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CHARACTERIZING THE MULTIPLICATIVE GROUP OF A REAL CLOSED FIELD IN TERMS OF ITS DIVISIBLE MAXIMAL SUBGROUP

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Abstract. Let F be a field and M be a maximal subgroup of the multiplicative group $F^* = F \setminus \{0\}$ of index p. It is proved that if M is divisible, then $Br(F)_p \neq 0$ if and only if $p = 2$ and F is Euclidean. Furthermore, it is shown that in this case F^* contains a divisible maximal subgroup if and only if F^* is isomorphic to the multiplicative group of a real closed field.

1. Introduction

Given the field of real numbers $\mathbb{R}$, denote by $\mathbb{R}^*$ and $\mathbb{R}^+$ the multiplicative group of real numbers and the multiplicative group of positive real numbers, respectively. We recall that a nontrivial multiplicative abelian group G is *divisible* if and only if G has no maximal subgroup if and only if $G = G^p$ for each prime p. It is easily seen that $\mathbb{R}^+$ is a divisible maximal subgroup of index 2 in $\mathbb{R}^*$ and $\mathbb{R}$ is Euclidean. The object of this note is to show that this property on the multiplicative group of a field F implies that $Br(F)_p \neq 0$ if and only if $p = 2$ and F is Euclidean, where $Br(F)_p$ is the p-primary component of the Brauer group of F. Furthermore, if R is a real closed field, then it is easily seen

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that (cf. Theorem A below) R^* contains a unique maximal subgroup which is divisible. Here, we also characterize the multiplicative group of a real closed field in terms of its divisible maximal subgroup. To be more precise, it is proved that F^* contains a divisible maximal subgroup if and only if F^* is isomorphic to the multiplicative group of a real closed field.

2. **Divisible maximal subgroups**

We begin our investigation with the following easy lemma.

Lemma 2.1. *Let G be a multiplicative abelian group and M be a maximal subgroup of G. If M is divisible, then M is the unique maximal subgroup of G.*

Proof. Assume that $M_1 \neq M$ is another maximal subgroup of G. Then, we have $G = MM_1$ and hence $G/M_1 \cong M/M \cap M_1$; i.e., $M \cap M_1$ is a maximal subgroup of M. Since M is divisible, then we conclude that $M \cap M_1 = 1$. Therefore, $G/M_1 \cong M \cong C_q$ for some prime number q, where C_q is the cyclic group of q elements. This last relation also leads to a contradiction since a finite group cannot be divisible, and so $M = M_1$ as required.

We shall also need the following theorem to prove our main result.

Theorem A ([1], p. 107)*If F is a real closed field, then $F^* \cong \mathbb{Z}_2 \times \mathbb{Q}^{|F|}$. Conversely, for any infinite cardinal λ, the group $\mathbb{Z}_2 \times \mathbb{Q}^{\lambda}$ is isomorphic to the multiplicative group of a suitable real closed field.*

Theorem 2.2. *Let M be a maximal subgroup of F^* of index p. Then, we have:*
(1) *If M is divisible, then $Br(F)_p \neq 0$ if and only if $p = 2$ and F is Euclidean.*
(2) *If F is Euclidean, then F^* contains a divisible maximal subgroup of index 2 if and only if F^* is isomorphic to the multiplicative group of a real closed field.*

Proof. (1) Assume that M is a maximal subgroup of F^* with $F^*/M \cong C_p$ for some prime p. We first claim that if p is odd, then there exists a unique cyclic field extension K/F of degree p such that $N(K^*) = F^*$, where N is the norm of K to F. To see this, we know, by Lemma 2.1 that M is the unique maximal subgroup of F^* such that $F^*/M \cong C_p$. Since M is divisible and maximal in F^*, then by (4.1.4) of [3], we have $F^* \cong M \times C_p$. This means that F contains a primitive p-th root of unity. Now, it is easily seen that there is an element $a \in F$ such that the equation $x^p - a = 0$ has no solutions in F. Since F has a primitive p-th root of unity, then we obtain a cyclic extension $K = F(b)$ of degree p over F with $b^p - a = 0$. By Kummer theory and the Prũfer-Baer Theorem (cf. [3], p. 105), the Galois group $Gal(K/F)$ is isomorphic with a subgroup of $F^*/F^{*p} \cong C_p$. Therefore, K/F is the only cyclic extension of degree p over F; i.e., we may take $a = \omega$ a primitive p-th root of unity. Since p is odd, then we have $N(b) = (-1)^{p+1}\omega = \omega$. Furthermore, because M is divisible, then we obtain $M = M^p \subset N(K^*)$ and hence $N(K^*) = F^*$, as claimed. Now, assume that $Gal(K/F)$ is generated by the automorphism σ of order $p = [K : F]$. Fix an element $\lambda \in F^*$ and a symbol y. We set $D = K1 \oplus Ky \oplus \cdots \oplus Ky^{p-1}$, and multiply elements of D by using distributive law, and the rules $y^p = \lambda$, $yk = \sigma(k)y$ for all $k \in K$. This way, we obtain the cyclic algebra $(K/F, \sigma, \lambda)$. Now, since $F^* = N(K^*)$, then we conclude that $\lambda \in N(K^*)$. Thus, by Corollary 4 of [2], p. 82, $[D]$ is trivial in $Br(F)$. If $Br(F)_p \neq 0$, then there is an F-central simple algebra, say A, such that $[A] \neq 0$ with $p[A] = 0$. By Theorem 1 of [2], p. 119, $[A] = \oplus[C_i]$, where C_i is a cyclic algebra of index p. From the above argument, we know that $[C_i] = 0$ for each i. Thus, $[A] = 0$ which is a contradiction. Thus, we must have $p = 2$. Now, we have $F^* \cong M \times C_2$, which shows that $-1 \notin M$. The equation $x^2 + 1 = 0$ over F has no root in F, since $a^2 = -1$ with $a \in F$ implies that $-1 \in M$, which is false. Now, consider the extension $L = F(i)$ with $i^2 = -1$. The above proof shows that $N_{L/F}(L^*) = M$. We claim that M defines a positive cone for F. It is clear that $M \cap -M = \emptyset$, $MM \subseteq M$, and $M \cup -M \cup \{0\} = F$. To show that $M + M \subseteq M$, take $\alpha, \beta \in M$. Since $M = F^{*2}$, then there exist $\lambda, \mu \in F^*$ such that $\alpha = \lambda^2, \beta = \mu^2$. Now, consider the element $x = \lambda + \mu i \in L$. We have $N_{L/F}(x) = \lambda^2 + \mu^2 = \alpha + \beta \in M$, since $N(L^*) = M$. Therefore, F is formally real and since $M = F^{*2}$, then we conclude that F is Euclidean. On the other hand, since F is Euclidean, then take the quaternion algebra Q over F. Because -1 is not a sum of squares, then we conclude that Q is a division algebra and hence $Br(F)_2 \neq 0$.

(2) One way is clear from Theorem A. If M is the divisible maximal subgroup of F^*, then from the proof of (1) we have $F^* \cong M \times C_2$. Since M is divisible from the theory of divisible abelian groups we know that M is a direct product of quasi-cyclic and full rational groups (cf. [1], p. 96). We claim that M contains no primitive p-th root of unity. Since -1 is not in M, then it suffices to consider $p > 2$. If ω is a primitive p-th root of unity, then for $p \neq 2$ we have $1^2 + \omega^2 + \cdots + \omega^{2(p-1)} = (\omega^{2p} - 1)/(\omega^2 - 1) = 0$, which is not possible in a formally real field (by (1)). Thus, we cannot have any copy of a quasi-cyclic group in our decomposition of M and hence $M \cong \mathbb{Q}^\lambda$ for some cardinal λ. Since $\mathbb{Q}$ is of torsion-free rank 1, λ is the torsion-free rank of F^*. Now, because $CharF = 0$, then we have $\mathbb{Q}^* \subset F^*$, and hence λ is infinite by Lemma 4.1.16 of [1], which asserts that $Q^* \cong \mathbb{Z}_2 \times \mathbb{Z}^{\aleph_0}$. Therefore, we have $M \cong \mathbb{Q}^\lambda$ for some infinite cardinal λ. Now, by Theorem A, we obtain the result. $\square$

We observe that in the conclusion of the theorem, F need not necessarily be real closed. In fact, if F is obtained from the rationals $\mathbb{Q}$ by iteratively adjoining roots of positive real algebraic numbers, then the positive cone of the resulting field F is such a maximal subgroup. But F is not real closed.

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