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Author(s):

S. Rashidi and N. Soltankhah

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ON THE POSSIBLE VOLUME OF μ-(v, k, t) TRADES

S. RASHIDI AND N. SOLTANKHAH*

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Abstract. A μ-way (v, k, t) *trade* of volume m consists of μ disjoint collections $T_1, T_2, \dots T_\mu$, each of m blocks, such that for every t-subset of v-set V the number of blocks containing this t-subset is the same in each T_i $(1 \le i \le \mu)$. In other words any pair of collections $\{T_i, T_j\}$, $1 \le i < j \le \mu$ is a (v, k, t) trade of volume m. In this paper we investigate the existence of μ-way (v, k, t) trades and prove the existence of: (i) 3-way $(v, k, 1)$ trades (Steiner trades) of each volume $m, m \ge 2$. (ii) 3-way $(v, k, 2)$ trades of each volume $m, m \ge 6$ except possibly $m = 7$. We establish the non-existence of 3-way $(v, 3, 2)$ trade of volume 7. It is shown that the volume of a 3-way $(v, k, 2)$ Steiner trade is at least $2k$ for $k \ge 4$. Also the spectrum of 3-way $(v, k, 2)$ Steiner trades for $k = 3$ and 4 are specified.
Keywords: μ-way (v, k, t) trade, 3-way $(v, k, 2)$ trade, one-solely.
MSC(2010): Primary 05B30; Secondary 05B05.

1. Introduction

Given a set of v treatments V, let k and t be two positive integers such that $t < k < v$. A (v, k, t) *trade* $T = \{T_1, T_2\}$ of volume m consists of two disjoint collections T_1 and T_2, each one containing m k-subsets of V, called *blocks*, such that every t-subset of V is contained in the same number of blocks in T_1 and T_2. A (v, k, t) trade is called (v, k, t) *Steiner* trade if any t-subset of V occurs at most once in $T_1(T_2)$.
A $t - (v, k, \lambda)$ *design* (V, B) is a collection of blocks such that each $t-$subset of V is contained in λ blocks.

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*Corresponding author.

When $m = 0$ the trade is said to be *void*. A (v, k, t) trade (design) is also a (v, k, t') trade (design), for all t' with $0 < t' < t$. In a (v, k, t) trade, both collections of blocks must cover the same set of elements. This set of elements is called the $foundation$ of the trade and is denoted by $found(T)$.

A $2-(v, 3, 1)$ design is called a *Steiner triple system* of order v and is often denoted by $STS(v)$. It is well known that a $STS(v)$ exists if and only if $v \equiv 1, 3 \pmod 6$.

A *Kirkman triple system* of order v that is often denoted by $KTS(v)$ is a Steiner triple system of order v (V, B) together with a partition R of the set of triples B into subsets $R_1, R_2, \dots, R_n$ called parallel classes such that each R_i $(i = 1, 2, \dots, n)$ is a partition of V.

A *partial triple system* (PTS) is a pair (V, P) where V is a finite nonempty n-set and P is a collection of 3-subsets of V, called blocks (or triples), such that every pair of distinct elements of V is contained in at most one block of P.

Two partial triple systems (V, P_1) and (V, P_2) are said to be disjoint and mutually balanced (DMB) if:

(i) $P_1 \cap P_2 = \phi$.

(ii) any given pair of distinct elements of V is contained in a block of P_1 if and only if it is contained in a block of P_2.

Milici and Quattrocchi (1986) used what is now known as Steiner trades and named them, DMB (disjoint and mutually balanced). The concept of trade was first introduced in 1960s by Hedayat [11]. Hedayat and Li applied the method of trade-off and trades for building BIBDs with repeated blocks (1979-1980). Papers by Hwang [12], Mahmoodian and Soltankhah [15], and Asgari and Soltankhah [3] deal with the existence and non-existence of (v, k, t) trades. The concept of trade was introduced for BIBDs first and then it was used in the Latin squares with Latin trade title (see [1]) and in the Graph theory with G-trade title (see [4]).

The definition of trades can be generalized, and here we introduce μ-way trades $(\mu \geq 2)$ as follows:

Definition 1.1. *A μ-way (v, k, t) trade of volume m consists of μ disjoint collections T_1, $T_2, \dots T_\mu$, each of m blocks, such that for every t-subset of v-set V the number of blocks containing this t-subset is the same in each T_i $(1 \leq i \leq \mu)$. In other words any pair of collections $\{T_i, T_j\}$, $1 \leq i < j \leq \mu$ is a (v, k, t) trade of volume m.*

Definition 1.2. *A μ-way (v,k,t) trade is called μ-way (v,k,t) Steiner trade if any t-subset of* found(T) *occurs at most once in T_1 $(T_j,\ j \geq 2)$.*
Example 1.3. *The following trades are 3-way $(8,3,2)$ Steiner trade and 3-way $(11,3,2)$ Steiner trade of volume 8 and 13, respectively:*

T_1	T_2	T_3
1, 2, 3	1, 2, 4	1, 2, 7
1, 4, 7	1, 3, 8	1, 3, 5
1, 5, 8	1, 5, 7	1, 4, 8
2, 4, 8	2, 3, 7	2, 4, 6
2, 6, 7	2, 6, 8	2, 3, 8
3, 5, 7	4, 6, 7	3, 6, 7
3, 6, 8	4, 5, 8	4, 5, 7
4, 5, 6	3, 5, 6	5, 6, 8

T_1	T_2	T_3
1, 2, 11	2, 3, 11	1, 3, 11
3, 10, 11	1, 10, 11	2, 10, 11
1, 3, 7	1, 2, 8	1, 2, 9
1, 10, 9	1, 3, 5	1, 10, 8
1, 5, 8	1, 7, 9	1, 5, 7
2, 3, 6	2, 10, 9	2, 3, 4
2, 10, 8	2, 4, 6	2, 6, 8
2, 4, 9	3, 10, 6	3, 10, 7
3, 4, 5	3, 4, 7	3, 5, 6
6, 8, 4	6, 5, 8	7, 9, 4
6, 10, 5	7, 5, 10	9, 5, 10
7, 9, 5	9, 4, 5	8, 4, 5
7, 10, 4	8, 4, 10	6, 4, 10

Trades are also intimately connected with the so-called *intersection* problem for combinatorial structures. This basically asks, given two combinatorial structures with the same parameters, and based on the same underlying set, such as a pair of block designs or a pair of latin rectangles, in how many ways may they intersect? So for two block designs, how many common blocks may there be? Of course, removing a set of m blocks from a design and replacing them with a distinct set of m blocks which nevertheless still make the whole collection of blocks a design with the same parameters, is utilising a trade of volume m to yield two designs with m blocks different, and so a known number of blocks in common.

The *intersection problem* has also been considered for more than just *pairs* of combinatorial structures; the intersection of μ combinatorial structures with $\mu > 2$ was dealt with in, for example, [17] for three Steiner triple systems and [2] for three latin squares. These correspond in the same manner to μ-way trades in the corresponding combinatorial structure.

So it is clear that if there exist three $t-(v,k,\lambda)$ designs (V,B) which intersect in the same set of m blocks, and which differ in the remaining blocks then we obtain a 3-way (v',k,t) trade of volume $b_v - m$ where

$b_v = |B|$. Conversely let $D = (V, B)$ be a $t - (v, k, \lambda)$ design and $T = \{T_1, T_2, T_3\}$ be a 3-way (v, k, t) trade of volume m. If $T_1 \subseteq B$, we say that D contains the trade T, and if we replace T_i $(i = 2, 3)$ with T_1, then we obtain new designs $D_i = (D - T_1) \cup T_i$ which are denoted by $D_i = D + T_i$ with the same parameters of D, and $|D_i \cap D| = |D_i \cap D_j| = b_v - m$ $(2 \le i, j \le 3)$. If there is not a 3-way (v', k, t) trade of volume m, then there does not exist three designs with intersection number $b_v - m$. It is important to understand the structure of μ-way trades and conditions for their existence and non-existence. Here, the following question is of interest.

Question 1.4. *For a given μ, what is the set of all possible volume sizes (the "volume spectrum") of a μ-way (v, k, t) trade?*

We now introduce some notations. Let $\mathcal{S}_\mu(t, k)$ ($\mathcal{S}_{\mu s}(t, k)$) denote the set of all possible volume sizes of a μ-way (v, k, t) trade (μ-way (v, k, t) Steiner trade).

This question has been answered for $\mu = 2$ until now as follows:

(1) [12] $\mathcal{S}_2(2, k) = \mathbb{N} \setminus \{1, 2, 3, 5\}$.

(2) [14] $\mathcal{S}_{2s}(2, 3) = \mathbb{N} \setminus \{1, 2, 3, 5\}$.

(3) [7] $\mathcal{S}_{2s}(2, 4) = \mathbb{N} \setminus \{1, 2, 3, 4, 5, 7\}$.

(4) [8] $\mathcal{S}_{2s}(2, 5) = \mathbb{N} \setminus \{1, 2, 3, 4, 5, 6, 7, 9, 11\}$.

(5) [8] $\mathcal{S}_{2s}(2, 6) = \mathbb{N} \setminus \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13\}$.

(6) [8] If $0 < m < 2k - 2$ or $m = 2k - 1$ then $m \notin \mathcal{S}_{2s}(2, k)$.

(7) [8] If $m = 0, m \ge 3k - 3$ or m is even and $2k - 2 \le m \le 3k - 4$ then $m \in \mathcal{S}_{2s}(2, k)$.

(8) [9] $2k + 1 \in \mathcal{S}_{2s}(2, k)$ precisely when $k \in \{3, 4, 7\}$.

(9) [13] If m is odd and $2k + 3 \le m \le 3k - 4$, then $\mathcal{S}_{2s}(2, k)$ does not contain m for $k \ge 7$.

(10) [10] $\mathcal{S}_{2s}(3, 4) = \mathbb{N} \setminus \{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13\}$.

In this paper for $\mu = 3$, we investigate this question and our results include the following.

Main results:
(1) $\mathcal{S}_3(1,k) = \mathcal{S}_{3s}(1,k) = \mathbb{N} \setminus \{1\},\ k \geq 2$.
(2) $\mathcal{S}_3(2,3) = \mathbb{N} \setminus \{1,2,3,4,5,7\}$.
(3) $\mathcal{S}_3(2,k) \setminus \{7\} = \mathbb{N} \setminus \{1,2,3,4,5\}$.
(4) $\mathcal{S}_{3s}(2,3) = \mathbb{N} \setminus \{1,2,3,4,5,7\}$.
(5) $\mathcal{S}_{3s}(2,4) = \mathbb{N} \setminus \{1,2,3,4,5,6,7\}$.
(6) $S_{3s}(2,k) \subseteq \mathbb{N} \setminus \{1,2,\ldots,2k-1\}$.

2. **Preliminary results**

We start this section with some notations and useful results. Let $T = \{T_1, \ldots, T_\mu\}$ be a μ-way (v,k,t) trade of volume m, and $x, y \in$ found(T). Then the number of blocks in T_i $(1 \leq i \leq \mu)$ which contains x is denoted by r_x and the number of blocks containing $\{x,y\}$ is denoted by λ_{xy}. The set of blocks in T_i $(1 \leq i \leq \mu)$ which contains $x \in$ found(T) is denoted by T_{ix} $(1 \leq i \leq \mu)$ and the set of remaining blocks by T'_{ix} $(1 \leq i \leq \mu)$. By applying a result in [12], we see that if $r_x < m$, then $T_x = \{T_{1x}, \ldots, T_{\mu x}\}$ is a μ-way $(v,k,t-1)$ trade of volume r_x, and furthermore $T'_x = \{T'_{1x}, \ldots, T'_{\mu x}\}$ is a μ-way $(v-1,k,t-1)$ trade of volume $m - r_x$. If we remove x from the blocks of T_x, then the result will be a μ-way $(v-1,k-1,t-1)$ trade which is called derived trade of T.
It is easy to show that if T is a Steiner trade then its derived trade is also a Steiner trade.
If $T = \{T_1, \ldots, T_\mu\}$ and $T^* = \{T_1^*, \ldots, T_\mu^*\}$ are two μ-way (v,k,t) trades. Then we define $T + T^* = \{T_1 \cup T_1^*, \ldots, T_\mu \cup T_\mu^*\}$. It is easy to see that $T + T^*$ is a μ-way (v,k,t) trade. If T and T^* are Steiner trades and found(T) $\cap$ found(T*) $= \phi$, then $T + T^*$ is also a Steiner trade.

Definition 2.1. *Let $T = \{T_1, T_2, \ldots, T_\mu\}$ be a μ-way (v,k,t) Steiner trade. We say T is t-solely balanced if T_i and T_j $(1 \leq i < j \leq \mu)$ contain no common $(t+1)-$subset.*

The following theorem will be used repeatedly in the sequel.
Theorem 2.2. *(i) Let $T = \{T_1, T_2, \ldots, T_\mu\}$ be a μ-way (v,k,t) trade of volume m. Then, based on T, a μ-way $(v+\mu, k+1, t+1)$ trade T^* of volume μm can be constructed.*
(ii) If T is t-solely balanced, then T^* is a Steiner trade.

Proof. (i) Let x_1, x_2 and x_μ be μ new elements. Then we can construct the blocks of $T^* = \{T_1^*, T_2^*, \ldots, T_\mu^*\}$ as follows.

T_1^*	T_2^*	$\ldots$	T_μ^*
x_1T_1	x_1T_2	$\ldots$	x_1T_μ
x_2T_2	x_2T_3	$\ldots$	x_2T_1
x_3T_3	x_3T_4	$\ldots$	x_3T_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x_\mu T_\mu$	$x_\mu T_1$	$\ldots$	$x_\mu T_{\mu-1}$

Clearly T^* is a μ-way $(v+\mu, k+1, t+1)$ trade of volume μm.
(ii) It is obvious. □

In the next example, we show the existence of a 3-way $(v,3,2)$ Steiner trade of volume 6 from a 3-way $(v,2,1)$ Steiner trade of volume 2.
Example 2.3. *Let $T = \{T_1, T_2, T_3\}$ be the 3-way $(v,2,1)$ Steiner trade of volume 2.*

T_1	T_2	T_3
12	13	14
34	24	23

Now we can construct $T^* = \{T_1^*, T_2^*, T_3^*\}$ by the method of the previous Theorem.

T_1^*	T_2^*	T_3^*
$x12$	$x13$	$x14$
$x34$	$x24$	$x23$
$13z$	$12y$	$12z$
$24z$	$34y$	$34z$
$14y$	$14z$	$13y$
$23y$	$23z$	$24y$

Remark 2.4. The 3-way $(v,3,2)$ Steiner trade of volume 6 is unique. This trade is isomorphic to the 3-way $(7,3,2)$ Steiner trade of volume 6 which is constructed in Example 2.3.
Let T be a 3-way $(v,3,2)$ Steiner trade of volume 6. First assume that, for each $x \in$ found(T), $r_x > 2$. So x must appear at least 3 times in T_1. Let the first block of T_{1x} be $x12$. So 1 and 2 must appear at least two times in T'_{1x}, since $r_1, r_2 \geq 3$. Hence x, 1 and 2 should each appear twice more in different blocks which contradicts the Steiner property of T. So there exists $x \in$ found(T) such that $r_x = 2$. We know $T_x \setminus \{x\}$ is a 3-way $(v,2,1)$ Steiner trade. Therefore T_x can be expressed as:

T_{1x}	T_{2x}	T_{3x}
$x12$	$x13$	$x14$
$x34$	$x24$	$x23$

Thus the pairs 13, 24, 14 and 23 must appear in distinct blocks of T_1. Since T is a 3-way $(v,3,2)$ Steiner trade, we conclude that a 3-way $(v,3,2)$ Steiner trade of volume 6 has the following structure.

T_1	T_2	T_3
$x12$	$x13$	$x14$
$x34$	$x24$	$x23$
$13z$	$12y$	$12z$
$24z$	$34y$	$34z$
$14y$	$14z$	$13y$
$23y$	$23z$	$24y$

Theorem 2.5. $\mathcal{S}_\mu(2,k) \subseteq \mathbb{N} \setminus \{1,2,3,4,5\},\ k \geq 3.$

Proof. We know $\mathcal{S}_2(2,k) = \mathbb{N} \setminus \{1,2,3,5\}$ (see [12]). So $\mathcal{S}_\mu(2,k) \subseteq \mathbb{N} \setminus \{1,2,3,5\}$.
The $(v,k,2)$ trade of volume 4 has unique structure (see [12]). If there exists a 3-way $(v,k,2)$ trade $T = \{T_1,T_2,T_3\}$ of volume 4, then (T_1,T_2), (T_2,T_3) and (T_1,T_3) are three $(v,k,2)$ trades of volume 4 and it is a contradiction, because the structure of $(v,k,2)$ trade of volume 4 is unique. □

3. 3-way Steiner trades

In this section we characterize $\mathcal{S}_{3s}(1,k)$, $\mathcal{S}_{3s}(2,3)$ and $\mathcal{S}_{3s}(2,4)$. First, we state some of the results in [16] which are needed in the sequel.
Let $D(v,k)$ be the maximum number of $STS(v)$s that can be constructed on a set with cardinality v such that any two $STS(v)$s intersect exactly in the same k blocks.
Theorem 3.1. [16] $D(v, b_v - 7) = 2$ *for every* $v \geq 7$*;* $v \neq 9$.
Theorem 3.2. [5] *Any partial Steiner triple system of order v can be embedded in a Steiner triple system of order w if* $w \equiv 1,3 \pmod 6$ *and* $w \geq 2v+1$.
Theorem 3.3. $7 \notin \mathcal{S}_{3s}(2,3)$.

Proof. Let $T = \{T_1,T_2,T_3\}$ be a 3-way $(v,3,2)$ Steiner trade of volume 7. It is obvious that T_1 is a partial Steiner triple system. So by Theorem 3.2, T_1 can be embedded in a $STS(v') = D$, where $v' \geq 2|\text{found(T)}|+1 \geq 13$. Then D, $D' = D + T_2$ and $D'' = D + T_3$ are three $STS(v)$s which intersect in the same set of $b_v - 7$ blocks. But this is impossible by Theorem 3.1. □

Theorem 3.4. $\mathcal{S}_{3s}(1, k) = \mathbb{N} \setminus \{1\},\ k \geq 2$.

Proof. We know the complete graph K_{2m} has $2m - 1$ disjoint 1-factors. If we take three 1-factors $F_1,\ F_2$ and F_3 as $T_1,\ T_2$ and T_3 respectively, then $T = \{T_1, T_2, T_3\}$ is a 3-way $(2m, 2, 1)$ trade of volume m.

For $k \geq 3$, let T be a 3-way $(v, 2, 1)$ Steiner trade of volume m and A be a $(k-2)m$-set disjoint from found(T). Set a partition of A to $(k-2)$ subsets $A_1, \ldots, A_m$. Then by adding A_i $(1 \leq i \leq m)$ to the i-th block of T, we obtain a 3-way $(v, k, 1)$ Steiner trade. □

Example 3.5. *A 3-way* $(4, 2, 1)$ *Steiner trade of volume 2 is.*

T_1	T_2	T_3
13	14	12
24	23	34

A 3-way $(8, 4, 1)$ *Steiner trade of volume 2 is.*

T_1	T_2	T_3
$x_1x_2 13$	$x_1x_2 14$	$x_1x_2 12$
$x_3x_4 24$	$x_3x_4 23$	$x_3x_4 34$

Theorem 3.6. $\mathcal{S}_{3s}(2, 3) = \mathbb{N} \setminus \{1, 2, 3, 4, 5, 7\}$.

Proof. By Theorem 2.2 (ii) and Theorem 3.4 there exists a 3-way $(v, 3, 2)$ Steiner trade of volume $3m$ $(m \geq 2)$. Note that the 3-way $(2m, 2, 1)$ Steiner trades of volume m constructed in Theorem 3.4 are 1-solely balanced. The existence of a 3-way $(v, 3, 2)$ Steiner trade of volumes $3m+1$ and $3m+2$, can be proved by using the following two recursive relations:

(i) $3m + 1 = 3(m - 3) + 10 \quad m - 3 \geq 2$;

(ii) $3m + 2 = 3(m - 2) + 8 \quad m - 2 \geq 2$.

These constructions, together with 3-way $(v, 3, 2)$ Steiner trades of volumes: 8, 10, 11 and 13 suffice to prove the existence.

We can see a 3-way $(v, 3, 2)$ Steiner trade of volume 8 and 13 in Example 1.3.

Consider three $STS(v)$s intersecting in $b_v - m$ blocks, where $b_v = \frac{v(v-1)}{6}$. The remaining set of blocks form a 3-way $(v', 3, 2)$ Steiner trade of volume m. We know that there exist three $STS(9)$s which intersect in $b_9 - 11 = 12 - 11 = 1$ block and three $STS(v)$s which intersect in $b_v - 10$ blocks for $v \geq 19$ (see [17]). So $\{10, 11\} \subseteq \mathcal{S}_{3s}(2, 3)$.

The non-existence of Steiner trades of volumes $1, 2, 3, 4, 5$ and 7 can be concluded from Theorems 2.5 and 3.3. □

Theorem 3.7. $\mathcal{S}_{3s}(2,k) \subseteq \mathbb{N} \setminus \{1,2,...,2k-1\}$ *for* $k \geq 4$.

Proof. Let $T = \{T_1, T_2, T_3\}$ be a 3-way $(v,k,2)$ Steiner trade of volume m. Let for each $x \in$ found(T) $r_x \geq 3$, and $a_1, \ldots, a_k$ be a block in T_1. Corresponding to each a_i, there exist two other blocks in T_1, which contain a_i $(1 \leq i \leq k)$ but not a_j $(j \neq i)$ (Since T is a Steiner trade). T_1 must contain at least $2k+1$ blocks.
Now let there exists $x \in$ found(T), such that $r_x = 2$, then T_x is a 3-way $(v,k,1)$ Steiner trade. So (T_{1x}, T_{2x}) has the following form from [12].

T_{1x}	T_{2x}
$S_1S_3S_5$	$S_1S_4S_5$
$S_2S_4S_5$	$S_2S_3S_5$

with $S_i \subseteq V$ for $i = 1, \cdots, 5$. $|S_1| = |S_2| \geq 1$, $|S_3| = |S_4| \geq 1$, $S_i \cap S_j = \phi$ for all $i \neq j$, and $|S_1| + |S_3| + |S_5| = k$.
Since T is a 3-way $(v,k,2)$ Steiner trade and $r_x = 2$, therefore $S_5 = \{x\}$. Without loss of generality, let
$S_1S_3 = a_2a_3a_4 \ldots a_k$ and $S_2S_4 = b_2b_3b_4 \ldots b_k$. So there exists i such that
$S_1S_4 = a_2 \ldots a_ib_{i+1} \ldots b_k$ and $S_2S_3 = b_2 \ldots b_ia_{i+1} \ldots a_k$. Then corresponding to each pair a_pb_q and b_pa_q in T_2, $2 \leq p \leq i$ and $i+1 \leq q \leq k$, there must exist $2(i-1)(k-i)$ blocks in T_1.
We know that there does not exist a repetitive block in T_3. So a_2 must appear in T_3 with some $b_j, j \notin \{i+1, \ldots, k\}$ or with some $a_j, j \in \{i+1, \ldots, k\}$ (one block of T_{3x} contains a_2 and $b_{i+1} \ldots b_k$). In the first case we have at least one block for a_2b_j in T_1. if the second case happen, we have $k-i$ blocks for a_jb_r $r \in \{i+1, \ldots, k\}$. Therefore in two cases, there exists at least another block in T_1. We have the same situation for b_2. Then we have:

$$|T_1| \geq 2 + 2(i-1)(k-i) + 2 \geq 2k - 2 + 2 = 2k.$$

So the volume of 3-way $(v,k,2)$ Steiner trade is at least $2k$. □

The 3-way $(v,k,1)$ Steiner trades $(k \geq 3)$, which were constructed in Theorem 3.4, are not 1-solely balanced. But for $k = 3$ by using the idea of Kirkman triple systems, in the following theorem we introduce a 3-way $(v,3,1)$ Steiner trade 1-solely balanced.
Theorem 3.8. *There exists a 3-way* $(v,3,1)$ *Steiner trade which is 1-solely balanced of volume* m $(m \geq 3)$.

Proof. We know that, there exists a $KTS(v)$ if and only if $v \equiv 3 \pmod 6$ [6]. For $m = 2k+1$, consider a $KTS(3m)$. Let P_1, P_2, P_3, be three parallel classes of $KTS(3m)$. We can construct a 3-way $(v,3,1)$ Steiner trade 1-solely balanced of volume m as follows.

T_1	T_2	T_3
P_1	P_2	P_3

For $m = 2k$, consider two 3-way $(v,3,1)$ Steiner trades 1-solely balanced T and T' of odd volumes with disjoint foundations, then $T + T'$ is a 3-way $(v,3,1)$ Steiner trade 1-solely balanced of volume m, except for $m = 4$, which we handle below.

T_1	T_2	T_3
123	147	158
456	25a	24c
789	8b6	7b3
abc	39c	69a

□

Theorem 3.9. $\mathcal{S}_{3s}(2,4) = \mathbb{N} \setminus \{1,2,3,4,5,6,7\}$.

Proof. By Theorem 3.7, $\mathcal{S}_{3s}(2,4) \subseteq \mathbb{N} \setminus \{1,2,3,4,5,6,7\}$.
By Theorem 3.8 and Theorem 2.2 (ii), there exists a 3-way $(v,4,2)$ Steiner trade of volume $3m$ for $m \geq 3$.
The existence of a 3-way $(v,4,2)$ Steiner trade of volumes $3m+1$ and $3m+2$, can be proved by using the following two recursive relations:
$3m+1 = 3(m-3)+10 \quad m-3 \geq 3$;
$3m+2 = 3(m-2)+8 \quad m-2 \geq 3$.
These constructions, together with the 3-way $(v,4,2)$ Steiner trades of volumes: 8, 10, 11, 13, 14, 16 suffice to prove the existence.
Later we handle the trades of volumes $m = 8, 10, 11, 13, 14$ and 16 (see appendix). □

Example 3.10. In this example we construct a 3-way $(9,3,1)$ trade of volume 3 from a KTS(9). Then we obtain a 3-way $(12,4,2)$ Steiner trade of volume 9 from it.

KTS(9):

P_1	P_2	P_3	P_4
123	147	159	168
456	258	267	249
789	369	348	357

the 3-way $(9,3,1)$ trade of volume 3:

T_1	T_2	T_3
123	147	159
456	258	267
789	369	348

the 3-way $(12,4,2)$ Steiner trade of volume 9:

T_1	T_2	T_3
$x123$	$x147$	$x159$
$x456$	$x258$	$x267$
$x789$	$x369$	$x348$
$y147$	$y159$	$y123$
$y258$	$y267$	$y456$
$y369$	$y348$	$y789$
$z159$	$z123$	$z147$
$z267$	$z456$	$z258$
$z348$	$z789$	$z369$

4. **3-way** $(v,k,2)$ **trades**

In the previous section we observed that $\mathcal{S}_{3s}(1,k)=\mathbb{N}\setminus\{1\}$ for $k\geq 2$. So $\mathcal{S}_3(1,k)=\mathbb{N}\setminus\{1\}$ for $k\geq 2$. In this section we investigate the spectra $\mathcal{S}_3(2,3)$ and $\mathcal{S}_3(2,k)$.

Theorem 4.1. *If there exists a 3-way $(v,3,2)$ trade of volume 7, then it is a 3-way $(v,3,2)$ Steiner trade.*

Proof. Let $T=\{T_1,T_2,T_3\}$ be a 3-way $(v,3,2)$ trade of volume 7. We prove that there does not exist any pair $x,y\in$ found(T) with $\lambda_{xy}\geq 2$. First, Suppose that $\lambda_{xy}\geq 3$.

T_1	T_2	T_3
xyz_1	xyz_4	xyz_7
xyz_2	xyz_5	xyz_8
xyz_3	xyz_6	xyz_9
$---$	$---$	$---$
$---$	$---$	$---$
$---$	$---$	$---$
$---$	$---$	$---$

The pairs xz_i for $i = 4 \ldots 9$ must appear in the blocks of T_1. So the element x must appear three times more in T_1 and therefore $r_x \geq 6$. But $r_x \neq 6, 7$ for all $x \in$ found(T). Because T_x and T'_x are trades of volume r_x and $m - r_x$. We know that there does not exist any trade of volume one.
If $\lambda_{xy} = 2$ then T has the following form.

T_1	T_2	T_3
xyz_1	xyz_3	xyz_6
xyz_2	xyz_4	xyz_5
$---$	$---$	$---$
$---$	$---$	$---$
$---$	$---$	$---$
$---$	$---$	$---$
$---$	$---$	$---$

The pairs xz_i for $i = 3, \ldots, 6$ must appear in the blocks of T_1. So $r_x, r_y \geq 4$.

T_1	T_2	T_3
xyz_1	xyz_3	xyz_6
xyz_2	xyz_4	xyz_5
$x--$	$x--$	$x--$
$x--$	$x--$	$x--$
$y--$	$y--$	$y--$
$y--$	$y--$	$y--$
$---$	$---$	$---$

It is obvious that the 3rd and 4th blocks of T_1, also the 4th and 5th blocks of T_1 contain the elements z_i for $i = 3, \ldots, 6$. Hence the 7th block of T_1 contains z_1 and z_2, because the order of each element is at least two.
Now there exists an empty place in the last block of T_1. By the previous

reason, there does not exist any new element in this place. If one of the elements x, y and z_i, $i = 1, \dots, 6$ appears in this place (Name it w), then the pair $z_1 w$ must appear in the blocks of $T_2(T_3)$ which is impossible. $\square$

Theorem 4.2. $\mathcal{S}_3(2,3) = \mathbb{N} \setminus \{1,2,3,4,5,7\}$.

Proof. The conclusion follows from Theorems 4.1, 2.5, and 3.6. $\square$

Theorem 4.3. $\mathcal{S}_3(2,k)$ *contains* $\mathbb{N} \setminus \{1,2,3,4,5\}$, *except possibly 7.*

Proof. We have a 3-way $(v,3,2)$ trade of volume m, $m \in \mathbb{N} \setminus \{1,2,3,4,5,7\}$ from Theorem 4.2. Let A be a $(k-3)m$-set disjoint from found(T). Set a partition of A to $(k-3)$ subsets $A_1, \dots, A_m$. Then by adding A_i $(1 \le i \le m)$ to the i-th block of T, we obtain a 3-way $(v,k,2)$ trade of volume m, $m \in \mathbb{N} \setminus \{1,2,3,4,5,7\}$. The non-existence of 3-way $(v,k,2)$ trades of volume m, $m \in \{1,2,3,4,5\}$ follows from Theorem 2.5. $\square$

5. Appendix

The following trades are necessary in the proof of Theorem 3.9.

$m = 8$:

T_1	T_2	T_3
$124a$	$125b$	$124b$
1568	1468	$156c$
$17bc$	$17ac$	$17a8$
$235b$	$234a$	$235a$
$346c$	$356c$	3468
$378a$	$378b$	$37cb$
$489b$	$589a$	$4c9a$
$59ac$	$49bc$	$59b8$

$m = 10$:

T_1	T_2	T_3
0139	0238	$089c$
$028c$	$091c$	0123
$124a$	$987a$	$824a$
$17bc$	$84bc$	$87b3$
$235b$	$935b$	$295b$
2679	9642	$267c$
$346c$	$376c$	9463
$378a$	$341a$	$971a$
$489b$	$127b$	$41cb$
$59ac$	$52ac$	$5ca3$

$m = 11:$

T_1	T_2	T_3
$028c$	$025c$	0286
0457	0468	$045b$
$06ab$	$07ab$	$07ac$
1568	1675	1578
$17bc$	$18bc$	$1bc6$
$235b$	$236b$	$235c$
2679	2789	$27b9$
$346c$	$347c$	3476
$378a$	$385a$	$3b8a$
$489b$	$459b$	$89c4$
$59ac$	$69ac$	$596a$

$m = 13:$

T_1	T_2	T_3
0139	$149c$	0739
$028c$	$248c$	$328c$
0457	$04b7$	3451
$06ab$	$46a5$	$36ab$
$124a$	$18a0$	$724b$
1568	$1b62$	7568
$17bc$	$157c$	$17ac$
$235b$	$835b$	$205a$
2679	8679	2691
$346c$	$306c$	$046c$
$378a$	$372a$	$018b$
$489b$	0295	$489a$
$59ac$	$b9ac$	$59bc$

$m = 14:$

T_1	T_2	T_3
0456	1456	2456
$28ad$	$08ed$	$18ad$
$37be$	$37ba$	$37fe$
$19cf$	$29cf$	$09cb$
0789	1789	2789
$15bd$	$25bd$	$05fd$
$24ce$	$04ca$	$14ce$
$36af$	$36ef$	$36ab$
$0abc$	$1abc$	$2afc$
$68e1$	$268a$	$068e$
$257f$	$057f$	$157b$
$0def$	$1dfa$	$2deb$
$147a$	$247e$	$047a$
$269b$	$069b$	$169f$

$m = 16:$

T_1	T_2	T_3
0456	1456	0856
$28ad$	$38ad$	$2bad$
$37be$	$07be$	$37fe$
$19cf$	$29cf$	$19c4$
0789	1789	$07b9$
$15bd$	$25bd$	$15fd$
$24ce$	$34ce$	$28ce$
$36af$	$06af$	$36a4$
$0abc$	$1abc$	$0afc$
$68e1$	$268e$	$16be$
$257f$	$357f$	2574
$349d$	$049d$	$38d9$
$0def$	$1def$	$0de4$
$147a$	$247a$	$187a$
$269b$	$369b$	$269f$
$358c$	$058c$	$35bc$

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(Saeedeh Rashidi) Department of Mathematics, Alzahra University, P.O. Box 19834, Tehran, Iran
E-mail address: s.rashidi@alzahra.ac.ir

(Nasrin Soltankhah) Department of Mathematics, Alzahra University, P.O. Box 19834, Tehran, Iran
E-mail address: soltan@alzahra.ac.ir