Title:
Fekete-Szegö coefficient functional for transforms of universally prestarlike functions

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FEKETE-SZEGÖ COEFFICIENT FUNCTIONAL FOR TRANSFORMS OF UNIVERSALLY PRESTARLIKE FUNCTIONS

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(Communicated by Javad Mashreghi)

Abstract. Universally prestarlike functions of order $\alpha \leq 1$ in the slit domain $\Lambda = \mathbb{C} \setminus [1, \infty)$ have been recently introduced by S. Ruscheweyh. This notion generalizes the corresponding one for functions in the unit disk $\Delta$ (and other circular domains in $\mathbb{C}$). In this paper, we obtain the Fekete-Szegö coefficient functional for transforms of such functions.

Keywords: Prestarlike functions, universally prestarlike functions, Fekete-Szegö functional.

MSC(2010): Primary: 30C45.

1. Introduction

Let $H(\Omega)$ denote the set of all analytic functions defined in a domain $\Omega$. For a domain $\Omega$ containing the origin, $H_0(\Omega)$ stands for the set of all function $f \in H(\Omega)$ with $f(0) = 1$. We also use the notation $H_1(\Omega) = \{zf : f \in H_0(\Omega)\}$. In the special case when $\Omega$ is the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, we use the abbreviations $H, H_0$ and $H_1$ respectively for $H(\Omega), H_0(\Omega)$ and $H_1(\Omega)$. A function $f \in H_1$ is called starlike of order $\alpha$ with $(0 \leq \alpha < 1)$ if $f$ satisfies the inequality

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\( (1.1) \quad \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in \Delta), \)

the set of all such functions is denoted by \( S_\alpha \). The convolution or Hadamard Product of two functions \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) is defined as

\[ (f \ast g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \]

A function \( f \in H_1 \) is called prestarlike of order \( \alpha \) if

\( (1.2) \quad \frac{z}{(1-z)^2 - 2\alpha} \ast f(z) \in S_\alpha. \)

The set of all such functions is denoted by \( \mathcal{R}_\alpha \). The notion of prestarlike functions has been extended from the unit disk to other disks and half planes containing the origin by Ruscheweyh and Salinas [6]. Let \( \Omega \) be one such disk or half plane. Then there are two unique parameters \( \gamma \in \mathbb{C} \setminus \{0\} \) and \( \rho \in [0, 1] \) such that

\( (1.3) \quad \Omega_{\gamma, \rho} = \{w_{\gamma, \rho}(z) : z \in \Delta\} \)

where, \( w_{\gamma, \rho}(z) = \frac{\gamma z}{1 - \rho z} \). Note that \( 1 \notin \Omega_{\gamma, \rho} \) if and only if \( |\gamma + \rho| \leq 1 \).

**Definition 1.1.** [5, 6, 7] Let \( \alpha \leq 1 \), and \( \Omega = \Omega_{\gamma, \rho} \) for some admissible pair \((\gamma, \rho)\). A function \( f \in H_1(\Omega_{\gamma, \rho}) \) is called prestarlike of order \( \alpha \) in \( \Omega_{\gamma, \rho} \) if

\( (1.4) \quad f_{\gamma, \rho}(z) = \frac{1}{\gamma} f(w_{\gamma, \rho}(z)) \in \mathcal{R}_\alpha. \)

The set of all such functions \( f \) is denoted by \( \mathcal{R}_\alpha(\Omega) \).

Let \( \Lambda \) be the slit domain \( \mathbb{C} \setminus [1, \infty) \) (the slit being along the positive real axis).

**Definition 1.2.** [5, 6, 7] Let \( \alpha \leq 1 \). A function \( f \in H_1(\Lambda) \) is called universally prestarlike of order \( \alpha \) if and only if \( f \) is prestarlike of order \( \alpha \) in all sets \( \Omega_{\gamma, \rho} \) with \( |\gamma + \rho| \leq 1 \). The set of all such functions is denoted by \( \mathcal{R}^u_\alpha \).
For a univalent function \( f(z) \) of the form
\[
(1.5) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]
the \( k^{th} \) root transform is defined as
\[
(1.6) \quad F(z) = \left[ f(z^k) \right]^{\frac{1}{k}} := z + \sum_{n=1}^{\infty} d_{kn+1} z^{kn+1}
\]
k \( \in \mathbb{N} = \{1, 2, \ldots\} \).

**Definition 1.3.** \([7, 9]\) Let \( \phi(z) \) be an analytic function with positive real part on \( \Delta \), which satisfies \( \phi(0) = 1 \), \( \phi'(0) > 0 \) and which maps the unit disc \( \Delta \) onto a region starlike with respect to 1 and symmetric with respect to the real axis. Then the class \( \mathcal{R}_u^u(\phi) \) consists of all analytic functions \( f \in H_1(\Lambda) \) satisfying
\[
(1.7) \quad \frac{D^{\beta} f(z)}{D^{2\alpha} f(z)} < \phi(z).
\]
where \( \prec \) denotes the subordination, and where \( (D^\beta f)(z) = \frac{z}{(1-z)^\beta} \ast f \), for \( \beta \geq 0 \). In particular, for \( \beta = n \in \mathbb{N} \), we have \( D^{n+1} f = \frac{z}{n} (z^{n-1} f)' \).

We let \( \mathcal{R}_u^u(A, B) \) denote the class \( \mathcal{R}_u^u(\phi) \) where \( \phi(z) = \frac{1 + Az}{1 + Bz} \) \( (-1 \leq B < A \leq 1) \). For suitable choices of \( A, B, \alpha \) the class \( \mathcal{R}_u^u(A, B) \) reduces to several well known classes of functions. For instance, \( \mathcal{R}_u^u(1, -1) \) is the class \( S^* = S_0 \) of starlike univalent functions.

In this section, sharp bounds for the Fekete-Szegö coefficient functional \( |d_{2k+1} - \mu d_{k+1}^2| \) associated with the \( k^{th} \) root transform of the functions belonging to the class \( \mathcal{R}_u^u(\phi) \) are found. In particular cases, these bounds reduce to results of \([1, 8]\).

**Remark 1.4.** \([7]\) Let \( F(z) = \sum_{k=0}^{\infty} a_k z^k = \int_0^1 \frac{d\mu(t)}{1-tz} \) where \( a_k = \int_0^1 t^k d\mu(t) \), and \( \mu(t) \) is a probability measure on \([0, 1]\). Let \( T \) denote the set of all such functions \( F \). They are analytic in the slit domain \( \Lambda \).

To prove our result we need the following theorems.
Theorem 1.5. [7] Let $0 \leq \alpha \leq 1$ and $f \in H_1(\Lambda)$. Then $f \in \mathcal{R}_\alpha^u$ if and only if

\[
D^{3-2\alpha} f
\]

satisfies

\[
D^{2-2\alpha} f \in T.
\]

This admits an explicit representation of the functions in $\mathcal{R}_\alpha^u$. If $f \in H_0$ has all its Taylor coefficients at the origin different from zero we write $f^{(-1)}$ for the (possibly formal but) unique solution of $f \ast f^{(-1)} = \frac{1}{1-z}$.

Theorem 1.6. [8] Let $f$ be a universally prestarlike function of order $\alpha \leq 1$, then the function $f(z)$ has a representation of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

for $n = 2, 3, \ldots$ where,

\[
a_n = \left\{ \sum_{k=1}^{n-1} \frac{C(\alpha, k) a_k b_{n-k}}{C'(\alpha, n) - C(\alpha, n)} \right\}
\]

\[
C(\alpha, n) = \frac{\prod_{k=2}^{n} (k-2\alpha)}{(n-1)!}, \quad C(\alpha, k) = \frac{\prod_{m=2}^{k} (m-2\alpha)}{(k-1)!}, \quad C(\alpha, 1)a_1 = 1
\]

\[
C'(\alpha, n) = \frac{\prod_{k=2}^{n} (k+1-2\alpha)}{(n-1)!}, \quad b_n = \int_0^1 t^n d\mu(t)
\]

and $\mu(t)$ is a probability measure on $[0, 1]$.

Let $\Omega_1$ be the class of analytic functions $\omega$, normalized by $\omega_1(0) = 0$, satisfying the condition $|\omega_1(z)| < 1$. The following two lemmas regarding the coefficients of functions in $\Omega_1$ are needed to prove our main results. The following lemma is a reformulation of the corresponding result for functions with positive real part due to Ma and Minda [4].

Lemma 1.7. [2] If $\omega \in \Omega_1$ and

\[
\omega(z) = \omega_1 z + \omega_2 z^2 + \cdots, (z \in \Delta)
\]
then,

$$|\omega_2 - t\omega_1^2| \leq \begin{cases} -t, & t \leq -1 \\ 1, & -1 \leq t \leq 1 \\ t, & t \geq 1. \end{cases}$$

For $t < -1$, or $t > 1$, the equality holds if and only if $\omega(z) = z$ or one of its rotations. For $-1 < t < 1$, the equality holds if and only if $\omega(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if $\omega(z) = z\frac{\lambda + z}{1 + \lambda z}$ ($0 \leq \lambda \leq 1$) or one of its rotations, while for $t = 1$, equality holds if and only if $\omega(z) = -z\frac{\lambda + z}{1 + \lambda z}$ ($0 \leq \lambda \leq 1$) or one of its rotations.

**Lemma 1.8.** [3] If $\omega \in \Omega_1$, then $|\omega_2 - t\omega_1^2| \leq \max\{1, |t|\}$, for any complex number $t$. The result is sharp for the function $\omega(z) = z^2$ or $z$.

We now estimate the sharp bound for the coefficient functional $|d_{2k+1} - \mu d_{k+1}^2|$ corresponding to the $k^{th}$ root transformation of universally prestarlike functions of order $\alpha$ with respect to $\phi$.

2. Coefficient bounds for the $k^{th}$ root transformation

**Theorem 2.1.** Let $\phi(z) = 1 + B_1z + B_2z^2 + \cdots$, and

$$\sigma_1 = \frac{-k}{(3 - 2\alpha)B_1} + \frac{kB_2}{(3 - 2\alpha)B_1^2} + \frac{k(2 - 2\alpha)}{(3 - 2\alpha)} - \frac{k}{2} + \frac{1}{2},$$

$$\sigma_2 = \frac{k}{(3 - 2\alpha)B_1} + \frac{kB_2}{(3 - 2\alpha)B_1^2} + \frac{k(2 - 2\alpha)}{(3 - 2\alpha)} - \frac{k}{2} + \frac{1}{2},$$

$$t = \frac{-B_2}{B_1} - (2 - 2\alpha)B_1 + (3 - 2\alpha)B_1 \left[\frac{1}{2} - \frac{1}{2k} + \frac{\mu}{k}\right].$$

If $f \in \mathcal{R}_\alpha^u(\phi)$ and $F$ is the $k^{th}$ root transformation of $f$ given by (1.5), then,

$$|d_{2k+1} - \mu d_{k+1}^2| \leq \begin{cases} -\frac{B_1}{(3 - 2\alpha)k}t, & \mu \leq \sigma_1 \\ \frac{B_1}{(3 - 2\alpha)k}, & \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{B_1}{(3 - 2\alpha)k}, & \mu \geq \sigma_2, \end{cases}$$
and when $\mu$ is a complex number
\[ |d_{2k+1} - \mu d_{k+1}^2| \leq \frac{B_1}{(3-2\alpha)^k} \max \{1, |t|\}. \]

Proof. If $f \in \mathcal{R}_n^\omega(\phi)$, then there is an analytic function $\omega \in \Omega_1$ of the required form such that
\[ \frac{D^{3-2\alpha}f(z)}{D^{2-2\alpha}f(z)} = \phi(\omega(z)). \]

We know that, \[ \frac{D^{3-2\alpha}f(z)}{D^{2-2\alpha}f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \] where \( b_n = \int_0^1 t^n d\mu(t) \) and $\mu(t)$ is a probability measure on $[0,1]$, and
\[ \phi(\omega(z)) = 1 + B_1 \omega_1 z + (B_1 \omega_2 + B_2 \omega_1^2) z^2 + \cdots. \]

Therefore,
\[ 1 + b_1 z + b_2 z^2 + \cdots = 1 + B_1 \omega_1 z + (B_1 \omega_2 + B_2 \omega_1^2) z^2 + \cdots. \]

Now, equating the coefficients of $z$ and $z^2$ we get
\[ b_1 = B_1 \omega_1, \quad b_2 = B_1 \omega_2 + B_2 \omega_1^2 \]

Now,
\[ \frac{D^{3-2\alpha}f(z)}{D^{2-2\alpha}f(z)} = 1 + \left[ \mathcal{C}'(\alpha, 2)a_2 - \mathcal{C}(\alpha, 2)a_2 \right] z + \]
\[ \left[ \mathcal{C}'(\alpha, 3)a_3 - \mathcal{C}(\alpha, 2)\mathcal{C}'(\alpha, 2)a_2^2 - \mathcal{C}(\alpha, 3)a_3 + (\mathcal{C}(\alpha, 2)a_2)^2 \right] z^2 + \cdots \]
\[ = 1 + b_1 z + b_2 z^2 + \cdots \]

where, \( \mathcal{C}(\alpha, n) = \frac{\prod_{k=2}^{n} (k-2\alpha)}{(n-1)!}, \quad \mathcal{C}'(\alpha, n) = \frac{\prod_{k=2}^{n} (k+1-2\alpha)}{(n-1)!}, \)

\[ b_n = \int_0^1 t^n d\mu(t) \] for $n = 2, 3, \ldots$ and $\mu(t)$ is a probability measure on $[0,1]$.

Equating the coefficients of $z$ and $z^2$ respectively and simplifying we get,
\[ a_2 = b_1; \quad a_3 = \frac{b_2 + (2 - 2\alpha)b_1^2}{(3 - 2\alpha)} \]

Now, using (2.2) in (2.3) we get,
\[ a_2 = B_1 \omega_1; \quad a_3 = \frac{B_1 \omega_2 + (B_2 + (2 - 2\alpha)B_1^2) \omega_1^2}{(3 - 2\alpha)} \]
Now, for a function $f$, a computation shows that

$$ (2.5) \quad [f(z^k)]^k = z + \frac{a_2}{k} z^{k+1} + \left( \frac{a_3}{k} - \frac{(k-1)a_2^2}{2k^2} \right) z^{2k+1} + \ldots $$

Now, by using (2.5) in (1.6) and equating the coefficients of $z$ and $z^2$ we get,

$$ (2.6) \quad d_{k+1} = \frac{a_2}{k} \quad ; \quad d_{2k+1} = \frac{a_3}{k} - \frac{(k-1)a_2^2}{2k^2} $$

Now, using (2.4) in (2.6) we get,

$$ d_{k+1} = \frac{B_1 \omega_1}{k} $$

and

$$ d_{2k+1} = \frac{1}{k} \left[ \frac{B_1 \omega_2 + (B_2 + (2 - 2\alpha)B_1^2) \omega_1^2}{(3 - 2\alpha)} - \frac{B_1^2 \omega_1^2}{2} + \frac{B_1^2 \omega_1^2}{2k} \right] $$

Now,

$$ d_{2k+1} - \mu d_{k+1}^2 = \frac{1}{k} \left[ \frac{B_1 \omega_2 + (B_2 + (2 - 2\alpha)B_1^2) \omega_1^2}{(3 - 2\alpha)} - \frac{B_1^2 \omega_1^2}{2} + \frac{B_1^2 \omega_1^2}{2k} \right] $$

and hence

$$ d_{2k+1} - \mu d_{k+1}^2 = \frac{B_1}{(3 - 2\alpha)k} [\omega_2 - \omega_1^2 t] $$

The first result is established by an application of Lemma (1.7)

If $t \leq -1$, then,

$$ \mu \leq \frac{-k}{(3 - 2\alpha)B_1} + \frac{kB_2}{(3 - 2\alpha)B_1^2} + \frac{k(2 - 2\alpha)}{(3 - 2\alpha)} - \frac{k}{2} + \frac{1}{2} \quad (\mu \leq \sigma_1), $$

and Lemma (1.7) gives:

$$ |d_{2k+1} - \mu d_{k+1}^2| \leq -\frac{B_1}{(3 - 2\alpha)k} t. $$

For $-1 \leq t \leq 1$, we have $\sigma_1 \leq \mu \leq \sigma_2$, where

$$ \sigma_1 = \frac{-k}{(3 - 2\alpha)B_1} + \frac{kB_2}{(3 - 2\alpha)B_1^2} + \frac{k(2 - 2\alpha)}{(3 - 2\alpha)} - \frac{k}{2} + \frac{1}{2}, $$

$$ \sigma_2 = \frac{k}{(3 - 2\alpha)B_1} + \frac{kB_2}{(3 - 2\alpha)B_1^2} + \frac{k(2 - 2\alpha)}{(3 - 2\alpha)} - \frac{k}{2} + \frac{1}{2}. $$
and Lemma (1.7) yields:

$$|d_{2k+1} - \mu d_{k+1}^2| \leq \frac{B_1}{(3-2\alpha)^{2k}}.$$  

For $t \geq 1$, we have,

$$\mu \geq \frac{k}{(3-2\alpha)B_1} + \frac{kB_2}{(3-2\alpha)B_1^2} + \frac{k(2-2\alpha)}{(3-2\alpha)} - \frac{k}{2} + \frac{1}{2} \quad (\mu \geq \sigma_2),$$

and it follows from Lemma (1.7) that

$$|d_{2k+1} - \mu d_{k+1}^2| \leq \frac{B_1}{(3-2\alpha)^{2k}}t.$$  

For the sharpness of the results in the above theorem we have the following:

1. If $\mu = \sigma_1$, then the equality holds in the Lemma (1.7) if and only if
   $$\omega(z) = z\frac{\lambda + z}{1 + \lambda z} \quad (0 \leq \lambda \leq 1) \text{ or one of its rotations.}$$

2. If $\mu = \sigma_2$, then
   $$\omega(z) = -z\frac{\lambda + z}{1 + \lambda z} \quad (0 \leq \lambda \leq 1) \text{ or one of its rotations.}$$

3. If $\sigma_1 < \mu < \sigma_2$, then
   $$\omega(z) = z^2.$$  

The second result follows by an application of an estimate of Lemma (1.8) \hfill \square

**Remark 2.2.** For $k = 1$, the $k^{th}$ root transformation of $f$ reduces to the given function $f$ itself. Thus the estimate given in the above theorem is an extension of the corresponding result for the Fekete-Szegö functional corresponding to universally prestarlike functions of order $\alpha$ with respect to $\phi$ which was already proved in [8].

**Remark 2.3.** Taking $\alpha = \frac{1}{2}$ the class $R_{\alpha}^u(\phi)$ becomes the class of starlike univalent functions with respect to $\phi$ and Theorem (2.1) reduces to Theorem (2.1) of Ali, Lee, V. Ravichandran, S. Supramaniam [1].

**Remark 2.4.** In view of the Alexander result that $f$ is a convex function with respect to $\phi$ if and only if $zf'$ is a starlike function with respect to $\phi$, the estimate for $|d_{2k+1} - \mu d_{k+1}^2|$ for a convex function with respect to $\phi$ can be obtained from the corresponding estimate for starlike function with respect to $\phi$. 
Remark 2.5. If $\alpha = \frac{1}{2}$ and $k = 1$ in Theorem (2.1) we get the Fekete-Szegő coefficient functional corresponding to starlike function with respect to [1].

References


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