ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 40 (2014), No. 6, pp. 1413–1431

Title:

On the planarity of a graph related to the join of subgroups of a finite group

Author(s):

H. Ahmadi and B. Taeri

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 40 (2014), No. 6, pp. 1413–1431
Online ISSN: 1735-8515

ON THE PLANARITY OF A GRAPH RELATED TO THE JOIN OF SUBGROUPS OF A FINITE GROUP

H. AHMADI AND B. TAERI*

(Communicated by Ali Reza Ashrafi)

Abstract. Let G be a finite group which is not a cyclic p-group, p a prime number. We define an undirected simple graph $\Delta(G)$ whose vertices are the proper subgroups of G, which are not contained in the Frattini subgroup of G and two vertices H and K are joined by an edge if and only if $G = \langle H, K\rangle$. In this paper we classify finite groups with planar graph. Our result shows that only few groups have planar graphs.
Keywords: Graph on group, plannar graph, finite group.
MSC(2010): Primary: 20D60; Secondary: 05C25.

1. Introduction

The study of algebraic structures, using the properties of graphs, becomes an exciting research topic leading to many fascinating results and questions. There are many papers on assigning a graph to subgroups of a group or submodules of a module or ideals of a ring and investigation of algebraic properties of groups or rings using the associated graph.

In 1964 Bosák [8] defined the intersection graph of semigroups. In 1969 Csákány and Pollák [9] studied the intersection graph of subgroups of a finite group. Some other works on the graphs associated to subgroups of a group or submodules of a module or ideals of a ring can be found in [2, 5, 10, 12].

Article electronically published on December 11, 2014.
Received: 22 April 2013, Accepted: 17 October 2013.
*Corresponding author.

Some authors studied the planarity of subgroup lattices of a finite group with two approaches: Platt and Schmidt [14, 17], visualized the Hasse diagram of subgroups of a finite group as a graph. Platt [14] showed that a finite lattice is planar if and only if the (undirected) graph obtained from its (Hasse) diagram by adding an edge between its least and greatest elements is a planar graph. Schmidt [17] classified all finite groups with planar subgroup lattices.

A similar work on the lattice of a finite group G was done by Bohanon and Reid [7]. They defined the subgroup graph of a group G, which is a graph whose vertices are the subgroups of the group G and two vertices H_1 and H_2 are joined by an edge if and only if H_1 is a maximal subgroup of H_2. They called a finite group with planar subgroup graph, a planar group and classified all finite planar groups.

For any finite group G different from a cyclic group of prime power order, the authors [1] defined an undirected simple graph $\Delta(G)$ whose vertices are the proper subgroups of G which are not contained in the Frattini subgroup of G and two vertices H_1 and H_2 are joined by an edge if $\langle H_1, H_2\rangle = G$. In [1] the authors studied elementary properties of $\Delta(G)$. They showed that $\Delta(G)$ is connected and determined its clique and chromatic number and obtain bounds for its diameter and girth. In this paper we continue this study and classify finite groups with planar graphs.

Throughout this paper all groups are finite different from a cyclic $p-$group, p a prime. For a group G, we denote by $\pi(G)$ the set of all prime divisors of $|G|$. For $p \in \pi(G)$, the set of all Sylow $p-$subgroups of G is denoted by $\mathrm{Syl}_p(G)$. By $\mathcal{M}(G)$ and $\Phi(G)$ we mean the set of all maximal subgroups of G and the Frattini subgroup of G, respectively. Note that $\Phi(G)$ is the intersection of all maximal subgroups of G. Also by $\mathrm{rank}(G)$ for an abelian group G, we mean the minimal number of generators of G. Clearly an abelian group G of rank r is the direct product of r cyclic subgroups.

A planar graph is a graph that can be embedded in the plane so that no two edges intersect geometrically except at a vertex, which both are incident. A subdivision of an edge $e = uv$ of a graph Δ is obtained by introducing a new vertex w in Δ, that is, by replacing the edge $e = uv$ of Δ by the path uwv of length 2. So the new vertex w is of degree 2 in the resulting graph. A subdivision of a graph Δ is a graph obtained from Δ by applying a finite number of subdivisions of edges in succession.

In Section 2 we study the planarity of this graph and with contribution of Kuratowski's theorem, we classify groups with planar graphs in four classes of cyclic, abelian non-cyclic, nilpotent non-abelian and non-nilpotent groups.

2. Planarity of $\Delta(G)$

First note that a group G has a unique maximal subgroup if and only if G is a cyclic $p-$group, for some prime number p. In this work all groups are finite with at least two maximal subgroups.

Definition 2.1. *For a group G we associate a graph $\Delta(G)$ to G whose vertex set is $\{H < G \mid H \nleq \Phi(G)\}$, the set of proper subgroups H which are not contained in $\Phi(G)$, with two vertices H_1 and H_2 are adjacent if and only if $G = \langle H_1, H_2 \rangle$.*

Remark 2.2. *It is well-known that $\Phi(G)$ is the set of non-generator elements of G and if $H \leq G$, then we have $H \leq \Phi(G)$ if and only if $\langle H, K \rangle \lneq G$, for each $1 \neq K \lneq G$. Thus to avoid isolated vertices we define the vertex set of $\Delta(G)$ to be the set of proper subgroups, which are not contained in $\Phi(G)$.*

In this section with contribution of the following well-known theorem, due to Kuratowski (see [4, Theorem 8.6.5]), we classify groups with planar graphs in Theorem 2.4.

Theorem 2.3. *A graph is planar if and only if it has no subdivisions of K_5 or $K_{3,3}$.*

Our main result is the following:

Theorem 2.4. *The graph $\Delta(G)$ is planar if and only if G is one of the following types:*

(1) $\mathbb{Z}_{p_1 p_2^m}$, $\mathbb{Z}_{p_1^2 p_2^m}$, $\mathbb{Z}_{p_1 p_2 p_3}$, $\mathbb{Z}_{p_1^2 p_2 p_3}$ *or* $\mathbb{Z}_{p_1^2 p_2^2 p_3}$, *where p_1, p_2, p_3 are distinct prime numbers and $m \geq 1$.*
(2) $\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_p$, *where p is an odd prime number and $n \geq 2$.*
(3) Q_8, $Q_8 \times \mathbb{Z}_p$ *or* $\langle a, b \mid a^{2^s} = b^2 = 1, [a, b] = a^{2^{s-1}} \rangle$, *where Q_8 is the quaternion group of order 8, p is an odd prime number and $s \geq 3$.*
(4) S_3, *the symmetric group on three symbols or* $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$.

We need the following straightforward lemmas:

Lemma 2.5. $\Delta(G)$ *is non-planar if one of the following holds:*

(1) G has a normal subgroup N such that $\Delta(\frac{G}{N})$ is non-planar.

(2) G has at least 5 maximal subgroups.

(3) G has at least 5 subgroups with mutually coprime index.

(4) G has distinct subgroups H_i and K_i, where $1 \leq i \leq 3$; such that $G = \langle H_i, K_j\rangle; 1 \leq i \leq j \leq 3$.

Proof. (1) If $\overline{\Gamma}$ is a subgraph of $\Delta(\frac{G}{N})$, then there exists a subgraph Γ of $\Delta(G)$ such that $\overline{\Gamma} \cong \Gamma$. To see this, let $V(\overline{\Gamma}) = \{\frac{H}{N} \mid H \lneq G\}$ be the vertex set of $\overline{\Gamma}$. Since $\frac{H}{N} \nleq \Phi(\frac{G}{N})$ and $\frac{\Phi(G)N}{N} \leq \Phi(\frac{G}{N})$, then $H \nleq \Phi(G)$. Now define $V(\Gamma) = \{H \lneq G \mid \frac{H}{N} \in V(\overline{\Gamma})\}$ and consider the natural bijection $\theta : V(\overline{\Gamma}) \to V(\Gamma)$ with $\theta(\frac{H}{N}) = H$. Also θ is a graph isomorphism, as $\langle \frac{H_1}{N}, \frac{H_2}{N}\rangle = \frac{\langle H_1, H_2\rangle}{N}$, for all $H_1, H_2 \in V(\Gamma)$. Therefore if $\Delta(\frac{G}{N})$ is non-planar, then $\Delta(G)$ is also non-planar.

(2) Let M_1 and M_2 be distinct maximal subgroups of G. Then $M_i \lneq \langle M_1, M_2\rangle \leq G$ and by the maximality of M_i we have $\langle M_1, M_2\rangle = G$. This shows that maximal subgroups of G are mutually adjacent, and the set of maximal subgroups of G, $\mathcal{M}(G)$, form a complete subgraph of $\Delta(G)$. Thus if $|\mathcal{M}(G)| \geq 5$, then K_5 is a subgraph of $\Delta(G)$ so it is non-planar.

(3) If H and K are subgroups of coprime indices of G, then $G = HK$ and so H and K are joined by an edge in $\Delta(G)$. Therefore the set of such subgroups form a complete subgraph of $\Delta(G)$.

Finally if (4) holds, then $K_{3,3}$ is a subgraph of $\Delta(G)$ and it is non-planar. □

Lemma 2.6. *If G has one of the types D_{2^s} $(s \geq 3)$, D_{18} or $\mathbb{Z}_3 \rtimes \mathbb{Z}_8$, then $\Delta(G)$ is non-planar, where D_{2n} denotes the dihedral group of order $2n$.*

Proof. Let $D_8 = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1\rangle$. Then $\Delta(D_8)$ contains a subdivision of K_5 as shown in Figure 1. So it is non-planar.
The non-planarity of $\Delta(D_{2^s})$, $s \geq 3$, can be seen by using induction on s and the fact $D_{2^s}/Z(D_{2^s}) \cong D_{2^{s-1}}$.

Now let $D_{18} = \langle a, b \mid a^9 = b^2 = (ab)^2 = 1\rangle$. Then $\Delta(D_{18})$ contains a subdivision of K_5 as shown in Figure 2. So it is non-planar.

Finally let $\mathbb{Z}_3 \rtimes \mathbb{Z}_8 = \langle a, b \mid a^3 = b^8 = 1, a^b = a^{-1}\rangle$. According to the Figure 3, $\Delta(\mathbb{Z}_3 \rtimes \mathbb{Z}_8)$ contains a subdivision of K_5 and it is non-planar. □

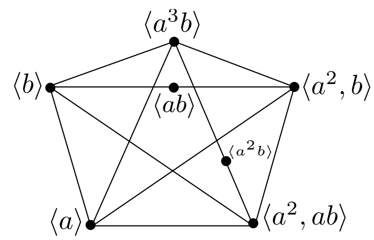

FIGURE 1. A subdivision of K_5 in $\Delta(D_8)$

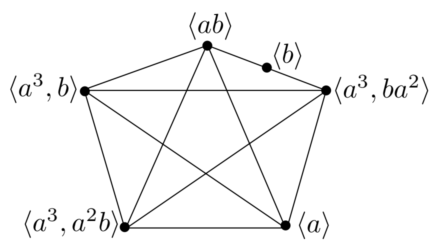

FIGURE 2. A subdivision of K_5 in $\Delta(D_{18})$

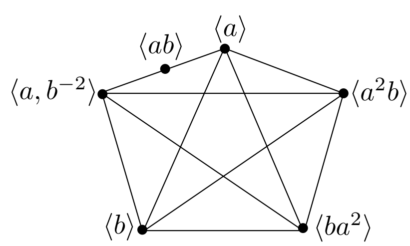

FIGURE 3. A subdivision of K_5 in $\Delta(\mathbb{Z}_3 \rtimes \mathbb{Z}_8)$

We divide the proof of Theorem 2.4, into several steps. First we consider abelian groups. We divide this case to cyclic and non-cyclic cases.

Proposition 2.7. *Let G be a cyclic group. Then $\Delta(G)$ is planar if and only if G is one of the types $\mathbb{Z}_{p_1p_2^n}$, $\mathbb{Z}_{p_1^2p_2^n}$, $\mathbb{Z}_{p_1p_2p_3}$, $\mathbb{Z}_{p_1^2p_2p_3}$ or $\mathbb{Z}_{p_1^2p_2^2p_3}$, where p_1, p_2, p_3 are distinct prime numbers and $n \geq 1$.*

Proof. First we prove that if G is one types mentioned above, then $\Delta(G)$ is planar. If $G \cong \mathbb{Z}_{p_1p_2^{n_2}}$, then $\Delta(G) = K_{1,n_2}$ and it is planar. If

$G = \langle a\rangle \times \langle b\rangle \cong \mathbb{Z}_{p_1^2 p_2^{n_2}}$, where $|a| = p_1^2$, $|b| = p_2^{n_2}$, then $\Delta(G)$ is the graph shown in Figure 4, which is planar.

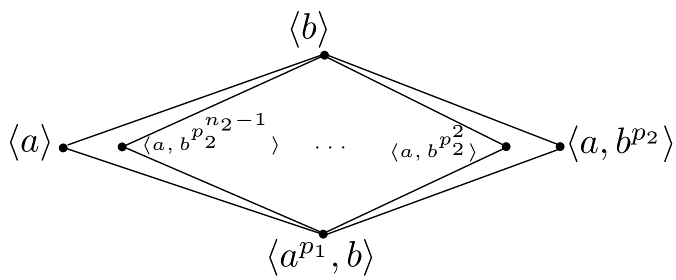

FIGURE 4. $\Delta(\mathbb{Z}_{p_1^2 p_2^{n_2}})$ is planar

If $G = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \cong \mathbb{Z}_{p_1p_2p_3}$, where $|a| = p_1$, $|b| = p_2$, $|c| = p_3$, then one can see that $\Delta(G)$ is the planar graph in Figure 5.

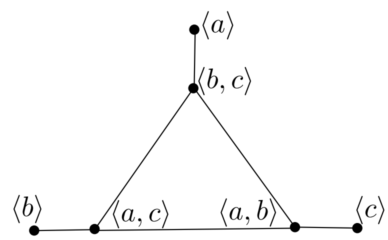

FIGURE 5. $\Delta(\mathbb{Z}_{p_1p_2p_3})$ is planar

If $G = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \cong \mathbb{Z}_{p_1^2p_2p_3}$, where $|a| = p_1^2$, $|b| = p_2$, $|c| = p_3$, again it is easy to see that $\Delta(G)$ is the planar graph in Figure 6.

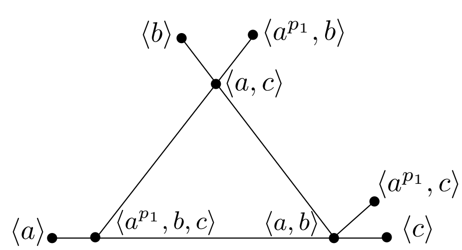

FIGURE 6. $\Delta(\mathbb{Z}_{p_1^2p_2p_3})$ is planar

If $G = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \cong \mathbb{Z}_{p_1^2 p_2^2 p_3}$, where $|a| = p_1^2$, $|b| = p_2^2$, $|c| = p_3$, then $\Delta(G)$ is the planar graph in Figure 7.

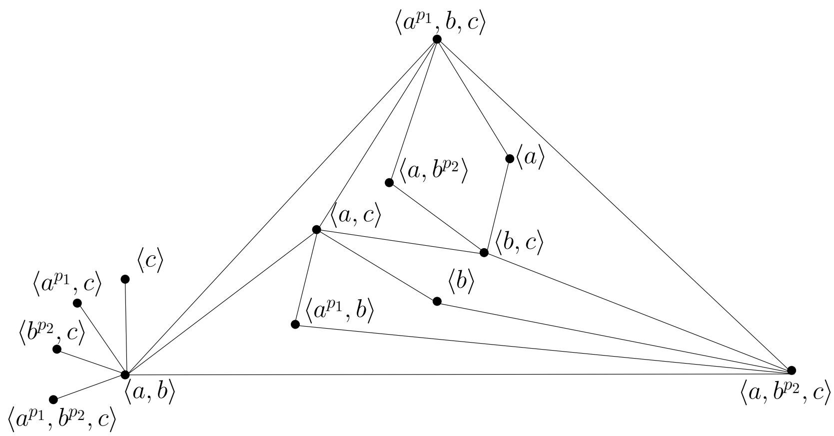

FIGURE 7. $\Delta(\mathbb{Z}_{p_1^2 p_2^2 p_3})$ is planar

To prove the converse suppose that $\Delta(G)$ is planar and $|G| = p_1^{n_1} \cdots p_k^{n_k}$, where p_is are distinct primes and $n_i \geq 1$, $1 \leq i \leq k$. If $|\pi(G)| \geq 5$, then by Lemma 2.5-(3), $\Delta(G)$ is non-planar. So $2 \leq |\pi(G)| \leq 4$.

If $|\pi(G)| = 4$, then there exists a normal subgroup N of G such that $\overline{G} := \frac{G}{N} = \langle \bar{a}\rangle \times \langle \bar{b}\rangle \times \langle \bar{c}\rangle \times \langle \bar{d}\rangle \cong \mathbb{Z}_{p_1p_2p_3p_4}$, where $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are distinct elements of $\overline{G}$ of orders p_1, p_2, p_3, p_4 respectively. Now $\Delta(\overline{G})$ contains a subdivision of $K_{3,3}$ as shown in Figure 8. Hence $\Delta(\overline{G})$ is non-planar and so is $\Delta(G)$. Therefore the case $|\pi(G)| = 4$ cannot happen.

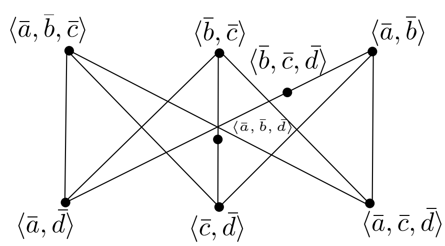

FIGURE 8. A subdivision of $K_{3,3}$ in $\Delta(\mathbb{Z}_{p_1p_2p_3p_4})$

Thus $|\pi(G)| = 2$ or 3.

Suppose that $|\pi(G)| = 2$ and $G = \langle a \rangle \times \langle b \rangle$, $|a| = p_1^{n_1}$ and $|b| = p_2^{n_2}$. If $n_i \geq 3$, for each $i \in \{1, 2\}$, then there exists a normal subgroup N of G such that $\overline{G} := \frac{G}{N} = \langle \bar{a} \rangle \times \langle \bar{b} \rangle$ and $|\bar{a}| = p_1^3$, $|\bar{b}| = p_2^3$. Now if we put $H_1 = \langle \bar{b} \rangle$, $H_2 = \langle \bar{a}^{p_1^2}, \bar{b} \rangle$, $H_3 = \langle \bar{a}^{p_1}, \bar{b} \rangle$, $K_1 = \langle \bar{a} \rangle$, $K_2 = \langle \bar{a}, \bar{b}^{p_2^2} \rangle$ and $K_3 = \langle \bar{a}, \bar{b}^{p_2} \rangle$, then by Lemma 2.5-(4), we conclude that $\Delta(G)$ is non-planar, a contradiction. Therefore $n_1 \leq 2$ or $n_2 \leq 2$. By symmetry we may assume that $n_1 \leq 2$. Thus $G \cong \mathbb{Z}_{p_1 p_2^{n_2}}$ or $\mathbb{Z}_{p_1^2 p_2^{n_2}}$, as desired.

Now let $|\pi(G)| = 3$. Then $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, where a, b, c are elements of orders $p_1^{n_1}, p_2^{n_2}$ and $p_3^{n_3}$, respectively. Suppose, if possible, that there exists $i \in \{1, 2, 3\}$ such that $n_i \geq 3$. We may assume that $i = 1$. Then there exists a normal subgroup N of G such that $\overline{G} := \frac{G}{N} = \langle \bar{a} \rangle \times \langle \bar{b} \rangle \times \langle \bar{c} \rangle$, with $|\bar{a}| = p_1^3$, $|\bar{b}| = p_2$, $|\bar{c}| = p_3$. If we put $H_1 = \langle \bar{a}^{p_1}, \bar{b}, \bar{c} \rangle$, $H_2 = \langle \bar{a}^{p_1^2}, \bar{b}, \bar{c} \rangle$, $H_3 = \langle \bar{b}, \bar{c} \rangle$, $K_1 = \langle \bar{a} \rangle$, $K_2 = \langle \bar{a}, \bar{c} \rangle$ and $K_3 = \langle \bar{a}, \bar{b} \rangle$, then by Lemma 2.5-(4), $\Delta(\overline{G})$ is non-planar and so is $\Delta(G)$, which is a contradiction.

Thus without loss of generality we may assume that

$$(n_1, n_2, n_3) \in \{(1, 1, 1), (2, 1, 1), (2, 2, 1), (2, 2, 2)\}.$$

If $(n_1, n_2, n_3) = (2, 2, 2)$, then $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_{p_1^2 p_2^2 p_3^2}$. Hence $\Delta(G)$ contains a subdivision of K_5, as shown in Figure 9, so it is non-planar.

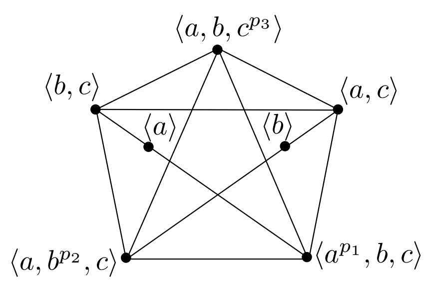

FIGURE 9. A subdivision of K_5 in $\Delta(\mathbb{Z}_{p_1^2 p_2^2 p_3^2})$

Therefore $(n_1, n_2, n_3) \in \{(1, 1, 1), (2, 1, 1), (2, 2, 1)\}$ and this completes the proof. $\square$

Now we want to consider abelian groups. First we need the following lemma, which classifies abelian $p-$groups with planar graphs. Recall that the number of maximal subgroups of an elementary abelian p-group of rank n is $\frac{p^n - 1}{p - 1}$.

Lemma 2.8. *Let P be an abelian $p-$group of order p^n where p is a prime number and $n \geq 2$. Then $\Delta(P)$ is planar if and only if $p \in \{2, 3\}$ and P is one of the types $\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

Proof. First we prove that if P is one of the types mentioned above, then $\Delta(P)$ is planar. If $P \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then since P has four maximal subgroups $\Delta(P) = K_4$, which is planar. If $P = \langle a\rangle \times \langle b\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}$, where a, b are of orders 2 and 2^{n-1} respectively, we have $\frac{P}{\Phi(P)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. So P has exactly three maximal subgroups $\langle b\rangle, \langle ab\rangle, \langle a, b^2\rangle$ and hence $\Phi(P) = \langle b^2\rangle$. If $n = 2$, then clearly $\Delta(P) = K_3$, which is planar. So suppose that $n \geq 3$ and let $H \in V(\Delta(P))$, i.e., $H \nleq \Phi(P)$, such that $H \notin \mathcal{M}(P)$. Then H is contained in a maximal subgroup of P. Obviously $H \nleq \langle b\rangle$ and $H \nleq \langle ab\rangle$, because the maximal subgroup of $\langle b\rangle$ (respectively, $\langle ab\rangle$) is contained in $\Phi(P)$. Hence $H \leq \langle a, b^2\rangle$. Now it follows easily that $\Delta(P)$ is the planar graph shown in Figure 10.

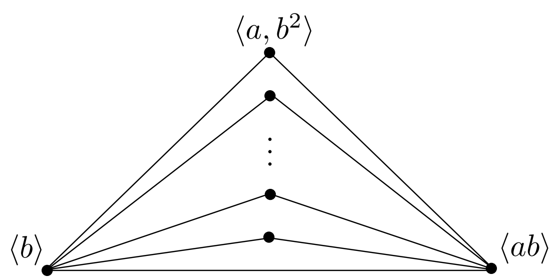

FIGURE 10. $\Delta(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}})$ is planar

Now to prove the converse suppose that $\Delta(P)$ is planar. First we claim that $p \in \{2, 3\}$. Suppose, by contrary that $p \geq 5$. Since $\operatorname{rank}(P) \geq 2$, there exists a normal subgroup N of P such that $\overline{P} := \frac{P}{N} = \langle \bar{a}\rangle \times \langle \bar{b}\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where $\bar{a}$ and $\bar{b}$ are elements of $\overline{P}$ both of order p. Thus $\overline{P}$ has $p+1$ maximal subgroups and so by Lemma 2.5-(2), $\Delta(\overline{P})$ is non-planar and so is $\Delta(P)$. Therefore $p \in \{2, 3\}$.

If $\operatorname{rank}(P) \geq 3$, then similarly we have $\overline{P} := \frac{P}{N} = \langle \bar{a}\rangle \times \langle \bar{b}\rangle \times \langle \bar{c}\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, for some normal subgroup N of P and distinct elements $\bar{a}$, $\bar{b}$, $\bar{c}$ of $\overline{P}$ all of order p. Clearly $|\mathcal{M}(\overline{P})| = 7$ or 13, for $p = 2$ or 3, respectively. Hence in both cases, Lemma 2.5-(2) implies that $\Delta(\overline{P})$ is non-planar. Thus $\Delta(P)$ is non-planar, which is a contradiction. This shows that $\operatorname{rank}(P) = 2$. Now we handle the cases $p = 2$ and $p = 3$ separately:

Case 1. $p = 2$. In this case we have $P \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}$ or $\overline{P} := \frac{P}{N} = \langle \bar{a} \rangle \times \langle \bar{b} \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_4$, for some normal subgroup N of P and distinct elements $\bar{a}$ and $\bar{b}$ of $\overline{P}$ both of order 4. In the second case if we put $H_1 = \langle \bar{a}, \bar{b}^2 \rangle$, $H_2 = \langle \bar{a}^2, \bar{a}\bar{b}^2 \rangle$, $H_3 = \langle \bar{a} \rangle$, $K_1 = \langle \bar{a}^{-1}\bar{b} \rangle$, $K_2 = \langle \bar{a}\bar{b} \rangle$ and $K_3 = \langle \bar{a}^2, \bar{b}\bar{a}^{-1} \rangle$, then Lemma 2.5-(4) implies that $\Delta(\overline{P})$ is non-planar and so is $\Delta(P)$. Thus we have the first case, as desired.
Case 2. $p = 3$. In this case we have $P \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\overline{P} := \frac{P}{N} = \langle \bar{a} \rangle \times \langle \bar{b} \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_9$, for some normal subgroup N of P and distinct elements $\bar{a}$ and $\bar{b}$ of $\overline{P}$ of orders 3 and 9 respectively. In the second case if we put $H_1 = \langle \bar{a}, \bar{b}^3 \rangle$, $H_2 = \langle \bar{a}\bar{b}^3 \rangle$, $H_3 = \langle \bar{a}\bar{b}^6 \rangle$, $K_1 = \langle \bar{a}\bar{b} \rangle$, $K_2 = \langle \bar{a}^2\bar{b} \rangle$ and $K_3 = \langle \bar{b} \rangle$, then by Lemma 2.5-(4), $\Delta(\overline{P})$ is non-planar and so is $\Delta(P)$. Thus the first case happens. This completes the proof. □

Now we are ready to characterize abelian non-cyclic groups with planar graphs.

Proposition 2.9. *Let G be an abelian non-cyclic group. Then $\Delta(G)$ is planar if and only if G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_p$, where p is an arbitrary odd prime and $n \geq 2$.*

Proof. By Lemma 2.8, the graphs of $\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ are planar. Now let $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_p$, where p is an arbitrary odd prime. Then $\Delta(G)$ is the planar graph shown in Figure 11.

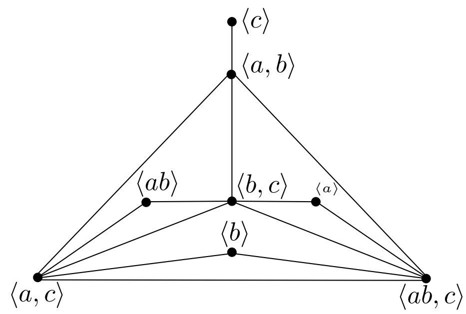

FIGURE 11. $\Delta(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_p)$ is planar

Now we prove the converse. Suppose that $\Delta(G)$ is planar. Since G is abelian, $G = G_1 \times G_2 \times \cdots \times G_k$, where G_i is the Sylow p_i-subgroup of G for $1 \leq i \leq k$. If $k \geq 4$, then $\overline{G} := \frac{G}{N} \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4}$, for some normal subgroup N of G and by Proposition 2.7, $\Delta(\overline{G})$ is non-planar and so is $\Delta(G)$. Hence $1 \leq k \leq 3$.

Suppose, if possible, that $k=3$. Then we can assume that $\operatorname{rank}(G_1)\geq 2$. So $\overline{G}:=\frac{G}{N}\cong\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\mathbb{Z}_{p_3}$ and by Proposition 2.7, $\Delta(\overline{G})$ is non-planar and so is $\Delta(G)$. Therefore $1\leq k\leq 2$.

If $k=2$, then $G=G_1\times G_2$. Since G is non-cyclic, we can assume that $\operatorname{rank}(G_1)=2$ and $\operatorname{rank}(G_2)=1$. Suppose that $G_1=\langle a\rangle\times\langle b\rangle$ and $G_2=\langle c\rangle$, where a,b,c are distinct elements of G of orders $p_1^{n_1},p_1^{n_2},p_2^{n_3}$ respectively. So we have $G=\langle a\rangle\times\langle b\rangle\times\langle c\rangle\cong\mathbb{Z}_{p_1^{n_1}}\times\mathbb{Z}_{p_1^{n_2}}\times\mathbb{Z}_{p_2^{n_3}}$. If $p_1\geq 5$, then $\overline{G}:=\frac{G}{N}\cong\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_1}$, for some normal subgroup N of G. Therefore $\overline{G}$ has p_1+1 maximal subgroups and Lemma 2.5-(2) implies that $\Delta(\overline{G})$ is non-planar and so is $\Delta(G)$. Thus $p_1\in\{2,3\}$. Without loss of generality, we may assume that $n_1\geq 3$. Again for some normal subgroup N of G we have $\overline{G}:=\frac{G}{N}\cong\mathbb{Z}_{p_1^3}\times\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}$ and similar to that of the proof of Proposition 2.7, $\Delta(\overline{G})$ is non-planar and so is $\Delta(G)$. Hence $n_i\leq 2$, for each $i\in\{1,2,3\}$ and G is isomorphic to one of the following types

$$\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2},\quad \mathbb{Z}_{p_1}\times\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2^2},\quad \mathbb{Z}_{p_1}\times\mathbb{Z}_{p_1^2}\times\mathbb{Z}_{p_2},$$
$$\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_1^2}\times\mathbb{Z}_{p_2^2},\quad \mathbb{Z}_{p_1^2}\times\mathbb{Z}_{p_1^2}\times\mathbb{Z}_{p_2},\quad \text{or}\quad \mathbb{Z}_{p_1^2}\times\mathbb{Z}_{p_1^2}\times\mathbb{Z}_{p_2^2},$$

where $p_1\in\{2,3\}$ and p_2 is an arbitrary prime different from p_1. Now we want to show that $p_1=2$ and $G\cong\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{p_2}$.

Case 1. $p_1=2$. If $G=\langle a\rangle\times\langle b\rangle\times\langle c\rangle\cong\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{p_2^2}$, then $\Delta(G)$ contains a subdivision of K_5, as shown in Figure 12 and it is non-planar. Also if G is one of the types $\mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_{p_2^2}$ or $\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_{p_2^2}$, then $\overline{G}:=$

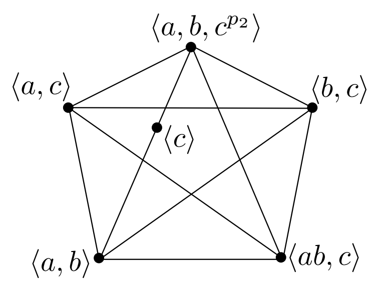

FIGURE 12. A subdivision of K_5 in $\Delta(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{p_2^2})$

$\frac{G}{N}\cong\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{p_2^2}$ for some normal subgroup N of G and by previous part $\Delta(\overline{G})$ is non-planar and so is $\Delta(G)$. Now let $G=\langle a\rangle\times\langle b\rangle\times\langle c\rangle\cong\mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_{p_2}$. If we put $H_1=\langle a,b\rangle$, $H_2=\langle a\rangle$, $H_3=\langle a^2,a^{-1}b\rangle$,

$K_1 = \langle b, c\rangle$, $K_2 = \langle a^2b, c\rangle$ and $K_3 = \langle a^2, b, c\rangle$, then Lemma 2.5-(4) implies that Then $\Delta(G)$ is non-planar.

Finally if $G = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_{p_2}$, then $\overline{G} := \frac{G}{N} \cong \mathbb{Z}_4 \times \mathbb{Z}_4$, for some normal subgroup N of G and by Lemma 2.8, $\Delta(\overline{G})$ is non-planar and so is $\Delta(G)$.

Case 2. $p_1 = 3$. In all types of G mentioned before Case 1, there exists a normal subgroup N of G such that $\overline{G} := \frac{G}{N} \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{p_2}$. Obviously $|\mathcal{M}(\overline{G})| \geq 5$, and by Lemma 2.5-(2), $\Delta(\overline{G})$ is non-planar and so is $\Delta(G)$.

Finally suppose that $k = 1$. Then G is an abelian p_1-group of order $p_1^{n_1}$ and since $\Delta(G)$ is planar, $p_1 \in \{2, 3\}$ and by Lemma 2.8, G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{n_1-1}}$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$, and the result follows. $\square$

Now we want to classify non-abelian groups with planar graphs. First we consider non-abelian $p-$groups. We need the classification of minimal non-abelian p-groups. Recall that a non-abelian p-group P is minimal non-abelian if all its proper subgroups are abelian groups. By the main result of [15], a p-group P is minimal non-abelian if and only if P is isomorphic to one of the following:

(i) $\langle a, b \mid a^{p^s} = b^{p^t} = 1, [a, b] = a^{p^{s-1}}\rangle$, $s > 1$, $t \geq 1$; $|P| = p^{s+t}$

(ii) $\langle a, b, c \mid a^{p^s} = b^{p^t} = c^p = 1, c = [a, b], [a, c] = [b, c] = 1\rangle$, $s, t \geq 1$, $|P| = p^{s+t+1}$

(ii) Q_8.

Lemma 2.10. *Let P be a non-abelian $p-$group of order p^n, where p is a prime number and $n \geq 3$. Then $\Delta(P)$ is planar if and only if $p = 2$ and $P \cong Q_8$ or $P \cong \langle a, b \mid a^{2^s} = b^2 = 1, [a, b] = a^{2^{s-1}}\rangle$, for some $s \geq 3$.*

Proof. First note that $\Delta(Q_8) \cong K_3$ is planar. Now suppose that

$$P = \langle a, b \mid a^{2^s} = b^2 = 1,\ [a, b] = a^{2^{s-1}}\rangle, \tag{1}$$

where $s \geq 3$. Clearly $P = \langle a\rangle \rtimes \langle b\rangle$ and $P' = \langle [a, b]\rangle$, the derived subgroup of P, and so

$$\overline{P} := \frac{P}{P'} = \langle \bar{a}, \bar{b} \mid \bar{a}^{2^{s-1}} = \bar{b}^2 = \bar{1},\ [\bar{a}, \bar{b}] = \bar{1}\rangle; \qquad \bar{a} := aP', \quad \bar{b} := bP'.$$

Hence $\overline{P} = \langle \bar{a}\rangle \times \langle \bar{b}\rangle \cong \mathbb{Z}_{2^{s-1}} \times \mathbb{Z}_2$. Thus $\mathcal{M}(\overline{P}) = \{\langle \bar{a}\rangle, \langle \bar{a}^2, \bar{b}\rangle, \langle \bar{a}\bar{b}\rangle\}$ from which it follows that $\mathcal{M}(P) = \{\langle a\rangle, \langle a^2, b\rangle, \langle ab\rangle\}$. Now one can easily check that $\Delta(P)$ is the same as the Figure 10 and so it is planar.

To prove the converse suppose that $\Delta(P)$ is planar. We consider three cases:

Case 1. $p = 2$. We have to prove that P is either Q_8 or P has the presentation (1). If this is not the case, choose a non-abelian 2-group P of least order such that $\Delta(P)$ is planar, $P \ncong Q_8$ and $P \ncong \langle a, b \mid a^{2^s} = b^2 = 1,\ [a, b] = a^{2^{s-1}} \rangle$. Take $x \in Z(P) \cap \Phi(P)$ such that $|x| = 2$ and put $\overline{P} := P/\langle x \rangle$. Note that $\overline{P}$ is not cyclic, $\Delta(\overline{P})$ is planar and $|\overline{P}| < |P|$. Thus the minimality of $|P|$ implies that $\overline{P} \cong Q_8$ or $\overline{P}$ has the presentation (1) or $\overline{P}$ is abelian. We show that the first and the second cases do not happen.

Suppose that $\overline{P} \cong Q_8$ and let M be a maximal subgroup of P. Thus, since $x \in \Phi(P) \leq M$, $\overline{M} := \frac{M}{\langle x \rangle}$ is a maximal subgroup of $\overline{P}$ and so is cyclic. Thus, as $x \in Z(P)$, M is abelian. Hence P is a minimal non-abelian 2-group of order 16. By the classification of minimal non-abelian $p-$groups, P has type (i) or (ii). If P has type (i), then $(s, t) \in \{(3, 1), (2, 2)\}$. When $(s, t) = (3, 1)$, P has the presentation (1) which is a contradiction; and when $(s, t) = (2, 2)$, P has a factor group isomorphic to D_8, which is impossible by Lemma 2.6. If P has type (ii), then $(s, t) = (2, 1)$ and P has a factor group isomorphic to D_8, a contradiction.

Now suppose that $\overline{P}$ has presentation (1). To obtain a contradiction we need a slightly long argument. We distinguish two subcases:

Subcase 1. P has no cyclic maximal subgroup. Then since $\mathcal{M}(\overline{P}) = \{\langle \bar{a} \rangle, \langle \bar{a}^2, \bar{b} \rangle, \langle \bar{a}\bar{b} \rangle\}$, we have $\mathcal{M}(P) = \{\langle a, x \rangle,\ \langle a^2, b, x \rangle,\ \langle ab, x \rangle\}$. If we put $H_1 = \langle a, x \rangle$, $H_2 = \langle ab, x \rangle$, $H_3 = \langle a \rangle$, $K_1 = \langle a^2, b, x \rangle$, $K_2 = \langle b \rangle$ and $K_3 = \langle a^2 b \rangle$, then by Lemma 2.5-(4), $\Delta(P)$ is non-planar, which is a contradiction. Thus this subcase cannot happen.

Subcase 2. P has a cyclic maximal subgroup. Since P is non-abelian of order 2^{s+2}, by [16, Theorem 5.3.4], P is isomorphic to one of the following groups:

(a) $\langle a, b \mid a^{2^{s+1}} = b^2 = 1, a^b = a^{1+2^s} \rangle$; $s \geq 1$.

(b) The dihedral group $D_{2^{s+2}}$ of order 2^{s+2}; $s \geq 1$.

(c) $Q_{2^{s+2}} = \langle a, b \mid a^{2^{s+1}} = 1, b^2 = a^{2^s}, a^b = a^{-1} \rangle$; $s \geq 1$, generalized quaternion group.

(d) $SD_{2^{s+2}} = \langle a, b \mid a^{2^{s+1}} = b^2 = 1, a^b = a^{2^s - 1} \rangle$; $s \geq 1$, semidihedral group.

In the case (a), P clearly has the presentation (1), a contradiction. The case (b) cannot happen, by Lemma 2.6. Also since $Q_{2^{s+2}}/Z(Q_{2^{s+2}}) \cong D_{2^{s+1}}$ and $SD_{2^{s+2}}/\langle a^{2^s} \rangle \cong D_{2^{s+1}}$, the cases (c) and (d) cannot hapen. Therefore we have shown that $\overline{P}$ cannot have presentation (1).

Hence the minimality of $|P|$ implies that $\overline{P}$ is abelian, and so by Lemma 2.8 we have

$$\overline{P} = \frac{P}{\langle x \rangle} \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}}, \tag{2}$$

where $m \geq 3$. Hence $P/\langle x \rangle$ is generated by 2 elements and so P is generated by 2 elements, as $x \in \Phi(P)$. Thus by Burnside basis Theorem (see [16, Theorem 5.3.2]) $\frac{P}{\Phi(P)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore $|\mathcal{M}(P)| = |\mathcal{M}(\frac{P}{\Phi(P)})| = 3$. Also it is easy to see that $P' = \langle x \rangle$ and so for each $a, b \in P$, we have $[a^2, b] = [a, b]^a [a, b] = [a, b]^2 = 1$, as $P' = \langle x \rangle \leq Z(P)$ and x has order 2. Therefore $a^2 \in Z(P)$ and $P^2 \leq Z(P)$. These imply that $\Phi(P) = P'P^2 \leq Z(P)$. Now let H be a maximal subgroup of P. Then, since $\frac{H}{\Phi(P)} \leq \frac{P}{\Phi(P)}$ is cyclic and $\Phi(P) \leq Z(P)$, H is abelian. Thus all proper subgroups of P are abelian, that is P is a minimal non-abelian $2-$group. By the classification of minimal non-abelian $p-$groups, P has type (i), (ii) or (iii). Since $P \ncong Q_8$ the case (iii) cannot happen. If P has type (i), then clearly $P = \langle a \rangle \rtimes \langle b \rangle$ and $P' = \langle [a, b] \rangle = \langle x \rangle$ and so

$$\overline{P} = \frac{P}{P'} = \langle \bar{a}, \bar{b} \mid \bar{a}^{2^{s-1}} = \bar{b}^{2^t} = \bar{1}, [\bar{a}, \bar{b}] = \bar{1} \rangle; \qquad \bar{a} := aP', \quad \bar{b} := bP'.$$

Hence $\overline{P} = \langle \bar{a} \rangle \times \langle \bar{b} \rangle \cong \mathbb{Z}_{2^{s-1}} \times \mathbb{Z}_{2^t}$. According to (2), $s = 2$ or $t = 1$.

If $t = 1$, then P has presentation (1), a contradiction. If $s = 2$, then $P = \langle a, b \mid a^4 = b^{2^t} = 1, [a, b] = a^2 \rangle$ and so $\frac{P}{\langle b^2 \rangle} \cong D_8$ (note that $[a, b^2] = 1$ and so $\langle b^2 \rangle \lhd P$), which has non-planar graph, a contradiction.

Now let P be of type (ii). Similar to that of type (i), we have $P' = \langle [a, b] \rangle$ and

$$\overline{P} = \langle \bar{a}, \bar{b} \mid \bar{a}^{2^s} = \bar{b}^{2^t} = \bar{1}, [\bar{a}, \bar{b}] = \bar{1} \rangle.$$

Hence $\overline{P} = \langle \bar{a} \rangle \times \langle \bar{b} \rangle$. According to (2), $s = 1$ or $t = 1$. By symmetry it suffices to assume that $t = 1$. Then we have:

$$P = \langle a, b, c \mid a^{2^s} = b^2 = c^2 = 1, c = [a, b], [a, c] = [b, c] = 1 \rangle.$$

Clearly $[a^2, b] = 1$ and so $\langle a^2 \rangle \lhd P$. It follows that $\frac{P}{\langle a^2 \rangle} \cong D_8$, which has non-planar graph, a contradiction.

Case 2. $p = 3$. We claim that there is no non-abelian $3-$group with planar graph. Suppose on the contrary that there exists a $3-$group P with planar graph $\Delta(P)$ and $|P|$ is minimal with respect to these properties. Choose $x \in Z(P)$ such that $|x| = 3$ and put $\overline{P} := \frac{P}{\langle x \rangle}$. Since $\Delta(\overline{P})$ is planar and $|\overline{P}| < |P|$, by the minimality of $|P|$, $\overline{P}$ is an abelian

3−group. By Lemma 2.8, we have $\overline{P} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Therefore $|P| = 27$. There exist two non-abelian 3-groups of order 27 such that their graphs contain $K_{3,3}$ as a subgraph, hence they are non-planar which gives a contradiction.

Case 3. Let $p \geq 5$. Since $E := \frac{P}{\Phi(P)} \cong \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p = \mathbb{Z}_p^m$, where $m \geq 2$, then there exists $A \triangleleft E$ such that $\overline{E} := \frac{E}{A} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, and hence by Lemma 2.5-(2), $\Delta(\overline{E})$ is non-planar and so is $\Delta(P)$, which is a contradiction. □

Now we classify non-abelian groups with planar graphs. We divide it into two cases; non-abelian nilpotent groups and non-nilpotent groups.

Theorem 2.11. *Let G be a non-abelian nilpotent group. Then $\Delta(G)$ is planar if and only if $G \cong Q_8$, $G \cong \langle a, b \mid a^{2^s} = b^2 = 1, [a, b] = a^{2^{s-1}} \rangle$ or $G \cong Q_8 \times \mathbb{Z}_p$, where p is an arbitrary odd prime and $s \geq 3$.*

Proof. Put $Q_8 = \langle u, v \mid u^2 = v^2, vu = u^3 v \rangle$.

If G has the first or the second type, then by Lemma 2.10, $\Delta(G)$ is planar. If G has the third type, $\Delta(G)$ is the planar graph as shown in the Figure 13

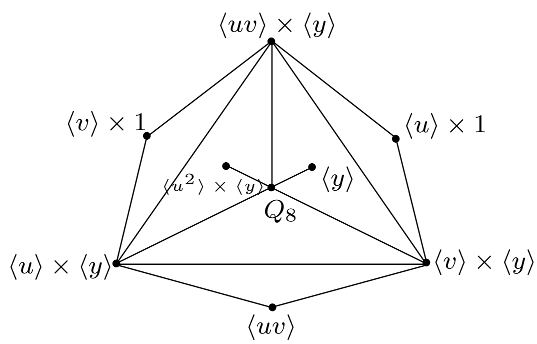

FIGURE 13. $\Delta(Q_8 \times \mathbb{Z}_p)$ is planar

To prove the converse, let $\Delta(G)$ be planar. Then it contains no K_5 as a subgraph. Therefore $|\mathcal{M}(G)| \leq 4$. If $|\mathcal{M}(G)| = 2$, then G is a cyclic group, which is a contradiction. If $|\mathcal{M}(G)| = 3$, then by [3, Lemma 1], G is a 2−group or a cyclic group of order $p^n q^m r^k$, where p, q, r are distinct primes and m, n, k are positive integers. By hypothesis the later case does not hold and since $\Delta(G)$ is planar, by Lemma 2.10, we have $G \cong Q_8$ or $G \cong P$, where $P := \langle a, b \mid a^{2^s} = b^2 = 1, [a, b] = a^{2^{s-1}} \rangle; s \geq 3$.

Finally let $|\mathcal{M}(G)| = 4$. By hypothesis G is non-abelian and [3, Theorem 4] implies that either G is a $3-$group such that $\frac{G}{\Phi(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $G = Q \times R$, where Q is a $2-$group and $R = \langle x \rangle$ is a cyclic Sylow $p-$subgroup of G of order p^m, where p is an odd prime number and $m \geq 1$. By Lemma 2.10, the first case cannot happen, so we have the second case. Now $\Delta(Q)$ is planar because of planarity of $\Delta(G)$. Hence by Lemma 2.10, $Q \cong P$ or $Q \cong Q_8$.

First suppose that $Q \cong P$, i.e., $G \cong P \times R$. If we put $H_1 = P$, $H_2 = \langle a \rangle \times R$, $H_3 = \langle ab \rangle \times R$, $K_1 = \langle a^2, b \rangle \times R$, $K_2 = \langle b \rangle \times R$ and $K_3 = \langle a^2 b \rangle \times R$, then Lemma 2.5-(4) implies that $\Delta(G)$ is non-planar.

Hence $Q \cong Q_8$, i.e., $G \cong Q_8 \times \mathbb{Z}_{p^m}$. If $m \geq 2$, then $\Delta(G)$ contains a subdivision of K_5, as shown in Figure 14, and so is non-planar.

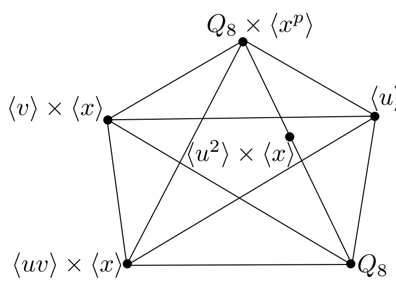

FIGURE 14. A subdivision of K_5 in $\Delta(Q_8 \times \langle x \rangle)$

Therefore $m = 1$ and so $G \cong Q_8 \times \mathbb{Z}_p$, as desired. □

Now we recall some well-known facts which are needed to prove Theorem 2.12. If G is a finite supersolvable group and p is the largest prime divisor of $|G|$, then the Sylow $p-$subgroup of G is normal (see [6, Theorems 6.2.2 and 6.2.5]).

For a finite non-nilpotent group G with $|\mathcal{M}(G)| = 4$, we have: G is a supersolvable group of order $2^n 3^m$; $n, m \geq 1$ and $\frac{G}{\Phi(G)} \cong S_3$. (see [3, Theorem 3]). In particular Sylow subgroups of G are cyclic. To see this, suppose that $P \in \mathrm{Syl}_3(G)$ and $Q \in \mathrm{Syl}_2(G)$. Since $P \lhd G$, $\Phi(P) \leq \Phi(G)$ and so $\Phi(P) \leq P \cap \Phi(G)$. Now if there exists a maximal subgroup M of P such that $P \cap \Phi(G) \not\leq M$, then $P = M(P \cap \Phi(G))$ and so $G = PQ = M(P \cap \Phi(G))Q = MQ$, which is a contradiction. Therefore $P \cap \Phi(G) \leq \Phi(P)$. Thus $\Phi(P) = P \cap \Phi(G)$. Now we have

$$\frac{P}{\Phi(P)} = \frac{P}{P \cap \Phi(G)} \cong \frac{P\Phi(G)}{\Phi(G)} \cong \mathbb{Z}_3,$$

and hence $\frac{P}{\Phi(P)}$ is cyclic and so is P. Also if N is a maximal subgroup of Q, then PN is a maximal subgroup of G. Thus $\Phi(G) \leq PN$ and $Q \cap \Phi(G) \leq Q \cap NP = N$. Hence $Q \cap \Phi(G) \leq \Phi(Q)$ and similarly Q is cyclic.

Now let $P = \langle a \rangle$ and $Q = \langle b \rangle$ and $\phi : Q \longrightarrow \mathrm{Aut}(P)$ be the permutation representation corresponding to the action of Q on P. Note that $\mathrm{Aut}(P) \cong \mathbb{Z}_2 \times \mathbb{Z}_{3^{m-1}}$ (see [13, Theorem 2.2.6]). Since $\mathrm{Aut}(P)$ is cyclic, it has a unique involution. On the other hand $\phi(b)$ is a 2-element of $\mathrm{Aut}(P)$. Therefore $\phi(b)$ is the unique involution of $\mathrm{Aut}(P)$. This implies that $a^b = a^{-1}$, that is the action of Q on P is inversion.

The following result completes the proof of Theorem 2.4.

Theorem 2.12. *Let G be a non-nilpotent group. Then $\Delta(G)$ is planar if and only if $G \cong S_3$ or $G \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4$.*

Proof. First note that $\Delta(S_3) \cong K_4$, which is a planar graph. By GAP [11], one can check that the group $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ has four cyclic maximal subgroups $M_i; 1 \leq i \leq 4$ such that $|M_i| = 4; 1 \leq i \leq 3$ and $|M_4| = 6$. Let K be the subgroup of M_4 of order 3. Thus $\Delta(G)$ is the planar graph which is shown in the Figure 15.

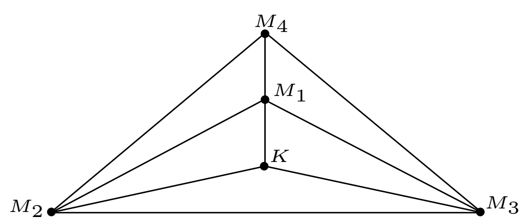

FIGURE 15. $\Delta(\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ is planar

Now let $\Delta(G)$ be planar. Thus by Lemma 2.5-(2), $|\mathcal{M}(G)| \leq 4$. Since G is non-cyclic, $|\mathcal{M}(G)| \neq 1, 2$. Also since G is non-nilpotent, [3, Lemma 1] implies that $|\mathcal{M}(G)| \neq 3$. Hence $|\mathcal{M}(G)| = 4$ and by above discussion G is a supersolvable group of order $2^n 3^m$, where $n, m \geq 1$, and $G = P \rtimes Q$, where $P = \langle a \rangle$ and $Q = \langle b \rangle$ are Sylow $3-$subgroup and Sylow $2-$subgroup of G, respectively; and $a^b = a^{-1}$. Thus we have

$$G = \langle a, b \mid a^{3^m} = b^{2^n} = 1,\ a^b = a^{-1} \rangle.$$

Also since $[a, b^2] = 1$ we have that $\langle b^2 \rangle \leq Z(G)$. Clearly $\langle a^9 \rangle \lhd G$. Now if $|P| \geq 9$, consider the normal subgroup $N := \langle a^9, b^2 \rangle$ of G and put $\overline{G} := \frac{G}{N}$, $\bar{a} := aN$ and $\bar{b} := bN$. Therefore $\overline{G} = \langle \bar{a}, \bar{b} \mid \bar{a}^9 = \bar{b}^2 = \bar{1}, \bar{a}^{\bar{b}} =$

$\bar{a}^{-1}\rangle$ and so $\overline{G} \cong D_{18}$. By Lemma 2.6, $\Delta(D_{18})$ is non-planar so is $\Delta(G)$. Thus $|P| = 3$.

Also if $|Q| \geq 8$, consider the normal subgroup $N := \langle b^8 \rangle$ of G and again with the above notations, we have $\overline{G} = \langle \bar{a}, \bar{b} \mid \bar{a}^3 = \bar{b}^8 = \bar{1}, \bar{a}^{\bar{b}} = \bar{a}^{-1}\rangle$ and so $\overline{G} \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_8$. By Lemma 2.6, $\Delta(\mathbb{Z}_3 \rtimes \mathbb{Z}_8)$ is non-planar, so is $\Delta(G)$. Hence $|Q| \in \{2, 4\}$, which implies that $G \cong S_3$ or $G \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4$. This completes the proof. □

Acknowledgments

The authors gratefully thank three anonymous referees for the constructive comments and recommendations which definitely helped to improve the readability and quality of the paper.

References

[1] H. Ahmadi and B. Taeri, A graph related to the join of subgroups of a finite group, *Rend. Semin. Mat. Univ. Padova* **131** (2014) 281–292.

[2] S. Akbari, H. A. Tavallayee and S. Khalashi Ghezelahmad, Intersection graph of submodules of a module, *J. Algebra Appl.* **11** (2012), no. 1, 8 Pages.

[3] A. Ashrafi, On the number of minimal and maximal subgroups of a finite group, *J. Sci. Univ. Tehran Int. Ed.* **3** (1998), no. 1, 15–20.

[4] R. Balakrishnan and K. Ranganathan, A Text Book of Graph Theory, Springer-Verlag, New York-Heidelberg-Berlin, 1991.

[5] A. Ballester-Bolinches, J. Cossey and R. Esteban-Romero, On a graph related to permutability in finite groups, *Ann. Mat. Pura Appl. (4)* **189** (2010), no. 4, 567–570.

[6] H. Bechtell, The Theory of Groups, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1971.

[7] J. P. Bohanon and L. Reid, Finite groups with planar subgroup lattices, *J. Algebraic Combin.* **23** (2006), no. 3, 207–223.

[8] J. Bosak, The graphs of semigroups, 11–125, Theory of Graphs and Applications, Academic Press, New York, 1964.

[9] B. Csákeány and G. Pollák, The graph of subgroups of a finite group, *Czechoslovak Math. J.* **19** (1969) 241–247.

[10] I. Chakrabarty, S. Ghosh, T. K. Mukherjee and M. K. Sen, Intersection graphs of ideals of rings, *Discrete Math.* **309** (2009), no. 17, 5381–5392.

[11] The GAP Group, GAP-Groups, *Algorithms and Programing*, Version 4.4; 2005, (http://www.gap-system.org).

[12] M. Herzog, P. Longobardi and M. Maj, On a graph related to the maximal subgroups of a group, *Bull. Aust. Math. Soc.* **81** (2010), no. 2, 317–328.

[13] H. Kurzweil and B. Stellmacher, The Theory of Finite Groups: An Introduction, Springer-Verlag New York, 2004.

[14] C. R. Platt, Planar lattices and planar graphs, *J. Combinatorial Theory Ser. B* **21** (1976), no. 1, 30–39.

[15] L. Rédei, Das schiefe product in der gruppen theorie comment, *Math. Helvet.* **20** (1947) 225–267.

[16] D. J. S. Robinson, A Course in the Theory of Groups, Springer-Verlag, New York-Heidelberg-Berlin, 1996.

[17] R. Schmidt, Planar subgroup lattices, *Algebra Universalis* **55** (2006), no. 1, 3–12.

(Hadi Ahmadi) Department of Mathematical Sceinces, Isfahan University of Technology, P.O. Box 84156-83111, Isfahan, Iran

E-mail address: `hadiahmadi@math.iut.ac.ir`

(Bijan Taeri) Department of Mathematical Sceinces, Isfahan University of Technology, P.O. Box 84156-838111, Isfahan, Iran

E-mail address: `b.taeri@cc.iut.ac.ir`