \cdot \phi_1(f)$ and go on inductively.

For $f = \sum_{|\beta|\geq 0} k_\beta \cdot \mathbf{x}^\beta$, we have (3.2). Applying ϕ_1 to f amounts to substituting z_j by $a_{1,j}\frac{\partial h}{\partial x_n}$, $j = 1,\dots,n-1$, and z_n by $\left(-\sum_{j=1}^{n-1} a_{1,j}h_{x_j}\right) + a_{1,n}$ in (3.2). Thus we find that

$$\phi_1(f) = f + \sum_{i=1}^{n-1} f_{x_i}\cdot(a_{1,i}h_{x_n}) + f_{x_n}\cdot\left(-\sum_{j=1}^{n-1} a_{1,j}h_{x_j} + a_{1,n}\right) + r$$

where

$$r = \sum_{|\alpha|\geq 2} w_\alpha \cdot (a_{1,1}h_{x_n})^{\alpha_1}(a_{1,2}h_{x_n})^{\alpha_2}\cdots(-\sum_{j=1}^{n-1} a_{1,j}h_{x_j} + a_{1,n})^{\alpha_n}.$$

Since $h_{x_n}(0) \neq 0$ we obtain

$$\begin{aligned} \mathrm{ord}\,(r) \geq & \mathrm{ord}(w_\alpha) + \sum_{i=1}^{n-1} \mathrm{ord}(a_{1,i}h_{x_n})\cdot\alpha_i \\ & + \mathrm{ord}(-\sum_{j=1}^{n-1} a_{1,j}h_{x_j} + a_{1,n})\cdot\alpha_n \\ & \geq o - \mid \alpha \mid + (Q+1)\cdot \mid \alpha \mid \\ & \geq o + 2\cdot Q = N+2, \quad r \in \mathcal{M}^{N+2}. \end{aligned}$$

Multiplying $\phi_1(f)$ by $v_1 = 1 + a_{1,0}$ and using (4.2) we get $g - v_1 \cdot \phi_1(f) = -\left(\sum_{i=1}^{n-1} f_{x_i}\cdot(a_{1,i}h_{x_n}) + f_{x_n}\cdot(-\sum_{j=1}^{n-1} a_{1,j}h_{x_j} + a_{1,n})\right)\cdot a_{1,0} - (1 + a_{1,0})r$.

Since $\mathrm{ord}\,[a_{1,0}\cdot(a_{1,i}h_{x_n})\cdot f_{x_i}] \geq Q + (Q+1) + (o-1) = N+2$ and $\mathrm{ord}\left[a_{1,0}\cdot(-\sum_{j=1}^{n} a_{1,j}h_{x_j} + a_{1,n})\cdot f_{x_n}\right] \geq Q + (Q+1) + (o-1) = N+2$, we have $g - v_1\cdot\phi_1(f) \in \mathcal{M}^{N+2}$. This proves (4.3).

Now, we prove $\phi_1(\langle h\rangle) = \langle h\rangle$.

We take a map $\psi :\ K[[\mathbf{x}]] \to K[[\mathbf{x}]],\quad x_i \mapsto x_i, (1 \leq i \leq n-1), x_n \mapsto h$. By Lemma 2.2, ψ is an isomorphism and ψ is the identity on $K[[x_1,\dots,x_{n-1}]]$. Because $K[[x_1,\dots,x_n]] = K[[x_1,\dots,x_{n-1}]][[x_n]]$ and the elements of $K[[x_1,\dots,x_n]]$ which are not in $\langle x_n\rangle$ are those with nonzero term in $K[[x_1,\dots,x_{n-1}]]$, ψ preserves this subset. Since ψ is an isomorphism, it follows that $\psi(\langle x_n\rangle) = \langle x_n\rangle$. In particular, the image $\psi(x_n) = h$ of the generator x_n of $\langle x_n\rangle$ is a generator of $\langle x_n\rangle$. We have $\langle x_n\rangle = \langle h\rangle$.

For any $g = g_n(x_1,\dots,x_n)x_n \in \langle x_n\rangle$,

$$\begin{aligned}\phi_1(g) = \phi_1(g_n)\cdot\phi_1(x_n) &= \phi_1(g_n)\cdot\left(x_n - \sum_{j=1}^{n-1}(a_{1,j}h_{x_j}) + a_{1,n}\right)\\ &= \phi_1(g_n)x_n - \phi_1(g_n)\cdot\left(\sum_{j=1}^{n-1}a_{1,j}h_{x_j}\right) + \phi_1(g_n)\cdot a_{1,n}.\end{aligned}$$

From the fact that $h_{x_j} = (H(x_1,\dots,x_n)\cdot x_n)_{x_j} = H_{x_j}\cdot x_n$, $j = 1,\dots,n-1$, $a_{1,n}\in\mathcal{M}^Q\cdot\mathcal{A}\subseteq\mathcal{A}$ and $\mathcal{A} = \langle x_n\rangle$, we obtain $\phi_1(g)\in\mathcal{A}$.

Therefore,

$$\phi_1(\langle h\rangle) = \phi_1(\langle x_n\rangle) = \langle x_n\rangle = \langle h\rangle. \tag{4.4}$$

Consequently, we can proceed inductively to construct sequences $\{b_{p,0}\}_{p\geq 1}$, and $\{b_{p,i}\}_{p\geq 1}$ for $i = 1,\dots,n$ with $b_{p,0}\in\mathcal{M}^{Q+p-1}$ and $b_{p,i}\in\mathcal{M}^{Q+p}$ for $i = 1,\dots,n$. By induction and Lemma 2.2, the generalizations of (4.3) and (4.4) hold, i.e. $g - u_p\cdot\varphi_p(f)\in\mathcal{M}^{N+1+p}$ and $\varphi_p(\langle h\rangle) = \langle h\rangle$. Again from Lemma 2.2, we obtain an automorphisms $(u,\varphi)\in\mathcal{K}_{\mathcal{A}}$ such that $g = u\cdot\varphi(f)$.

Case (2): Suppose $h = x_n + h_1(x_1,\dots,x_{n-1})$.

Because $\psi:\ K[[\mathbf{x}]]\to K[[\mathbf{x}]],\quad x_i\mapsto x_i, (1\leq i\leq n-1), x_n\mapsto h$ is an isomorphism, there is an inverse map $\psi^{-1}:\ K[[\mathbf{x}]]\to K[[\mathbf{x}]], x_i\mapsto x_i, x_n\mapsto x_n - h_1(x_1,\dots,x_{n-1})$.

Now let $Q = N - k\geq 1$, by assumption

$$\begin{aligned}g - f\in\mathcal{M}^{N+1} =& \mathcal{M}^Q\cdot\langle f\rangle + \mathcal{M}^Q\cdot\mathcal{A}\cdot\langle f_{x_n}\rangle +\\ &+\mathcal{M}^{Q+1}\cdot\langle\{f_{x_i}\cdot h_{x_n} - f_{x_n}\cdot h_{x_i};\ 1\leq i\leq n-1\}\rangle.\end{aligned}$$

There exist $a_{1,0} \in \mathcal{M}^Q,\ a_{1,i} \in \mathcal{M}^{Q+1}$, and $a_{1,n} \in \mathcal{M}^Q \cdot \mathcal{A} \subset \mathcal{M}^{Q+1},\ 1 \leq i \leq n-1$ such that

$$g - f = a_{1,0} \cdot f + \sum_{1\leq i\leq n-1} a_{1,i}\left(h_{x_i} \cdot f_{x_n} - h_{x_n} \cdot f_{x_i}\right) + a_{1,n} \cdot f_{x_n}$$
$$= a_{1,0} \cdot f + \sum_{i=1}^{n-1} a_{1,i} \cdot \left((h_1)_{x_i} \cdot f_{x_n} - f_{x_i}\right) + a_{1,n} \cdot f_{x_n},$$

where $h_{x_n} = 1$ and $h_{x_i} = (h_1)_{x_i}$. One easily deduces that

$$\psi^{-1}(g) - \psi^{-1}(f) = \psi^{-1}(a_{1,0}) \cdot \psi^{-1}(f) +$$
$$+ \sum_{i=1}^{n-1} \psi^{-1}(a_{1,i}) \cdot \left[\psi^{-1}\left((h_1)_{x_i}\right) \cdot \psi^{-1}(f_{x_n}) - \psi^{-1}\left(f_{x_i}\right)\right]$$
$$+ \psi^{-1}\left(a_{1,n}\right) \cdot \psi^{-1}(f_{x_n}) = \psi^{-1}(a_{1,0}) \cdot \psi^{-1}(f) -$$
$$- \sum_{i=1}^{n-1} \psi^{-1}(a_{1,i}) \cdot [(-(h_1)_{x_i}) \cdot f_{x_n}(x_1, \ldots,\ x_{n-1},\ x_n - h_1)$$
$$+ f_{x_i}(x_1, \ldots,\ x_{n-1},\ x_n - h_1)] +$$
$$+ \psi^{-1}(a_{1,n}) \cdot f_{x_n}(x_1, \ldots,\ x_{n-1},\ x_n - h_1) = \psi^{-1}(a_{1,0}) \cdot \psi^{-1}(f) -$$
$$- \sum_{i=1}^{n-1} \psi^{-1}(a_{1,i}) \cdot \left(\psi^{-1}(f_{x_i}) - h_{x_i} \cdot \psi^{-1}(f_{x_n})\right) + \psi^{-1}(a_{1,n}) \cdot \psi^{-1}(f_{x_n}),$$

i.e.,

$$\psi^{-1}(g) - \psi^{-1}(f) = \psi^{-1}(a_{1,0}) \cdot \psi^{-1}(f) - \tag{4.5}$$
$$- \textstyle\sum_{i=1}^{n-1} \psi^{-1}(a_{1,i}) \cdot \left(\psi^{-1}(f_{x_i}) - h_{x_i} \cdot \psi^{-1}(f_{x_n})\right) + \psi^{-1}(a_{1,n}) \cdot \psi^{-1}(f_{x_n}).$$

Let $b_{1,0} \doteq \psi^{-1}\left(a_{1,0}\right),\ b_{1,i} \doteq -\psi^{-1}\left(a_{1,i}\right),\ b_{1,n} \doteq \psi^{-1}\left(a_{1,n}\right)$, then

$$\psi^{-1}(g) - \psi^{-1}(f) = b_{1,0} \cdot \psi^{-1}(f) + \sum_{i=1}^{n-1} b_{1,i} \cdot \left(\psi^{-1}(f)\right)_{x_i}$$
$$+ b_{1,n} \cdot \left(\psi^{-1}(f)\right)_{x_n},$$

where $b_{1,0} = \psi^{-1}\left(a_{1,0}\right) \in \mathcal{M}^Q,\ b_{1,i} = -\psi^{-1}\left(a_{1,i}\right) \in \mathcal{M}^{Q+1},\ (i = 1, \ldots,\ n-1)$, and $b_{1,n} = \psi^{-1}\left(a_{1,n}\right) \in \mathcal{M}^Q \cdot \langle x_n \rangle$.

Therefore, we have

$$\psi^{-1}(g) - \psi^{-1}(f) \in \psi^{-1}\left(\mathcal{M}^{N+1}\right) = \mathcal{M}^{N+1}$$

and

$$\psi^{-1}(g)-\psi^{-1}(f)\in\mathcal{M}^{Q}\cdot\left\langle\psi^{-1}(f)\right\rangle+\mathcal{M}^{Q}\cdot\langle x_n\rangle\cdot\left\langle\psi^{-1}(f)_{x_n}\right\rangle + \mathcal{M}^{Q+1}\cdot\left\langle\psi^{-1}(f)_{x_1},\ \ldots,\ \psi^{-1}(f)_{x_{n-1}}\right\rangle.$$

Let $\widetilde{v_1}=1+b_{1,0}=1+\psi^{-1}(a_{1,0})\in K[[\mathbf{x}]]^*$ and $\widetilde{\phi_1}:K[[\mathbf{x}]]\to K[[\mathbf{x}]]: x_i\mapsto x_i+b_{1,i}=x_i-\psi^{-1}(a_{1,i}),\ (i=1,\ldots,n-1),\ x_n\mapsto x_n+b_{1,n}=x_n+\psi^{-1}(a_{1,n})$, where $a_{1,i}\in\mathcal{M}^{Q+1}$ and $a_{1,n}\in\mathcal{M}^{Q}\cdot\mathcal{A}$.

We want to show that

$$\psi^{-1}(g)-\widetilde{v_1}\cdot\widetilde{\phi_1}\left(\psi^{-1}(f)\right)\in\mathcal{M}^{N+2} \tag{4.6}$$

and

$$\psi^{-1}(g)-\widetilde{v_1}\cdot\widetilde{\phi_1}\left(\psi^{-1}(f)\right)\in\mathcal{M}^{Q+1}\cdot\left\langle\psi^{-1}(f)\right\rangle+ \mathcal{M}^{Q+1}\cdot\langle x_n\rangle\cdot\left\langle\left(\psi^{-1}(f)\right)_{x_n}\right\rangle+ \mathcal{M}^{Q+2}\cdot\left\langle\left(\psi^{-1}(f)\right)_{x_1},\ \ldots,\ \left(\psi^{-1}(f)\right)_{x_{n-1}}\right\rangle. \tag{4.7}$$

In fact, for $\psi^{-1}(f)=\sum_{|\beta|\geq 0}l_\beta\cdot\mathbf{x}^\beta$,

$$\psi^{-1}(f)\left((x_1+z_1),\ldots,(x_n+z_n)\right) = \sum_{|\beta|\geq 0}l_\beta\cdot\sum_{\gamma_1=0}^{\beta_1}\cdots\sum_{\gamma_n=0}^{\beta_n}d_{\beta,\gamma}\mathbf{x}^{\beta-\gamma}\cdot\mathbf{z}^\gamma=\sum_{\alpha\in\mathrm{N}^n}u_\alpha\cdot\mathbf{z}^\alpha, \tag{4.8}$$

where $u_\alpha=\sum_{|\beta|\geq 0,\ \beta\geq\alpha}l_\beta\cdot d_{\beta,\alpha}\cdot\mathbf{x}^{\beta-\alpha}$, it follows that $\mathrm{ord}(u_\alpha)\geq o-|\alpha|$.

Applying $\widetilde{\phi_1}$ to $\psi^{-1}(f)$ amounts to substituting z_j by $-\psi^{-1}(a_{1,j}),\ j=1,\ldots,n-1$, and z_n by $\psi^{-1}(a_{1,n})$ in (4.8) so we get

$$\widetilde{\phi_1}\left(\psi^{-1}(f)\right)=\psi^{-1}(f)+\sum_{i=1}^{n-1}\left[\psi^{-1}(f_{x_i})-\psi^{-1}(f_{x_n})\cdot(h_1)_{x_i}\right]\cdot \cdot\left(-\psi^{-1}(a_{1,i})\right)+\psi^{-1}(a_{1,n})\cdot\psi^{-1}(f_{x_n})+R,$$

where

$$R=\sum_{|\alpha|\geq 2}d_\alpha\cdot\left(-\psi^{-1}(a_{1,1})\right)^{\alpha_1}\cdots\left(-\psi^{-1}(a_{1,n-1})\right)^{\alpha_{n-1}}\cdot\left(\psi^{-1}(a_{1,n})\right)^{\alpha_n}.$$

Multiplying $\widetilde{\phi_1}(f)$ by $\widetilde{v_1} = 1 + \psi^{-1}(a_{1,0})$ and using (4.5) we get

$$\begin{aligned}
&\psi^{-1}(g) - \widetilde{v_1} \cdot \widetilde{\phi_1}\left(\psi^{-1}((f)\right) \\
=&\psi^{-1}(g) - \left(1 + \psi^{-1}(a_{1,0})\right) \cdot \\
&[\psi^{-1}(f) + \sum_{i=1}^{n-1} \left(\psi^{-1}(f_{x_i}) - \psi^{-1}(f_{x_n}) \cdot h_{x_i}\right) \cdot \left(-\psi^{-1}(a_{1,i})\right) \\
&+ \psi^{-1}(a_{1,n}) \cdot \psi^{-1}(f_{x_n}) + R] \\
=&\sum_{i=1}^{n-1} \left[\psi^{-1}(f_{x_i}) - \psi^{-1}(f_{x_n}) \cdot (h_1)_{x_i}\right] \cdot \left(-\psi^{-1}(a_{1,i})\right) \cdot \psi^{-1}(a_{1,0}) \\
&+ \psi^{-1}(a_{1,n}) \cdot \psi^{-1}(f_{x_n}) \cdot \psi^{-1}(a_{1,0}) + \left(1 + \psi^{-1}(a_{1,0})\right) \cdot R
\end{aligned}$$

Because $\text{ord}(h_1) \geq 1$ and $\text{ord}\left(\psi^{-1}(f_{x_i})\right) \geq o - 1, (i = 1, \cdots, n-1)$,

$$\begin{aligned}
&\text{ord}\left(\psi^{-1}(f_{x_i}) \cdot (-\psi^{-1}(a_{1,i})) \cdot \psi^{-1}(a_{1,0})\right) \\
\geq &o - 1 + (Q+1) + Q = N + 2,\ (i = 1, \ldots, n-1), \\
&\text{ord}\left(\psi^{-1}(f_{x_n}) \cdot (h_{x_i}) \cdot \left(-\psi^{-1}(a_{1,i})\right) \cdot \psi^{-1}(a_{1,0})\right) \\
\geq &o - 1 + (Q+1) + Q = N + 2, (i = 1, \ldots, n-1),
\end{aligned}$$

$$\text{ord}\left(\psi^{-1}(a_{1,n}) \cdot \psi^{-1}(f_{x_n}) \cdot \psi^{-1}(a_{1,0})\right) \geq N + 2$$

and

$$\begin{aligned}
\text{ord}(R) &= \text{ord}(d_\alpha) + \sum_{i=1}^{n} \text{ord}\left(\psi^{-1}(a_{1,i})\right) \cdot \alpha_i \\
&\geq o - \mid \alpha \mid + (Q+1) \cdot \mid \alpha \mid \geq N + 2,
\end{aligned}$$

so $R \in \mathcal{M}^{N+2}$ and

$$\psi^{-1}(g) - \widetilde{v_1} \cdot \widetilde{\phi_1}\left(\psi^{-1}((f)\right) \in \mathcal{M}^{N+2}.$$

Hence we have proved (4.6).

Moreover, we have

$$\widetilde{\phi_1}(x_n) = (\widetilde{\phi_1})_n = x_n + \psi^{-1}(a_{1,n}) \in \langle x_n \rangle.$$

Again by applying ψ to (4.6), we get

$$\psi\left(\psi^{-1}(g)\right) - \psi(\widetilde{v_1}) \cdot \psi\left(\widetilde{\phi_1}\left(\psi^{-1}((f)\right)\right) \in \psi(\mathcal{M}^{N+2}) = \mathcal{M}^{N+2},$$

i.e.

$$g - \psi\left(\widetilde{v_1}\right) \cdot \psi \circ \widetilde{\phi_1} \circ \psi^{-1}(f) \in \mathcal{M}^{N+2}.$$

Moreover

$$\psi \circ \widetilde{\phi_1} \circ \psi^{-1}(h) = \psi\left[\widetilde{\phi_1}(x_n)\right] = \psi\left[\widetilde{\phi_1}(x_n)\right] \in \psi\left(\langle x_n \rangle\right) = \langle h \rangle.$$

Consequently, let $\phi_1 = \psi \circ \widetilde{\phi_1} \circ \psi^{-1}$ and $v_1 = \psi\left(\widetilde{v_1}\right)$, then

$$g - v_1 \cdot \phi_1(f) \in \mathcal{M}^{N+2}.$$

Since by assumption

$$\begin{aligned}\mathcal{M}^{N+2} =& \mathcal{M}^Q \cdot \langle f \rangle + \mathcal{M}^Q \cdot \mathcal{A} \cdot \langle f_{x_n} \rangle \\ &+ \mathcal{M}^{Q+1} \cdot \langle \{ f_{x_i} \cdot h_{x_n} - f_{x_n} \cdot h_{x_i} |\ 1 \le i \le n-1 \} \rangle,\end{aligned}$$

there exist $d_{1,0} \in \mathcal{M}^Q,\ d_{1,i} \in \mathcal{M}^{Q+1}$, and $d_{1,n} \in \mathcal{M}^Q \cdot \mathcal{A} \subset \mathcal{M}^{Q+1}$, ($1 \le i \le n-1$) such that

$$\begin{aligned}g - v_1 \cdot \phi_1(f) &= d_{1,0} \cdot f + \sum_{1 \le j < n} d_{1,j} \cdot (h_{x_j} \cdot f_{x_n} - h_{x_n} \cdot f_{x_j}) + d_{1,n} \cdot f_{x_n} \\ &= d_{1,0} \cdot f + \sum_{j=1}^{n-1} d_{1,j} \cdot \left((h_1)_{x_j} \cdot f_{x_n} - f_{x_j}\right) + d_{1,n} \cdot f_{x_n}.\end{aligned}$$

The proof of the following formula is similar to that of (4.5):

$$\begin{aligned}\psi^{-1}(g) - \psi^{-1}\left(v_1 \cdot \phi_1(f)\right) \in& \mathcal{M}^{Q+1}\left\langle \psi^{-1}(f) \right\rangle + \mathcal{M}^{Q+1} \langle x_n \rangle \left\langle \left(\psi^{-1}(f)\right)_{x_n} \right\rangle \\ &+ \mathcal{M}^{Q+2}\left\langle \left(\psi^{-1}(f)\right)_{x_1},\ \ldots,\ \left(\psi^{-1}(f)\right)_{x_{n-1}} \right\rangle.\end{aligned}$$

Because

$$\psi^{-1}(g) - \psi^{-1}\left(v_1 \cdot \phi_1(f)\right) = \psi^{-1}(g) - \widetilde{v_1} \cdot \widetilde{\phi_1}\left(\psi^{-1}(f)\right),$$

we have proved (4.7).

Now we can proceed inductively to construct sequences $b_{p,0} \doteq \{\psi^{-1}(a_{p,0})\}_{p \ge 1}$, and $b_{p,i} \doteq \left\{\psi^{-1}(a_{p,i})\right\}_{p \ge 1}$ for $i = 1, \ldots, n$, with $b_{p,0} \in \mathcal{M}^{Q+p-1}$, $b_{p,i} \in \mathcal{M}^{Q+p}$ for $i = 1, \ldots, n-1$, and $b_{p,n} \in \mathcal{M}^{Q+p-1} \cdot \langle x_n \rangle$.

By induction and Lemma 2.2, we can generalize (4.6) as:

$$\psi^{-1}(g) - \widetilde{u_p} \cdot \widetilde{\varphi_p}\left(\psi^{-1}(f)\right) \in \mathcal{M}^{N+1+p}.$$

In the same way we also generalize (4.7) as:

$$\psi^{-1}(g) - \widetilde{u_p} \cdot \widetilde{\varphi_p}\left(\psi^{-1}(f)\right) \in \mathcal{M}^{Q+p} \cdot \left\langle \psi^{-1}(f) \right\rangle +$$
$$\mathcal{M}^{Q+p}\langle x_n \rangle \cdot \left\langle \left(\psi^{-1}(f)\right)_{x_n} \right\rangle + \mathcal{M}^{Q+p+1} \left\langle \left(\psi^{-1}(f)\right)_{x_1}, \ldots, \left(\psi^{-1}(f)\right)_{x_{n-1}} \right\rangle.$$

Meanwhile we have $\widetilde{\varphi_p}\left(\langle x_n \rangle\right) = \langle x_n \rangle$. Again by Lemma 2.2, we obtain $(\widetilde{u},\ \widetilde{\varphi}) \in \mathcal{K}$ such that

$$\psi^{-1}(g) = \widetilde{u} \cdot \widetilde{\varphi}\left(\psi^{-1}(f)\right), \text{ and } \widetilde{\varphi}\left(\langle x_n \rangle\right) = \langle x_n \rangle.$$

Therefore,

$$g = \psi(\widetilde{u}) \cdot \psi\left(\widetilde{\varphi}\left(\psi^{-1}(f)\right)\right) = \psi(\widetilde{u}) \cdot (\psi \circ \widetilde{\varphi} \circ \psi^{-1})(f),$$

and

$$\psi \circ \widetilde{\varphi} \circ \psi^{-1}(h) = \psi\left(\widetilde{\varphi}\left(\psi^{-1}(h)\right)\right) = \psi\left(\widetilde{\varphi}(x_n)\right) \in \psi\left(\langle x_n \rangle\right) = \langle h \rangle.$$

Let $u = \psi(\widetilde{u})$ and $\varphi = \psi \circ \widetilde{\varphi} \circ \psi^{-1}$. Then we get $g = u \cdot \varphi(f)$ with $(u, \varphi) \in \mathcal{K}_{\mathcal{A}}$.

Case (3): Let $h = H_1(x_1, \ldots, x_n) \cdot x_n + h_1(x_1, \ldots, x_{n-1})$.

Combining the case (1) and the case (2), we get $(u, \varphi) \in \mathcal{K}_{\mathcal{A}}$ such that $g = u \cdot \varphi(f)$.

The proof for right equivalence goes along the same lines.

Let o =ord(f), the condition

$$\mathcal{M}^{k+2} \subseteq \mathcal{M} \cdot j_{\mathcal{A}}(f) \subseteq \mathcal{M}^{o+1}$$

implies that $k \geq o - 1$ and that for any g with

$$g - f \in \mathcal{M}^{N+1} = \mathcal{M}^{Q-1} \cdot \mathcal{M}^{k+2} \subseteq \mathcal{M}^{Q} \cdot j_{\mathcal{A}}(f),$$

where $N = 2k - o + 2 \geq k + 1$ and $Q = N - k \geq 1$, there are $a_{1,i} \in \mathcal{M}^{Q+1}$ with

$$\begin{aligned} g - f &= \sum_{i=1}^{n-1} a_{1,i}(h_{x_i} \cdot f_{x_n} - h_{x_n} \cdot f_{x_i}) + a_{1,n} f_{x_n} \\ &= \sum_{i=1}^{n-1} (-a_{1,i} h_{x_n}) \cdot f_{x_i} + \sum_{i=1}^{n-1} (a_{1,i} h_{x_i}) \cdot f_{x_n} + a_{1,n} \cdot f_{x_n}. \end{aligned}$$

We can then define ϕ_1 as above and see that

$$g - \phi_1(f) = r \in \mathcal{M}^{N+2}.$$

Going on by induction and applying Lemma 2.2, we get an automorphism $\varphi \in \mathcal{R}_{\mathcal{A}}$ such that $g = \varphi(f)$. $\square$

Corollary 4.6. *Let $0 \neq f \in \mathcal{M}^2 \subseteq K[[\mathbf{x}]]$.*

(1) If $\mu_{\mathcal{A}}(f) < \infty$, then f is $(2\mu_{\mathcal{A}}(f) - ord(f)) - \mathcal{R}_{\mathcal{A}}-$determined.

(2) If $\tau_{\mathcal{A}}(f) < \infty$, then f is $(2\tau_{\mathcal{A}}(f) - ord(f)) - \mathcal{K}_{\mathcal{A}}-$determined.

Combining Theorem 4.3 and Corollary 4.6, we obtain:

Theorem 4.7. *Let $0 \neq f \in \mathcal{M} \subset K[[\mathbf{x}]]$ be a power series.*

(1) f is a relative $\mathcal{A}-$isolated singularity if and only if f is finitely $\mathcal{R}_{\mathcal{A}}-$determined.

(2) $\mathcal{R}_f$ is a relative $\mathcal{A}-$isolated hypersurface singularity if and only if f is finitely $\mathcal{K}_{\mathcal{A}}-$determined.

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