ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

# Bulletin of the Iranian Mathematical Society

Vol. 40 (2014), No. 6, pp. 1373–1385

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 40 (2014), No. 6, pp. 1373–1385
Online ISSN: 1735-8515

# TRANSLATION INVARIANT SURFACES IN THE 3-DIMENSIONAL HEISENBERG GROUP

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(Communicated by Mohammad Bagher Kashani)

Abstract. In this paper, we study translation invariant surfaces in the 3-dimensional Heisenberg group $\mathrm{Nil}_3$. In particular, we completely classify translation invariant surfaces in $\mathrm{Nil}_3$ whose position vector $x$ satisfies the equation $\Delta x = Ax$, where $\Delta$ is the Laplacian operator of the surface and $A$ is a $3 \times 3$-real matrix.
**Keywords:** Heisenberg group, finite type surface, invariant surface.
**MSC(2010):** Primary: 53C30; Secondary: 53B25.

## 1. Introduction

In late 1970's Chen [4] introduced the notion of finite type immersion in the $m$-dimensional Euclidean space $\mathbb{R}^m$. A submanifold $M$ of the $m$-dimensional Euclidean space $\mathbb{R}^m$ is said to be of finite type if its position vector field $x$ can be expressed as a finite sum of the eigenvectors of the Laplacian operator $\Delta$ of $M$, that is, $x = x_0 + \sum_{i=1}^{k} x_i$, where $x_0$ is a constant map, $x_1, \cdots, x_k$ non-constant maps such that $\Delta x_i = \lambda_i x_i, \lambda_i \in \mathbb{R}, i = 1, 2, \cdots, k$. If $\lambda_1, \lambda_2, \cdots, \lambda_k$ are different, then $M$ is said to be of $k$-type. The classification of 1-type submanifolds of Euclidean space was done by T. Takahashi [14]. He proved that the submanifolds in $\mathbb{R}^m$ satisfy the differential equation

$$\Delta x = \lambda x, \tag{1.1}$$

Article electronically published on December 11, 2014.
Received: 2 April 2013, Accepted: 8 October 2013.
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for some real number $\lambda$, if and only if either the submanifold is a minimal submanifold of $\mathbb{R}^m$ ($\lambda = 0$) or it is a minimal submanifold of a hypersphere of $\mathbb{R}^m$ centered at the origin ($\lambda \neq 0$).

As a generalization of Takahashi's condition (1.1), Garay [7] studied hypersurfaces in $\mathbb{R}^m$ whose coordinate functions are eigenfunctions of the Laplacian operator of the hypersurface, but not necessarily associated to the same eigenvalue. Specifically, he considered hypersurfaces in $\mathbb{R}^m$ satisfying the differential equation

$$\Delta x = Ax, \tag{1.2}$$

where $A \in \mathrm{Diag}(m, \mathbb{R})$ is an $m \times m$- diagonal matrix, and proved that such hypersurfaces are minimal in $\mathbb{R}^m$ and open pieces of either round hyperspheres or generalized right spherical cylinders. Garay called such submanifolds coordinate finite type. Related to this, Dillen, Pas and Verstraelen [5] observed that Garay's condition (1.2) is not coordinate invariant and they proposed the study of submanifolds of $\mathbb{R}^m$ satisfying the following equation:

$$\Delta x = Ax + B, \tag{1.3}$$

where $A \in \mathrm{Mat}(m, \mathbb{R})$ is a $m \times m$ matrix and $B \in \mathbb{R}^m$. On the other hand, the class of submanifolds satisfying (1.2) and the class of submanifolds satisfying (1.3) are the same if the submanifolds are hypersurfaces of Euclidean space [9]. Also, the above mentioned study can be extendeded the notion of an immersion of submanifolds into pseudo-Euclidean space (see [1, 2]). Recently, many geometers are studying an extension of Takahashi theorem for the linearized operators of the higher order mean curvatures of hypersurfaces (see [3, 11–13]).

A homogenous space is a Riemannian manifold $M$ such that for every two points $p$ and $q$ in $M$, there exists an isometry of $M$ mapping $p$ into $q$. This means that the space looks the same at every point. Remark that $M$ is homogeneous if the action of the isometry of $M$ is transitive. Homogenous geometries have main roles in the modern theory of manifolds. Homogenous spaces are, in a sense, the magnificent examples of Riemannian manifolds and have applications in physics [8]. To underline their importance from the mathematical point of view we roughly cite the famous Thurston conjecture. This conjecture asserts that every compact orientable 3-dimensional manifold has a canonical decomposition into pieces, each of which admits a canonical geometric structure from among the eight maximal simple connected homogenous Riemannian 3-dimensional geometries [15]. One of the eight model spaces is the 3-dimensional Heisenberg group $\mathrm{Nil}_3$.

In this paper, we shall classify translation invariant surfaces in the 3-dimensional Heisenberg group $\mathrm{Nil}_3$ satisfying the equation (1.2)

## 2. **Preliminaries**

Let $\mathrm{Nil}_3$ denote the 3-dimensional Heisenberg group. This is a two-step nilpotent Lie group which can be seen as the subgroup of $3 \times 3$-matrices given by

$$\mathrm{Nil}_3 = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \Big| \; a, b, c \in \mathbb{R} \right\} \subset GL(3, \mathbb{R}).$$

We denote the corresponding Lie algebra by

$$\mathcal{L}(\mathrm{Nil}_3) = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \Big| \; x, y, z \in \mathbb{R} \right\}.$$

Using the exponential map $\exp : \mathcal{L}(\mathrm{Nil}_3) \to \mathrm{Nil}_3$,

$$\exp(A) = I + A + \frac{A^2}{2} = \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

we can view $\mathrm{Nil}_3$ as $\mathbb{R}^3$ equipped with the group structure $*$ given by

$$(x, y, z) * (\bar{x}, \bar{y}, \bar{z}) = \left( x + \bar{x}, y + \bar{y}, z + \bar{z} + \frac{1}{2}x\bar{y} - \frac{1}{2}y\bar{x} \right). \tag{2.1}$$

The identity of the group is $0 = (0, 0, 0)$ and the inverse of $p = (a, b, c)$ is $\hat{p} = (-a, -b, -c)$. The left-multiplication by $p$ in $\mathrm{Nil}_3$, $L_p : q \mapsto p * q$, has tangent map

$$T_q L_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2}b & \frac{1}{2}a & 1 \end{pmatrix} \tag{2.2}$$

in the canonical coordinates $(x, y, z)$ of $\mathbb{R}^3$ (they are often referred to as exponential coordinates).

Let $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ denote the canonical vector fields in $\mathbb{R}^3$. Then from (2.2) we have that an orthonormal basis of the left-invariant vector fields in $\mathrm{Nil}_3$ is given in exponential coordinates by

$$(2.3)\qquad \begin{aligned} e_1 &= T_0L_{(x,y,z)}(\frac{\partial}{\partial x}) = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z},\\ e_2 &= T_0L_{(x,y,z)}(\frac{\partial}{\partial y}) = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z},\\ e_3 &= T_0L_{(x,y,z)}(\frac{\partial}{\partial z}) = \frac{\partial}{\partial z}, \end{aligned}$$

and the left-invariant metric $\tilde{g}$ in $\mathrm{Nil}_3$ is given by

$$(2.4)\qquad \tilde{g} = dx^2 + dy^2 + \left(dz + \frac{1}{2}(ydx - xdy)\right)^2.$$

On the other hand, the Lie brackets is given by

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = 0$$

and the Levi-Civita connection $\tilde{\nabla}$ of $\mathrm{Nil}_3$ is expressed as

$$(2.5)\qquad \begin{aligned} &\tilde{\nabla}_{e_1}e_1 = 0, && \tilde{\nabla}_{e_1}e_2 = \frac{1}{2}e_3, && \tilde{\nabla}_{e_1}e_3 = -\frac{1}{2}e_2,\\ &\tilde{\nabla}_{e_2}e_1 = -\frac{1}{2}e_3, && \tilde{\nabla}_{e_2}e_2 = 0, && \tilde{\nabla}_{e_2}e_3 = \frac{1}{2}e_1,\\ &\tilde{\nabla}_{e_3}e_1 = -\frac{1}{2}e_2, && \tilde{\nabla}_{e_3}e_2 = \frac{1}{2}e_1, && \tilde{\nabla}_{e_3}e_3 = 0. \end{aligned}$$

The following properties are well-known and can be found for example in [6]. Equipped with the left-invariant metric $\tilde{g}$, the Heisenberg group $\mathrm{Nil}_3$ is a homogenous Riemannian manifold whose group of isometrics $\mathcal{I}(\mathrm{Nil}_3)$ has dimension 4. Also, the identity component $\mathcal{I}_0(\mathrm{Nil}_3)$ of $\mathcal{I}(\mathrm{Nil}_3)$ is isometric to the semi-direct product of $\mathrm{Nil}_3$ and SO(2). In particular, a basis of Killing vector fields is given by

$$\begin{aligned} E_1 &= \frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial z}, \quad E_2 = \frac{\partial}{\partial y} - \frac{x}{2}\frac{\partial}{\partial z},\\ E_3 &= \frac{\partial}{\partial z}, \quad E_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \end{aligned}$$

One can check that $E_1, E_2, E_3$ are infinitesimal translations of the 1-parameter groups of isometries defined by

$$G_1 := \{(s,0,0) \mid s \in \mathbb{R}\}, G_2 := \{(0,s,0) \mid s \in \mathbb{R}\}, G_3 := \{(0,0,s) \mid s \in \mathbb{R}\},$$

respectively. Here these groups act on $\mathrm{Nil}_3$ by left translation. The vector field $E_4$ generates the group of rotations around the $z$-axis. Thus, $G_4$ is defined with SO(2).

**Theorem 2.1.** *([6]). The 1-dimensional subgroups of $\mathcal{I}_0(Nil_3)$ are:*

(1) *the 1-parameter subgroups generated by the linear combinations*

$$X = a_1E_1 + a_2E_2 + a_3E_3 + bE_4 \tag{2.6}$$

*with $b \neq 0$. In particular, the group generated by $X = bE_4$ is the group of rotations around the $z$-axis.*

(2) *the 1-parameter subgroups generated by the linear combinations*

$$X = a_1E_1 + a_2E_2 + a_3E_3 \tag{2.7}$$

*with $a_1^2 + a_2^2 + a_3^2 \neq 0$.*

A surface in $\mathrm{Nil}_3$ is said to be translation invariant if it is invariant under the action of 1-parameter subgroup generated by the Killing vector field given by (2.7).

**Lemma 2.2.** *([6]). Let $\Sigma$ be a surface in $\mathrm{Nil}_3$ invariant under the 1-parameter subgroup generated by a Killing vector fields of the form:*

$$a_1E_1 + a_2E_2 + a_3E_3, \quad a_1^2 + a_2^2 \neq 0.$$

*Then, $\Sigma$ is isometric to a surface invariant under the 1-parameter subgroup $G_1 = \{(s, 0, 0) \in Nil_3 \mid s \in \mathbb{R}\}$.*

Thus, for the study of translation type surfaces, we may restrict our attention to

(1) surfaces invariant under $G_1 = \{(s, 0, 0) \mid s \in \mathbb{R}\}$ or

(2) surfaces invariant under $G_3 = \{(0, 0, s) \mid s \in \mathbb{R}\}$.

First, let $\Sigma_1$ be a surface invariant under the 1-parameter subgroup $G_1$. Then the parametrization of $\Sigma_1$ is given by

$$\begin{aligned} x(s,t) &= (s, 0, 0) * (0, t, v(t)) \\ &= \left(s, t, v(t) + \frac{st}{2}\right). \end{aligned} \tag{2.8}$$

It is called $G_1$-translation invariant surface.

Next, let $\Sigma_3$ be a surface invariant under the 1-parameter subgroup $G_3$. Then, $\Sigma_3$ is locally expressed as

$$\begin{aligned} x(s,t) &= (0, 0, s) * (t, v(t), 0) \\ &= (t, v(t), s) \end{aligned} \tag{2.9}$$

which is called $G_3$-translation invariant surface.

It is well known that in terms of local coordinates $\{x_i\}$ of a surface $\Sigma$ the Laplacian operator $\Delta$ on $\Sigma$ is given by

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}} \sum_{i,j} \frac{\partial}{\partial x^i}(\sqrt{|\mathcal{G}|} g^{ij} \frac{\partial}{\partial x^j}), \tag{2.10}$$

where $\mathcal{G} = \det(g_{ij}), (g^{ij}) = (g_{ij})^{-1}$ and $(g_{ij})$ are the components of the induced metric of $\Sigma$ with respect to $\{x_i\}$.

## 3. $G_1$-translation invariant surfaces satisfying $\Delta x = Ax$

Let $\Sigma_1$ be a $G_1$-translation invariant surface in the 3-dimensional Heisenberg group $\mathrm{Nil}_3$. Then, $\Sigma_1$ is parametrized by

$$x(s,t) = \left(s, t, v(t) + \frac{st}{2}\right). \tag{3.1}$$

In this case, the natural frame $\{x_s, x_t\}$ is given by

$$\begin{aligned} \frac{\partial x}{\partial s} &:= x_s = e_1 + te_3, \\ \frac{\partial x}{\partial t} &:= x_t = e_2 + v'(t)e_3, \end{aligned}$$

from these the components of the induced metric of the surface are

$$g_{11} = 1 + t^2, \quad g_{12} = tv', \quad g_{22} = 1 + v'^2.$$

Let $U$ be a unit normal vector of $\Sigma_1$. Then it is defined by $\frac{x_s \times x_t}{||x_s \times x_t||} = \frac{1}{w^{\frac{1}{2}}}(x_s \times x_t)$ and hence we get

$$U = \frac{1}{w^{\frac{1}{2}}}(-te_1 - v'e_2 + e_3).$$

On the other hand,

$$\begin{aligned} x_{ss} &= \tilde{\nabla}_{x_s} x_s = -te_2, \\ x_{st} &= \tilde{\nabla}_{x_s} x_t = \frac{t}{2}e_1 - \frac{v'}{2}e_2 + \frac{1}{2}e_3, \\ x_{tt} &= \tilde{\nabla}_{x_t} x_t = v'e_1 + v''e_3. \end{aligned} \tag{3.2}$$

By (2.10), the Laplacian operator $\Delta$ of $\Sigma_1$ can be expressed as

$$\begin{aligned}\Delta = &\frac{1}{w^2}[v'w + tv''w - tv'(t + v'v'')]\frac{\partial}{\partial s}\\ &+ \frac{1}{w^2}[(1+t^2)(t+v'v'') - 2tw]\frac{\partial}{\partial t}\\ &+ \frac{1}{w}(2tv')\frac{\partial^2}{\partial s\partial t} - \frac{1}{w}(1+v'^2)\frac{\partial^2}{\partial s^2} - \frac{1}{w}(1+t^2)\frac{\partial^2}{\partial t^2}.\end{aligned}\tag{3.3}$$

By a straightforward computation, the Laplacian operator $\Delta x$ of $x$ with the help of (3.2) and (3.3) turns out to be

$$\begin{aligned}\Delta x = &-\frac{t}{w^2}(-v'' - t^2v'' + tv')e_1 + \frac{1}{w^2}(v'v'' + v'v''t^2 - tv'^2)e_2\\ &+ \frac{1}{w^2}(tv' - v'' - t^2v'')e_3.\end{aligned}\tag{3.4}$$

Suppose $\Sigma_1$ satisfies the condition (1.2), that is, $\Delta x = Ax$ for some matrix $A = (a_{ij})$, where $i, j = 1, 2, 3$. Then, from (3.1) and (3.4), using the fact that $w$ does not depend on $s$ we obtain the following equations:

$$-\frac{t}{w^2}(-v'' - t^2v'' + tv') = a_{11}s + a_{12}t + a_{13}(v + \frac{st}{2}),\tag{3.5}$$

$$\frac{1}{w^2}(v'v'' + v'v''t^2 - tv'^2) = a_{21}s + a_{22}t + a_{23}(v + \frac{st}{2}),\tag{3.6}$$

$$\begin{aligned}\frac{1}{w^2}(tv' - v'' - t^2v'') = &\frac{t}{2}\left(a_{11}s + a_{12}t + a_{13}(v + \frac{st}{2})\right)\\ &- \frac{s}{2}\left(a_{21}s + a_{22}t + a_{23}(v + \frac{st}{2})\right)\\ &+ a_{31}s + a_{32}t + a_{33}(v + \frac{st}{2}).\end{aligned}\tag{3.7}$$

Differentiating (3.5) and (3.6) with respect to $s$, we have $a_{11} + \frac{t}{2}a_{13} = 0$ and $a_{21} + \frac{t}{2}a_{23} = 0$, respectively. From these, $a_{11} = a_{13} = a_{21} = a_{23} = 0$. In this case, differentiating (3.7) with respect to $s$, we have $(a_{33} - a_{22})t + 2a_{31} = 0$. It follows that $a_{31} = 0$ and $a_{22} = a_{33}$. We put $a_{22} = a_{33} = \lambda$. Then, (3.5), (3.6) and (3.7) can be written as the forms:

$$v'' + t^2v'' - tv' = a_{12}w^2,\tag{3.8}$$

$$v'v'' + v'v''t^2 - tv'^2 = \lambda t w^2,\tag{3.9}$$

$$tv' - v'' - t^2v'' = \frac{1}{2}a_{12}t^2w^2 + a_{32}tw^2 + \lambda vw^2. \tag{3.10}$$

Combining (3.8) and (3.9), we find

$$a_{12}v' - \lambda t = 0. \tag{3.11}$$

Again, combining (3.9) and (3.10), we get

$$\frac{1}{2}a_{12}v't^2 + \lambda t + a_{32}v't + \lambda vv' = 0. \tag{3.12}$$

From (3.11) and (3.12), we have the following equation:

$$v'\left(\frac{1}{2}a_{12}t^2 + a_{32}t + a_{12} + \lambda v\right) = 0. \tag{3.13}$$

First of all, if $v' = 0$, then from (3.8), (3.9)) and (3.10) we can obtain $a_{12} = a_{32} = \lambda = 0$. Thus, $A = 0$. In this case, $\Sigma_1$ is parametrized by

$$x(s,t) = (s, t, \frac{st}{2} + c_1), \tag{3.14}$$

where $c_1$ is a constant.

Next, we suppose $v' \neq 0$. Then from (3.13) we have

$$\lambda v = -\frac{1}{2}a_{12}t^2 - a_{32}t - a_{12}. \tag{3.15}$$

(i) If $\lambda = 0$, then from (3.15) we have $a_{12} = a_{32} = 0$, that is, $A = 0$. In this case, (3.10) becomes

$$(t^2 + 1)v'' = tv'$$

and its general solution is given by

$$v = \frac{c_1}{2}\left(t\sqrt{t^2+1} + \ln(t + \sqrt{t^2+1}) + c_2\right) \tag{3.16}$$

where $0 \neq c_1, c_2$ are constants of integration.

(ii) If $\lambda \neq 0$, then from (3.15) we have

$$v = -\frac{a_{12}}{2\lambda}t^2 - \frac{a_{32}}{\lambda}t - \frac{a_{12}}{\lambda}. \tag{3.17}$$

Substituting it in (3.5), we get $a_{12} = a_{32} = 0$. It is a contradiction.

Thus, we have the following:

**Theorem 3.1.** *Let $\Sigma_1$ be a $G_1$-translation invariant surface in the 3-dimensional Heisenberg group $Nil_3$. Then, $\Sigma_1$ satisfies the equation $\Delta x = Ax, A \in Mat(3, \mathbb{R})$ if and only if the surface can be parametrized as*

$$x(s,t) = \left(s, t, v(t) + \frac{st}{2}\right),$$

*where*
(1) *either* $v(t) = c_1$ *with* $c_1 \in \mathbb{R}$,
(2) *or* $v = \frac{c_1}{2}\left(t\sqrt{t^2+1} + \ln(t+\sqrt{t^2+1}) + c_2\right)$ *with* $0 \neq c_1, c_2 \in \mathbb{R}$.

**Remark 3.2.** *The surfaces given in Theorem 3.1 are minimal and those surfaces was studied by C. Figueroa, F. Mercuri and R. Pedrosa ( [6]).*

## 4. $G_3$-translation invariant surfaces satisfying $\Delta x = Ax$

Let $\Sigma_3$ be a $G_3$-translation invariant surface in the 3-dimensional Heisenberg group $\text{Nil}_3$. Then, the parametrization of $\Sigma_3$ is given by

$$x(s,t) = (t, v(t), s). \tag{4.1}$$

From which, we have

$$x_s = e_3, \quad x_t = e_1 + v'e_2 + \frac{1}{2}(v - tv')e_3. \tag{4.2}$$

Therefore, the components of the induced metric of $\Sigma_3$ are

$$g_{11} = 1, \quad g_{12} = \frac{1}{2}(v - tv'), \quad g_{22} = 1 + {v'}^2 + \frac{1}{4}(v - tv')^2.$$

On the other hand, the values of $\tilde{\nabla}_{x_i} x_j$ are

$$\begin{aligned} &\tilde{\nabla}_{x_s} x_s = 0, \\ &\tilde{\nabla}_{x_s} x_t = \frac{1}{2}v'e_1 - \frac{1}{2}e_2, \\ &\tilde{\nabla}_{x_t} x_t = \frac{1}{2}v'(v - tv')e_1 + \left(v'' - \frac{1}{2}(v - tv')\right)e_2 - \frac{1}{2}tv''e_3. \end{aligned} \tag{4.3}$$

It is easy to show that the Laplacian operator $\Delta$ of $\Sigma_3$ can be expressed as

$$\begin{aligned} \Delta = &-\frac{1}{2(1+{v'}^2)^2}[v'v''(v - tv') + tv''(1+{v'}^2)]\frac{\partial}{\partial s} \\ &+ \frac{1}{(1+{v'}^2)^2}(v'v'')\frac{\partial}{\partial t} + \frac{1}{1+{v'}^2}(v - tv')\frac{\partial^2}{\partial s \partial t} \\ &- \frac{1}{1+{v'}^2}[1 + {v'}^2 + \frac{1}{4}(v - tv')^2]\frac{\partial^2}{\partial s^2} - \frac{1}{1+{v'}^2}\frac{\partial^2}{\partial t^2}. \end{aligned} \tag{4.4}$$

From (4.1)-(4.4), we can obtain by a direct computation

$$\Delta x = \frac{v''}{(1+{v'}^2)^2}(v'e_1 - e_2). \tag{4.5}$$

Suppose $\Sigma_3$ satisfies the equation $\Delta x = Ax$ for some matrix $A = (a_{ij})$, where $i, j = 1, 2, 3$. The case $v'' = 0$ will be treated separately. First of all, let us suppose that $v'' \neq 0$ on an open interval. Then, from (4.1) and (4.5) we have the following equations:

$$\frac{1}{(1+v'^2)^2} v'v'' = a_{11}t + a_{12}v + a_{13}s, \tag{4.6}$$

$$-\frac{1}{(1+v'^2)^2} v'' = a_{21}t + a_{22}v + a_{23}s, \tag{4.7}$$

$$0 = \frac{v}{2}(a_{11}t+a_{12}v+a_{13}s) - \frac{t}{2}(a_{21}t+a_{22}v+a_{23}s) + a_{31}t + a_{32}v + a_{33}s, \tag{4.8}$$

which imply that $a_{13} = a_{23} = a_{33} = 0$. In this case, substituting (4.6) and (4.7) into (4.8), we get

$$v''(vv'+t) + 2(a_{31}t + a_{32}v)(1+v'^2)^2 = 0 \tag{4.9}$$

and combining (4.6) and (4.7) we have

$$(a_{21}t + a_{22}v)v' + (a_{11}t + a_{12}v) = 0. \tag{4.10}$$

Now, we have to solve the ordinary differential equation (4.9) But, it is not easy. So, we give examples of translation invariant surfaces by distinguishing some special cases:

1. If $a_{12} = a_{21} = a_{31} = a_{32} = 0$, then from (4.9) we get $vv' + t = 0$ and its general solution is $v(t) = \sqrt{c - t^2}$, $c \in \mathbb{R}^+$. In this case, the matrix $A$ becomes $\left(\begin{smallmatrix} 1/c & 0 & 0 \\ 0 & 1/c & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$.

2. Assume $a_{12} = 0$. Then, from (4.8) we obtain

$$v(t) = \frac{a_{21}t^2 - 2a_{31}t}{(a_{11} - a_{22})t + 2a_{32}}, \tag{4.11}$$

if $a_{11} - a_{22} \neq 0$ and $a_{32} \neq 0$. Differentiating (4.11) with respect to $t$ and combining (4.10), it is transformed into a polynomial equation in $t$. Therefore, all coefficients of the polynomial equation must be zero. So we obtain the following equations:

$$a_{11}\left(a_{11} - a_{22})((a_{11} - a_{22})^2 + a_{21}^2\right) = 0, \tag{4.12}$$

$$\begin{aligned} &a_{11}a_{21}a_{22}a_{31} - a_{21}a_{22}^2a_{31} - 3a_{11}a_{21}^2a_{32} - 3a_{11}^3a_{32} \\ &- 3a_{11}a_{22}^2a_{32} + 6a_{11}^2a_{22}a_{32} + a_{21}^2a_{22}a_{32} = 0, \end{aligned} \tag{4.13}$$

$$2a_{21}^2a_{32} + 3a_{11}^2a_{32} - 3a_{11}a_{22}a_{32} - a_{11}a_{21}a_{31} - 2a_{21}a_{22}a_{31} = 0, \tag{4.14}$$

$$a_{11}a_{32}^2 - a_{21}a_{31}a_{32} + a_{22}a_{31}^2 = 0. \tag{4.15}$$

From (4.12), we have $a_{11} = 0$. In such case, equations (4.13), (4.14) and (4.15) are rewritten as

$$\begin{aligned} &a_{21}a_{22}(a_{21}a_{32} - a_{22}a_{31}) = 0, \\ &a_{21}(a_{21}a_{32} - a_{22}a_{31}) = 0, \\ &a_{31}(a_{21}a_{32} - a_{22}a_{31}) = 0, \end{aligned} \tag{4.16}$$

which imply that $a_{21}a_{32} - a_{22}a_{31} = 0$. Thus we have

$$v(t) = \frac{a_{22}a_{31}t^2 - 2a_{31}a_{32}t}{a_{32}(2a_{32} - a_{22}t)}.$$

3. Assume now that $a_{22} \neq 0$. In such case we can make the change of variable $u = a_{22}v + a_{21}t$ and equation (4.10) is reduced to an equation of the type

$$u' = P + Q\frac{t}{u}, \tag{4.17}$$

where $P = a_{21} - a_{12}$ and $Q = a_{12}a_{21} - a_{11}a_{22}$. If $P = 0$, the general solution of (4.17) is given by

$$u = \pm\sqrt{Qt^2 + c_1},$$

where $c_1$ is a constant of integration. From this, we have

$$v(t) = \pm\frac{1}{a_{22}}\sqrt{Qt^2 + c_1} - \frac{a_{12}}{a_{22}}t. \tag{4.18}$$

If $c_1 = 0$, then the function $v(t)$ is linear, which is a contradiction. So, $c_1$ is a non-zero constant. In such case, (4.8) becomes

$$\begin{aligned} &\left((a_{22}(a_{11} - a_{22}) + 2a_{12}^2)t + 2a_{22}a_{32}\right)\sqrt{Qt^2 + c_1} \\ &= 2a_{12}(a_{22}^2 - a_{12}^2)t^2 - 2a_{22}(a_{22}a_{31} + a_{12}a_{32})t - c_1a_{12}. \end{aligned} \tag{4.19}$$

(i) If $a_{12} = a_{22} \neq 0$, then from the coefficients of the polynomial equation (4.19) we have

$$a_{22}^3(a_{11} - a_{22})(a_{11} + a_{22})^2 = 0,$$
$$4a_{22}^3a_{32}(a_{11} - a_{22})(a_{11} + a_{22}) = 0,$$
$$a_{22}^2(c_1a_{22}^2 - 4a_{22}^2a_{31}^2 - 8a_{22}^2a_{31}a_{32} + 2c_1a_{11}a_{22} - 4a_{11}a_{22}a_{32}^2 + c_1a_{11}^2 = 0,$$
$$4c_1a_{22}^2(a_{11}a_{32} - a_{22}a_{31}) = 0,$$

$$c_1 a_{22}^2 (c_1 - 4a_{32}^2) = 0.$$

If $a_{11} = a_{22}$, then $Q = 0$, it is a contradiction. Thus, we have $a_{11} = -a_{22}$, $c_1 = 4a_{32}^2$ and the function $v(t)$ is given by

$$v(t) = \pm \frac{1}{a_{22}} \sqrt{2a_{22}^2 t^2 + 4a_{32}^2} - t,$$

where $a_{22}, a_{32} \in \mathbb{R} - \{0\}$.

(ii) We consider the case $a_{12} = -a_{22} \neq 0$. Then by applying the same algebraic method as above, we also obtain

$$v(t) = \pm \frac{1}{a_{22}} \sqrt{2a_{22}^2 t^2 + 4a_{32}^2} + t,$$

Return to the remained case $v'' = 0$, that is, $v(t) = at + b$, $a, b \in \mathbb{R}$. Then from (4.5) $\Delta x = 0$, it follows that $A = 0$.

Thus, we have the following:

**Theorem 4.1.** *Let $\Sigma_3$ be a $G_3$-translation invariant surface in the 3-dimensional Heisenberg group $Nil_3$. Then, $\Sigma_3$ is a coordinate finite surface if and only if the surface can be parametrized as*

$$x(s, t) = (t, v(t), s),$$

*where*

(1) *either $v(t) = at + b$ with $a, b \in \mathbb{R}$,*
(2) *or $v(t) = \sqrt{c - t^2}$ with $c \in \mathbb{R}^+$.*

**Proposition 4.2.** *The following surface*

$$x(s, t) = \left(t, \frac{ct^2 - 2dt}{a - bt}, s\right) \text{ or}$$

$$x(s, t) = \left(t, \pm \frac{1}{a} \sqrt{2a^2 t^2 + 4b^2} \pm t, s\right), a, b, c, d \in \mathbb{R} - \{0\}$$

*is one of $G_3$-translation invariant surfaces in the 3-dimensional Heisenberg group $Nil_3$ satisfying $\Delta x = Ax, A \in Mat(3, \mathbb{R})$.*


## Acknowledgments

The first author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012R1A1A2003994) and the second author was supported by the Education Research Institute, Busan National University of Education in 2014.

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