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FEKETE-SZEGÖ COEFFICIENT FUNCTIONAL FOR TRANSFORMS OF UNIVERSALLY PRESTARLIKE FUNCTIONS

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Abstract. Universally prestarlike functions of order $\alpha \leq 1$ in the slit domain $\Lambda = \mathbb{C} \setminus [1, \infty)$ have been recently introduced by S. Ruscheweyh.This notion generalizes the corresponding one for functions in the unit disk Δ (and other circular domains in $\mathbb{C}$). In this paper, we obtain the Fekete-Szegö coefficient functional for transforms of such functions.
Keywords: Prestarlike functions, universally prestarlike functions, Fekete-Szegö finctional.
MSC(2010): Primary: 30C45.

1. Introduction

Let $H(\Omega)$ denote the set of all analytic functions defined in a domain Ω. For a domain Ω containing the origin, $H_0(\Omega)$ stands for the set of all function $f \in H(\Omega)$ with $f(0) = 1$. We also use the notation $H_1(\Omega) = \{zf : f \in H_0(\Omega)\}$. In the special case when Ω is the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, we use the abbreviations H, H_0 and H_1 respectively for $H(\Omega), H_0(\Omega)$ and $H_1(\Omega)$. A function $f \in H_1$ is called starlike of order α with $(0 \leq \alpha < 1)$ if f satisfies the inequality

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$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \Delta), \tag{1.1}$$

the set of all such functions is denoted by S_α. The convolution or Hadamard Product of two functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

A function $f \in H_1$ is called prestarlike of order α if

$$\frac{z}{(1-z)^{2-2\alpha}} * f(z) \in S_\alpha. \tag{1.2}$$

The set of all such functions is denoted by $\mathcal{R}_\alpha$. The notion of prestarlike functions has been extended from the unit disk to other disks and half planes containing the origin by Ruscheweyh and Salinas [6]. Let Ω be one such disk or half plane. Then there are two unique parameters $\gamma \in \mathbb{C} \setminus \{0\}$ and $\rho \in [0, 1]$ such that

$$\Omega_{\gamma,\rho} = \{w_{\gamma,\rho}(z) : z \in \Delta\} \tag{1.3}$$

where, $w_{\gamma,\rho}(z) = \dfrac{\gamma z}{1 - \rho z}$. Note that $1 \notin \Omega_{\gamma,\rho}$ if and only if $|\gamma + \rho| \leq 1$.

Definition 1.1. [5, 6, 7] *Let $\alpha \leq 1$, and $\Omega = \Omega_{\gamma,\rho}$ for some admissible pair (γ, ρ). A function $f \in H_1(\Omega_{\gamma,\rho})$ is called prestarlike of order α in $\Omega_{\gamma,\rho}$ if*

$$f_{\gamma,\rho}(z) = \frac{1}{\gamma} f(w_{\gamma,\rho}(z)) \in \mathcal{R}_\alpha. \tag{1.4}$$

The set of all such functions f is denoted by $\mathcal{R}_\alpha(\Omega)$.

Let Λ be the slit domain $\mathbb{C} \setminus [1, \infty)$(the slit being along the positive real axis).

Definition 1.2. [5, 6, 7] *Let $\alpha \leq 1$. A function $f \in H_1(\Lambda)$ is called universally prestarlike of order α if and only if f is prestarlike of order α in all sets $\Omega_{\gamma,\rho}$ with $|\gamma + \rho| \leq 1$. The set of all such functions is denoted by $\mathcal{R}^u_\alpha$.*

For a univalent function $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.5}$$

the k^{th} root transform is defined as

$$F(z) = [f(z^k)]^{\frac{1}{k}} := z + \sum_{n=1}^{\infty} d_{kn+1} z^{kn+1} \tag{1.6}$$

$k \in N = \{1, 2, \ldots\}$.

Definition 1.3. [7, 9] *Let $\phi(z)$ be an analytic function with positive real part on Δ, which satisfies $\phi(0) = 1$, $\phi'(0) > 0$ and which maps the unit disc Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Then the class $\mathcal{R}_\alpha^u(\phi)$ consists of all analytic functions $f \in H_1(\Lambda)$ satisfying*

$$\frac{D^{3-2\alpha} f(z)}{D^{2-2\alpha} f(z)} \prec \phi(z). \tag{1.7}$$

where $\prec$ denotes the subordination, and where $(D^\beta f)(z) = \frac{z}{(1-z)^\beta} \star f$, for $\beta \geq 0$. In particular, for $\beta = n \in \mathrm{N}$, we have $D^{n+1} f = \frac{z}{n!}(z^{n-1} f)^{(n)}$.

We let $\mathcal{R}_\alpha^u(A, B)$ denote the class $\mathcal{R}_\alpha^u(\phi)$ where $\phi(z) = \dfrac{1 + Az}{1 + Bz}$ $(-1 \leq B < A \leq 1)$. For suitable choices of A,B,α the class $\mathcal{R}_\alpha^u(A, B)$ reduces to several well known classes of functions. For instance, $\mathcal{R}_{\frac{1}{2}}^u(1, -1)$ is the class $S^* = S_0$ of starlike univalent functions.
In this section, sharp bounds for the Fekete-Szegö coefficient functional $|d_{2k+1} - \mu d_{k+1}^2|$ associated with the k^{th} root transform of the functions belonging to the class $\mathcal{R}_\alpha^u(\phi)$ are found. In particular cases, these bounds reduce to results of [1, 8].

Remark 1.4. [7] *Let* $F(z) = \sum_{k=0}^{\infty} a_k z^k = \int_0^1 \frac{d\mu(t)}{1 - tz}$ where $a_k = \int_0^1 t^k d\mu(t)$, *and $\mu(t)$ is a probability measure on $[0, 1]$. Let T denote the set of all such functions F. They are analytic in the slit domain Λ.*

To prove our result we need the following theorems.

Theorem 1.5. [7] *Let* $0 \leq \alpha \leq 1$ *and* $f \in H_1(\Lambda)$. *Then* $f \in \mathcal{R}_\alpha^u$ *if and only if*

$$\frac{D^{3-2\alpha}f}{D^{2-2\alpha}f} \in T. \tag{1.8}$$

This admits an explicit representation of the functions in $\mathcal{R}_\alpha^u$. If $f \in H_0$ has all its Taylor coefficients at the origin different from zero we write $f^{(-1)}$ for the (possibly formal but) unique solution of $f * f^{(-1)} = \frac{1}{1-z}$.

Theorem 1.6. [8] *Let f be a universally prestarlike function of order* $\alpha \leq 1$, *then the function $f(z)$ has a representation of the form*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

for $n = 2, 3, \ldots$ *where,*

$$a_n = \left\{ \frac{\sum_{k=1}^{n-1} \mathbb{C}(\alpha, k) a_k b_{n-k}}{\mathbb{C}'(\alpha, n) - \mathbb{C}(\alpha, n)} \right\} \tag{1.9}$$

$$\mathbb{C}(\alpha, n) = \frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!}, \ \mathbb{C}(\alpha, k) = \frac{\prod_{m=2}^{k}(m-2\alpha)}{(k-1)!}, \mathbb{C}(\alpha, 1)a_1 = 1$$

$$\mathbb{C}'(\alpha, n) = \frac{\prod_{k=2}^{n}(k+1-2\alpha)}{(n-1)!}, \ b_n = \int_0^1 t^n d\mu(t)$$

and $\mu(t)$ *is a probability measure on* $[0, 1]$.

Let Ω_1 be the class of analytic functions ω, normalized by $\omega_1(0) = 0$, satisfying the condition $|\omega_1(z)| < 1$. The following two lemmas regarding the coefficients of functions in Ω_1 are needed to prove our main results. The following lemma is a reformulation of the corresponding result for functions with positive real part due to Ma and Minda [4].

Lemma 1.7. [2] *If* $\omega \in \Omega_1$ *and*

$$\omega(z) = \omega_1 z + \omega_2 z^2 + \cdots, (z \in \Delta) \tag{1.10}$$

then,

$$|\omega_2 - t\omega_1^2| \leq \begin{cases} -t, & t \leq -1 \\ 1, & -1 \leq t \leq 1 \\ t, & t \geq 1. \end{cases}$$

For $t < -1$, or $t > 1$, the equality holds if and only if $\omega(z) = z$ or one of its rotations. For $-1 < t < 1$, the equality holds if and only if $\omega(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if $\omega(z) = z\dfrac{\lambda + z}{1 + \lambda z}(0 \leq \lambda \leq 1)$ or one of its rotations, while for $t = 1$, equality holds if and only if $\omega(z) = -z\dfrac{\lambda + z}{1 + \lambda z}(0 \leq \lambda \leq 1)$ or one of its rotations.

Lemma 1.8. [3] *If $\omega \in \Omega_1$, then $|\omega_2 - t\omega_1^2| \leq max\{1, |t|\}$, for any complex number t. The result is sharp for the function $\omega(z) = z^2$ or z.*

We now estimate the sharp bound for the coefficient functional $|d_{2k+1} - \mu d_{k+1}^2|$ corresponding to the k^{th} root transformation of universally prestarlike functions of order α with respect to ϕ.

2. Coefficient bounds for the k^{th} root transformation

Theorem 2.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, and*

$$\sigma_1 = \frac{-k}{(3-2\alpha)B_1} + \frac{kB_2}{(3-2\alpha)B_1^2} + \frac{k(2-2\alpha)}{(3-2\alpha)} - \frac{k}{2} + \frac{1}{2}$$

$$\sigma_2 = \frac{k}{(3-2\alpha)B_1} + \frac{kB_2}{(3-2\alpha)B_1^2} + \frac{k(2-2\alpha)}{(3-2\alpha)} - \frac{k}{2} + \frac{1}{2}.$$

$$t = \frac{-B_2}{B_1} - (2-2\alpha)B_1 + (3-2\alpha)B_1\left[\frac{1}{2} - \frac{1}{2k} + \frac{\mu}{k}\right]$$

If $f \in \mathcal{R}_\alpha^u(\phi)$ and F is the k^{th} root transformation of f given by (1.5), then,

$$|d_{2k+1} - \mu d_{k+1}^2| \leq \begin{cases} -\dfrac{B_1}{(3-2\alpha)k}t, & \mu \leq \sigma_1 \\ \dfrac{B_1}{(3-2\alpha)k}, & \sigma_1 \leq \mu \leq \sigma_2 \\ \dfrac{B_1}{(3-2\alpha)k}t, & \mu \geq \sigma_2, \end{cases}$$

and when μ is a complex number

$$\left|d_{2k+1} - \mu d_{k+1}^2\right| \leq \frac{B_1}{(3-2\alpha)k} max\left\{1, |t|\right\}.$$

Proof. If $f \in \mathcal{R}_\alpha^u(\phi)$, then there is an analytic function $\omega \in \Omega_1$ of the required form such that

$$\frac{D^{3-2\alpha}f(z)}{D^{2-2\alpha}f(z)} = \phi(\omega(z)). \tag{2.1}$$

We know that, $\dfrac{D^{3-2\alpha}f(z)}{D^{2-2\alpha}f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n$ where $b_n = \int_0^1 t^n d\mu(t)$ and $\mu(t)$ is a probability measure on $[0,1]$, and

$$\phi(\omega(z)) = 1 + B_1\omega_1 z + (B_1\omega_2 + B_2\omega_1^2)z^2 + \cdots.$$

Therefore,

$$1 + b_1 z + b_2 z^2 + \cdots = 1 + B_1\omega_1 z + (B_1\omega_2 + B_2\omega_1^2)z^2 + \cdots$$

Now, equating the coefficients of z and z^2 we get

$$b_1 = B_1\omega_1,\ b_2 = B_1\omega_2 + B_2\omega_1^2 \tag{2.2}$$

Now, $\dfrac{D^{3-2\alpha}f(z)}{D^{2-2\alpha}f(z)} = 1 + \left[\mathbb{C}'(\alpha,2)a_2 - \mathbb{C}(\alpha,2)a_2\right] z +$
$\left[\mathbb{C}'(\alpha,3)a_3 - \mathbb{C}(\alpha,2)\mathbb{C}'(\alpha,2)a_2^2 - \mathbb{C}(\alpha,3)a_3 + (\mathbb{C}(\alpha,2)a_2)^2\right] z^2 + \cdots$

$$= 1 + b_1 z + b_2 z^2 + \cdots$$

where, $\mathbb{C}(\alpha,n) = \dfrac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!}$, $\mathbb{C}'(\alpha,n) = \dfrac{\prod_{k=2}^{n}(k+1-2\alpha)}{(n-1)!}$,
$b_n = \int_0^1 t^n d\mu(t)$ for $n = 2,3,\ldots$ and $\mu(t)$ is a probability measure on $[0,1]$.
Equating the coefficients of z and z^2 respectively and simplifying we get,

$$a_2 = b_1 \quad ; \quad a_3 = \frac{b_2 + (2-2\alpha)b_1^2}{(3-2\alpha)} \tag{2.3}$$

Now, using (2.2) in (2.3) we get,

$$a_2 = B_1\omega_1 \quad ; \quad a_3 = \frac{B_1\omega_2 + \left(B_2 + (2-2\alpha)B_1^2\right)\omega_1^2}{(3-2\alpha)} \tag{2.4}$$

Now, for a function f, a computation shows that

$$[f(z^k)]^{\frac{1}{k}} = z + \frac{a_2}{k} z^{k+1} + \left(\frac{a_3}{k} - \frac{(k-1)a_2^2}{2k^2} \right) z^{2k+1} + \cdots \tag{2.5}$$

Now , by using (2.5) in (1.6) and equating the coefficients of z and z^2 we get,

$$d_{k+1} = \frac{a_2}{k} \qquad ; \qquad d_{2k+1} = \frac{a_3}{k} - \frac{(k-1)a_2^2}{2k^2} \tag{2.6}$$

Now, using (2.4) in (2.6) we get,

$$d_{k+1} = \frac{B_1 \omega_1}{k}$$

and

$$d_{2k+1} = \frac{1}{k} \left[\frac{B_1 \omega_2 + \left(B_2 + (2-2\alpha) B_1^2 \right) \omega_1^2}{(3-2\alpha)} - \frac{B_1^2 \omega_1^2}{2} + \frac{B_1^2 \omega_1^2}{2k} \right]$$

Now,

$$d_{2k+1} - \mu d_{k+1}^2 = \frac{1}{k} \left[\frac{B_1 \omega_2 + \left(B_2 + (2-2\alpha) B_1^2 \right) \omega_1^2}{(3-2\alpha)} - \frac{B_1^2 \omega_1^2}{2} + \frac{B_1^2 \omega_1^2}{2k} \right] - \mu \frac{B_1^2 \omega_1^2}{k^2}$$

and hence

$$d_{2k+1} - \mu d_{k+1}^2 = \frac{B_1}{(3-2\alpha)k} [\omega_2 - \omega_1^2 t]$$

The first result is established by an application of Lemma (1.7)
If $t \leq -1$, then,

$$\mu \leq \frac{-k}{(3-2\alpha)B_1} + \frac{kB_2}{(3-2\alpha)B_1^2} + \frac{k(2-2\alpha)}{(3-2\alpha)} - \frac{k}{2} + \frac{1}{2} \qquad (\mu \leq \sigma_1),$$

and Lemma (1.7) gives:

$$|d_{2k+1} - \mu d_{k+1}^2| \leq -\frac{B_1}{(3-2\alpha)k} t.$$

For $-1 \leq t \leq 1$, we have $\sigma_1 \leq \mu \leq \sigma_2$, where

$$\sigma_1 = \frac{-k}{(3-2\alpha)B_1} + \frac{kB_2}{(3-2\alpha)B_1^2} + \frac{k(2-2\alpha)}{(3-2\alpha)} - \frac{k}{2} + \frac{1}{2},$$

$$\sigma_2 = \frac{k}{(3-2\alpha)B_1} + \frac{kB_2}{(3-2\alpha)B_1^2} + \frac{k(2-2\alpha)}{(3-2\alpha)} - \frac{k}{2} + \frac{1}{2}.$$

and Lemma (1.7) yields:

$$|d_{2k+1} - \mu d_{k+1}^2| \leq \frac{B_1}{(3-2\alpha)k}.$$

For $t \geq 1$, we have,

$$\mu \geq \frac{k}{(3-2\alpha)B_1} + \frac{kB_2}{(3-2\alpha)B_1^2} + \frac{k(2-2\alpha)}{(3-2\alpha)} - \frac{k}{2} + \frac{1}{2} \qquad (\mu \geq \sigma_2),$$

and it follows from Lemma (1.7) that

$$|d_{2k+1} - \mu d_{k+1}^2| \leq \frac{B_1}{(3-2\alpha)k}t.$$

For the sharpness of the results in the above theorem we have the following:

(1) If $\mu = \sigma_1$, then the equality holds in the Lemma (1.7) if and only if
$\omega(z) = z\dfrac{\lambda + z}{1 + \lambda z}(0 \leq \lambda \leq 1)$ or one of its rotations.

(2) If $\mu = \sigma_2$, then $\omega(z) = -z\dfrac{\lambda + z}{1 + \lambda z}(0 \leq \lambda \leq 1)$ or one of its rotations.

(3) If $\sigma_1 < \mu < \sigma_2$, then $\omega(z) = z^2$.

The second result follows by an application of Lemma (1.8) □

Remark 2.2. *For $k = 1$, the k^{th} root transformation of f reduces to the given function f itself. Thus the estimate given in the above theorem is an extension of the corresponding result for the Fekete-Szegö functional corresponding to universally prestarlike functions of order α with respect to ϕ which was already proved in* [8].

Remark 2.3. *Taking $\alpha = \dfrac{1}{2}$ the class $\mathcal{R}_\alpha^u(\phi)$ becomes the class of starlike univalent functions with respect to ϕ and Theorem (2.1) reduces to Theorem (2.1) of Ali, Lee, V. Ravichandran, S. Supramaniam* [1].

Remark 2.4. *In view of the Alexander result that f is a convex functions with respect to ϕ if and only if zf' is a starlike function with respect to ϕ, the estimate for $|d_{2k+1} - \mu d_{k+1}^2|$ for a convex function with respect to ϕ can be obtained from the corresponding estimate for starlike function with respect to ϕ.*

Remark 2.5. *If* $\alpha = \frac{1}{2}$ *and* $k = 1$ *in Theorem (2.1) we get the Fekete-Szegö coefficient functional corresponding to starlike function with respect to* [1].

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