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# ON SPECIAL SUBMODULE OF MODULES

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(Communicated by Teo Mora)

Abstract. Let $R$ be a domain with quotiont field $K$, and let $N$ be a submodule of an $R$-module $M$. We say that $N$ is powerful (strongly primary) if $x, y \in K$ and $xyM \subseteq N$, then $x \in R$ or $y \in R$ ($xM \subseteq N$ or $y^n M \subseteq N$ for some $n \geq 1$). We show that a submodule with either of these properties is comparable to every prime submodule of $M$, also we show that an $R$-module $M$ admits a powerful submodule if and only if it admits a strongly primary submodule. Finally we study finitely generated torsion free modules over domain each of whose prime submodules are strongly primary.
**Keywords:** Prime submodule, strongly prime submodule, primary submodule, strongly prime submodule, power submodule.
**MSC(2010):** Primary: 65F05; Secondary: 46L05, 11Y50.

## 1. Introduction

Throughout this work $R$ will denote an integral domain with quotient field $K$. Recall from [3] that a prime ideal $P$ of $R$ is said to be strongly prime if, $xy \in P$ for elements $x, y \in K$, we have $x \in P$ or $y \in P$. In this paper, we consider two generalizations of this concept . We define a non-zero submodule $N$ of an $R$-module $M$ to be powerful if, $xyM \subseteq N$ for elements $x, y \in K$, we have $x \in R$ or $y \in R$. It is easy to see that $R$ is powerful if and only if $R$ is a valuation domain. In the first section, we show that a powerful prime submodule is strongly prime. We also show that if $N$ is a proper powerful submodule of $M$, then $\sqrt{(N : M)}$ is a prime ideal in general and strongly prime when $R$ is seminormal. Moreover, a powerful submodule $N$ is comparable to every non-zero

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prime submodule of $M$, from which it follows that the prime submodule contained in $\sqrt{(N:M)}$ is linearly ordered.

As another generalization of the notion of "strongly prime" in Sec.2, we define a submodule $N$ of $M$ to be strongly prime if, $xyM \subseteq N$ with $x, y \in K$, we have $xM \subseteq N$ or $y^n M \subseteq N$ for some $n \geq 1$. Simple examples show that "powerful" and strongly primary are different notions. A strongly proper primary submodule of $M$ is clearly primary, and we observe that the converse is true over a valuation domain.

## 2. **Powerful submodules**

In this section, we want to consider the concept of a powerful submodule which is a generalization of the concept of strongly prime submodule. We shall prove that when a submodule $N$ is prime, $N$ is strongly prime if and only if $N$ is a powerful submodule.

**Definition 2.1.** *Let $R$ be an integral domain with quotient field $K$ and $M$ be an $R$-module. We define a non-zero submodule $N$ of $M$ to be powerful if, $xyM \subseteq N$, for elements $x, y \in K$, we have $x \in R$ or $y \in R$.*

We begin with a simple, but useful theorem on the concept of a powerful submodule.

**Theorem 2.2.** *A submodule $N$ of $M$ is powerful, if and only if $x^{-1}(N:M) \subseteq R$ for every $x \in K \setminus R$.*

*Proof.* Assume that $N$ is powerful, and take $x \in K \setminus R$. Then, for $a \in (N:M)$ we have, $xx^{-1}a = a \in (N:M)$, whence $x^{-1}a \in R$. For the converse, take $yzM \subseteq N$, $y, z \in K$. Suppose $y \notin R$. Then, $z = y^{-1}yz \in y^{-1}(N:M) \subseteq R$, as desired. □

**Corollary 2.3.** *$R$ is powerful if and only if $R$ is a valuation domain.*

**Theorem 2.4.** *Let $N$ be a powerful submodule of an $R$-module $M$ and $Q$ be a proper submodule of $N$. Then $\frac{N}{Q}$ is a powerful submodule of $\frac{M}{Q}$. In particular, if $P = (Q:M)$ is a prime ideal of $R$, then $\frac{N}{Q}$ is a powerful $\frac{R}{P}$-submodule of $\frac{M}{Q}$.*

*Proof.* Since we have $(\frac{N}{Q} : \frac{M}{Q}) = Ann(\frac{\frac{M}{Q}}{\frac{N}{Q}}) \cong Ann(\frac{M}{N}) = (N:M)$, and $N$ is a powerful submodule, it follows that $\frac{N}{Q}$ is a powerful submodule of $\frac{M}{Q}$ as $R$-module. □

**Definition 2.5.** *A submodule $N$ of an $R$-module $M$ is said to be prime in case $am \in N$, where $m \in M$ and $a \in R$, implies that $m \in N$ or $a \in (N : M)$. Also, $N$ is said to be a strongly prime submodule in case $N$ is a prime submodule and whenever, $xy \in (N : M)$ for elements $x, y \in K$, we have $x \in (N : M)$ or $y \in (N : M)$.*

**Theorem 2.6.** *Let $N$ be a prime submodule of an $R$-module $M$. Then, $N$ is a strongly prime submodule if and only if $N$ is a powerful submodule of $M$.*

*Proof.* Suppose that $N$ is a strongly prime submodule. If $x \in K \setminus R$ and $n \in (N : M)$ then $n = nxx^{-1} \in (N : M)$, whence $nx^{-1} \in (N : M)$ or $x \in (N : M)$. Since $x \notin R$ we must have $nx^{-1} \in (N : M)$. Thus $x^{-1}(N : M) \subseteq (N : M)$.

For the converse, let $N$ be a prime and powerful submodule of $M$ and $xy \in (N : M)$ whenever $x, y \in K$. Since $(N : M)$ is an ideal of $R$, hence $x^2y^2 \in (N : M)$. We may assume that $x \notin R$ and $y \in R$. If $x^2 \in R$, then since $x \notin R$, $x^2 \notin (N : M)$, and by the fact that $x^2y^2 \in (N : M)$ it follows that $y^2 \in (N : M)$, whence $y \in (N : M)$. If $x^2 \notin R$, Then, since $(\frac{y^2}{xy})x^2 \in (N : M)$, we have $\frac{y^2}{xy} \in R$. Hence $y^2 = (\frac{y^2}{xy})xy \in (N : M)$, and again we have $y \in (N : M)$. □

**Lemma 2.7.** *Let $Q \prec N$ be a submodule of an $R$-module $M$ and $N$ be a strongly prime submodule of $M$. Then $\frac{N}{Q}$ is a powerful submodule of $\frac{M}{Q}$.*

*Proof.* Use Theorem 2.6 and 2.4. □

**Lemma 2.8.** *If $Q \prec N$ is a submodule of an $R$-module $M$ and $N$ is a strongly prime, then $Q$ is powerful submodule of $M$.*

*Proof.* We have $(Q : M) \subseteq (N : M) \subseteq R$, therefore $xy \in (Q : M)$ for elements $x, y \in K$, implies that $xy \in (N : M)$. Since $N$ is strongly prime, $x \in (N : M)$ or $y \in (N : M)$, whence $x \in R$ or $y \in R$, and it follows that $Q$ is a powerful submodule of $M$. □

**Corollary 2.9.** *If $Q \prec N$ is a submodule of an $R$-module $M$ and $N$ is powerful, then $Q$ is also powerful.*

*Proof.* Since $(Q : M) \subseteq (N : M)$, $xy \in (Q : M)$ implies that $xy \in (N : M)$. Now since $N$ is powerful, we have $x \in R$ or $y \in R$. □

**Theorem 2.10.** *If $Q \prec N$ is a prime submodule of an $R$-module $M$ and $N$ is strongly prime then, $Q$ is strongly prime.*

*Proof.* This follows from Theorem 2.6 and Corollary 2.9. □

**Theorem 2.11.** *Let $N$ be a powerful submodule of an $R$-module $M$ and $Q$ an arbitrary submodule of $M$, then we have:*

(1) $(Q : M) \subseteq (N : M)$ *or* $(N : M)^2 \subseteq (Q : M)$.
(2) *If $Q$ is a prime submodule of an $R$-module $M$, then $(N : M)$ and $(Q : M)$ are comparable.*

*Proof.* To prove (1), we suppose that $Q$ is a submodule of $M$ and $(Q : M) \nsubseteq (N : M)$. Choose $a \in (Q : M) \setminus (N : M)$ and take $b, c \in (N : M)$. Then, $(\frac{bc}{a})(\frac{a}{b}) \in (N : M)$, and since $N$ is powerful with $\frac{a}{b} \notin R$, we have $\frac{bc}{a} \in R$. Hence $bc \in aR \subseteq (Q : M)$, as desired.

Statement (2): Since $Q$ is a prime submodule, hence $(Q : M)$ is a prime ideal in $R$. By (1), we have $(Q : M) \subseteq (N : M)$ or $(N : M)^2 \subseteq (Q : M)$. Since $(Q : M)$ is a prime ideal, $(Q : M) \subseteq (N : M)$ or $(N : M) \subseteq (Q : M)$. □

**Definition 2.12.** *Let $R$ be an integral domain and $M$ a torsion free $R$-module. Then, $M$ is said to be Fully strongly prime (FSP) if each prime submodule of $M$ is strongly prime.*

**Theorem 2.13.** *Let $R$ be an integral domain and $M$ be a finitely generated torsion free $R$-module, then $M$ is a FSP -module if and only if some maximal submodule of $M$ is powerful.*

*Proof.* Let $M$ be a FSP-module and $N$ be a maximal submodule of $M$. Therefore $N$ is a prime submodule and by Theorem 2.6 $N$ is powerful.

For the converse let some maximal submodule of $M$ as $N$ be a powerful submodule and also suppose that $Q$ is an arbitrary non-zero prime submodule of $M$. Now by 2.11(2), we have $(Q : M) \subseteq (N : M)$ or $(N : M) \subseteq (Q : M)$. Now if $(Q : M) \subseteq (N : M)$, then by Corollary 2.9, $Q$ is powerful and therefore by 2.6, $Q$ is strongly prime. And if $(N : M) \subseteq (Q : M)$, Since $(N : M)$ a maximal ideal in $R$, it follows that $(N : M) = (Q : M)$. Thus if $xy \in (Q : M)$ for every $x, y \in K$, then $xy \in (N : M)$. So $x \in (N : M)$ or $y \in (N : M)$ and hence $x \in (Q : M)$ or $y \in (Q : M)$. Therefore $Q$ is a strongly prime submodule of $M$. □

**Lemma 2.14.** *Let $\{N_\alpha\}$ be a family of powerful ideals of $R$, then $\sum N_\alpha$ is a powerful ideal.*

*Proof.* Use Theorem 2.2 □

**Question:** If $\{N_\alpha\}$ is a family of powerful submodules of an $R$-module $M$, is then $\sum N_\alpha$ a powerful submodule of $M$?

**Theorem 2.15.** *If $R$ contains a powerful ideal, then $R$ contains the largest unique powerful ideal.*

*Proof.* Let $\{N_\alpha\}$ be all powerful ideals of $R$, then $\sum N_\alpha$ is the largest unique powerful ideal of $R$ by Lemma 2.14. □

**Theorem 2.16.** *If $N$ is a proper powerful submodule of $R$-module $M$, then $P = \cap_k (N : M)^k$ is a strongly prime ideal.*

*Proof.* By Theorem 2.6 and Corollary 2.9, it suffices to show that $P$ is a prime ideal. Take $xy \in P$ with $x \notin P$. Then $x \notin (N : M)^n$ for some $n > 0$, whence by theorem 2.11 (1), $(N : M)^2 n \subseteq xR$. Hence for each $k > 0$, we have $xy \in P \subseteq (N : M)^{2n+k} \subseteq x(N : M)^k$. Thus $y \in (N : M)^k$ for each $k > 0$. It follows that $y \in P$. □

**Theorem 2.17.** *Let $N$ be a powerful submodule of an $R$-module $M$. If $x, y \in K$ and $xy \in \sqrt{(N : M)}$, then there is a positive integer $m$ such that either $x^m \in (N : M)$ or $y^m \in (N : M)$. In particular, if $N$ is a proper powerful submodule of an $R$-module $M$, then $\sqrt{(N : M)}$ is a prime ideal in $R$.*

*Proof.* Let $xy \in \sqrt{(N : M)}$, then $(xy)^n \in (N : M)$ for some $n > 0$. Hence $(\frac{x^{3n}}{x^n y^n})(\frac{y^{3n}}{x^n y^n}) = x^n y^n \in (N : M)$. Since $N$ is powerful, so that either $\frac{x^{3n}}{x^n y^n} \in R$ or $\frac{y^{3n}}{x^n y^n} \in R$, either $x^{3n} \in (N : M)$ or $y^{3n} \in (N : M)$. □

In spite of Theorem 2, the radical of a powerful ideal need not be powerful, as the following example shows.

**Example 2.18.** *Let $V = K + M$ be a discrete valuation domain with dim V=1, where $K$ is a field and $M = tV$ is a maximal ideal of $V$, and let $R = K + M^2$.*
*Claim: $M^3$ is a powerful ideal of $R$. To see this, take $xy \in M^3$, with $x, y \in K$ (the common quotient field of $R$ and $V$). We may write $x = ut^n$, $y = vt^m$, where $u, v$ are units of $V$ and $n, m$ are integers. Since $xy \in M^3$, we must have $n + m \geq 3$. Hence either $n \geq 2$ or $m \geq 2$, say $n \geq 2$. Then $x = ut^n \in M^2 \subseteq R$. This proves the claim. However Rad$(M^3) = M^2$ is not powerful since $t^2 \in M^2$ but $t \notin R$.*

**Corollary 2.19.** *If $N$ is a powerful submodule of an $R$-module $M$, then $\sqrt{(N : M)}$ is a powerful ideal.*

**Remark 2.20.** *We prove below that, over a seminormal domain, the radical of a powerful submodule is powerful. First, we need a lemma.*

**Lemma 2.21.** *Let $N$ be a powerful submodule of an $R$-module $M$. If $x \in K$ and $x^n \in (N : M)$ for some $n > 0$, then $x^{n+k} \in R$ for each $k \geq 0$.*

*Proof.* take $e = min\{m \geq 1 | x^m \in R\}$. Let $k$ be a positive integer and write $k = qe + r$ with $0 \leq r < e$. If $r = 0$, then it is easy to see that $x^{n+k} \in R$. Suppose that $r > 0$. We have $x^{e-r}x^{qe+n+r} = x^n x^{(q+1)e} \in (N : M)$. Since $x^{e-r} \notin R$, we have $x^{n+k} = x^{qe+n+r} \in R$ as desired. □

**Definition 2.22.** *Let $M$ be an $R$-module and $N$ be a submodule of $M$. We say that $N$ is radical if $x \in R$, $m \in M$ and $x^n m \in N$ for some $n > 0$ imply that $xm \in N$. A radical submodule $N$ of $M$ is said to be strongly radical, if $x \in K$, $m \in M$ and $x^n m \in N$ for some $n > 0$, imply that $xm \in N$. Also, $R$ is called seminormal if $x \in R$ whenever, $x^n \in R$ for all sufficiently large $n$.*

**Theorem 2.23.** *Let $N$ be a proper powerful submodule of an $R$-module $M$. Then $\sqrt{(N : M)}$ is a powerful ideal (and therefore strongly prime) if and only if $\sqrt{(N : M)}$ is strongly radical. In particular, if $R$ is seminormal, then $\sqrt{(N : M)}$ is strongly prime.*

*Proof.* It is easy to see that a powerful radical submodule must be strongly radical. Suppose that $\sqrt{(N : M)}$ is strongly radical, and take $xy \in \sqrt{(N : M)}$ with $x, y \in K$. Then by Theorem we have $x^m \in (N : M)$ or $y^m \in (N : M)$ for some $m > 0$. We may suppose that $x^m \in (N : M)$. Then, $x^m \in \sqrt{(N : M)}$, whence $x \in R$, as desired. The "in particular" □

Statement now follows from Lemma .

**Theorem 2.24.** *Let $R$ be an integral domain and $K = S^{-1}R$ ($S = R - 0$), then $R$ is a valuation domain if and only if $R$ as an ideal of $R$ is powerful.*

*Proof.* Take $R$ as a valuation domain and $xy \in R$ for $x, y \in K$, if $x \notin R$ then $x^{-1} \in R$. Therefore $x^{-1}xy \in R$, and hence $y \in R$. For the converse, we have $1 = x^{-1}x \in R$, hence $x \in R$ or $x^{-1} \in R$. □

**Theorem 2.25.** *Let $T$ be a domain, $R$ a subring of $T$, and $I$ a powerful ideal of $R$. Then $IT$ is a powerful ideal of $T$. In particular if $IT = T$ then $T$ is a valuation domain.*

*Proof.* Suppose that $x \in K - T$, then $x \notin R$ and therefore $x^{-1}I \subseteq R$ and hence $x^{-1}IT \subseteq T$, so $IT$ is a powerful ideal of $T$. Now if $IT = T$, then $x^{-1}T \subseteq T$ implies that $T$ is valuation domain. □

**Remark 2.26.** *Let $I$ be a powerful ideal of $R$ and suppose that $P \subseteq I$ is a non-zero finitely generated prime ideal of $R$. Then $R$ is a $FSP$-module as an $R$-module.*

*Proof.* If $P$ is not maximal, then $R$ contains a non-unit $x$ with $x \notin P$. Since $P$ is strongly prime (by Theorem 2.6 and Corollary 2.9) and $xx^{-1}P \subseteq P$ with $x \notin P$, we have $x^{-1}P \subseteq P$. Since $x^{-1}P \subseteq R$, Hence $x^{-1}$ is integral over $R$, which is impossible. Thus $P$ is maximal, and it follows that $R$ is a $FSP$- module by Theorem 2.13. □

**Corollary 2.27.** *Let $I$ be a powerful ideal and $m = \sqrt{I}$ be a maximal ideal of $R$. Then $R$ is a local ring with maximal ideal $m$.*

*Proof.* It follows from Theorem 2.11. □

**Corollary 2.28.** *Let $R$ be an integral domain, $I$ a powerful ideal of $R$ and $R \subseteq R'$ where $R'$ is an overing of $R$ such that $R' \neq K$ ($K = S^{-1}R$, $S = R - \{0\}$). Then*

(1) *If $IR' = R'$, then $P = N \cap R$, where $N$ is the maximal ideal of $R'$, is a common ideal which is powerful in both rings.*
(2) *If $IR' \neq R'$, then $I^2R'$ is a common ideal, and $I^3R'$ is powerful in both rings.*

*Proof.* For (1), recall that $R'$ is a valuation domain by Theorem 2.25. By Theorem 2.11 $I$ is comparable to $P$. The fact $IR' = R'$, then implies that $P \subseteq I$, whence $P$ is powerful, and therefore strongly prime in $R$. Note that $PR'$ is powerful in $R'$ by Theorem 2.25. We claim that $PR' = P$, to verify this, let $x \in R' - R$. Clearly $x^{-1} \notin P$. Hence, since $x^{-1}xP \subseteq P$ and $P$ is strongly prime, we have $xP \subseteq P$, as claimed.

(2) Let $x \in R' - R$. Then, by hypothesis, $x^{-1} \notin I$, whence $I^2 \subseteq x^{-1}R$ by Theorem 2.11. Hence, again $xI^2 \subseteq R$. Thus $I^2R'$ is an ideal of $R$. Since $I^3R' \subseteq I$, $I^3R'$ is powerful in $R$ by Theorem 2.9, and $I^3R'$ is powerful in $R'$ by Theorem 2.25. □

**Corollary 2.29.** *Suppose that $R'$ is an overing of $R$ and that $R$ and $R'$ share the non-zero ideal $J$. If $J$ is powerful in $R'$, then $J^3$ is a powerful ideal of $R$.*

*Proof.* Let $x \in K - R$. If $x \notin R'$, then $x^{-1}J \subseteq R'$ by Theorem 2.2. In this case , we have $x^{-1}J^3 \subseteq J^2R' \subseteq R$. Now assume $x \in R'$. Since $x \notin J$, we have $J^2 \subseteq xR'$ by Theorem 2.11. Hence $x^{-1}J^3 \subseteq JR' = J \subseteq R$, and the proof is complete. □

**Remark 2.30.** *In Corollary 2.29, if $R'$ is a valuation domain, then $J^2$ is powerful in $R$. However, for general, $R'$, the third power is best possible, as the following example shows.*

**Example 2.31.** *Let $K = Q(\sqrt{2})$ and $V = K[[X]] = K + M$, $M = XK[[X]]$. Then let $R' = Q + M$, $J = XR'$, and $R = Q + J$. Then $R$ and $R'$ share the ideal $J$, and since $R'$ is a FSP, $J$ is powerful in $R'$. However, $J^2$ is not powerful in $R$, since $\sqrt{2}X \times \sqrt{2}X = 2X^2 \in J^2$, but $\sqrt{2}X \notin R$.*

## 3. **Strongly primary submodules**

In this section we shall show that if $R$ is a seminormal ring, then strongly primary submodules are powerful and if $N$ is a strongly primary submodule, then $(N : M)$ is comparable to every radical ideal of $R$.

**Definition 3.1.** *A submodule $N$ of an $R$-module $M$ is said to be primary in case $am \in N$, where $m \in M$, $a \in R$, implies that $m \in N$ or $a^n \in (N : M)$ for some positive integer number $n \geq 1$ . Also $N$ is said to be a strongly primary submodule in case $N$ is a primary submodule and if $xy \in (N : M)$ for elements $x, y \in K$, we have $x \in (N : M)$ or $y^n \in (N : M)$ for some positive integer number $n \geq 1$.*

**Theorem 3.2.** *Let $M$ be an $R$-module and $R$ be a valuation domain. Then a primary submodule of $M$ is strongly primary.*

*Proof.* Let $K$ be a quotient field of $R$ and $N$ be a primary submodule of $M$. Also let $x, y \in K$ with $xy \in (N : M)$, and suppose that $x \notin (N : M)$. If $x \notin R$, then $x^{-1} \in R$, and we have $y = x^{-1}xy \in (N : M)$. Hence we may as well assume that $x \in R$. Since $x = y^{-1}xy \notin (N : M)$, it follows that $y \in R$. Now , since $x, y \in R$ with $N$ primary, we have $y^n \in (N : M)$ for some positive integer $n \geq 1$, as desired. □

**Remark 3.3.** *Professor R. Gilmer ([3], Exercise 2, P.293), showed that if $R$ is a valuation domain with $dim R > 1$, then there are ideals which are not primary. Since every ideal of a valuation domain is powerful, this shows that powerful ideals need not be (strongly) primary. Conversely, strongly primary ideals need not be powerful: In Example 2.18, $M^2$ is strongly primary but not powerful in $R$.*

**Notation 3.4.** *For a subset $S$ of an $R$-module $M$, we define $E(S)$ by $E(S) = \{x \in K | (\forall n \geq 1)(\exists m \in M) x^n m \notin S\}$.*

**Lemma 3.5.** *A non-zero primary submodule $N$ of $M$ is strongly primary if and only if $x^{-1}N \subseteq N$ for each $x \in E(N)$.*

*Proof.* If $N$ is strongly primary and $x \in E(N)$, then the equation $xx^{-1}N = N$ implies that $x^{-1}N \subseteq N$. Conversely, if $yz \in (N : M)$ with $y, z \in K$ and $z \in E(N)$, then the hypothesis yields $y = z^{-1}yz \in z^{-1}(N : M) \subseteq (N : M)$, as desired. □

**Theorem 3.6.** *Let $R$ be a seminormal domain. If $N$ is a proper strongly primary submodule of an $R$-module $M$, then $N$ is powerful, and $\sqrt{(N : M)}$ is strongly prime. In particular, a prime submodule of $M$ is strongly prime if and only if it is strongly primary.*

*Proof.* Let $x \in K - R$. We shall show that $x^{-1}(N : M) \subseteq (N : M)$ (whence $x^{-1}(N : M) \subseteq R$). By Lemma 3.5, it suffices to show that $x^n \notin (N : M)$ for all $n \geq 1$. Suppose, on the contrary, that $x^r \in (N : M)$, with $r$ minimal. It is then easy to see that $x^{-k} \notin (N : M)$ for each $k \geq 0$, that is $x^{-1} \in E(N : M)$. By Lemma 3.5, this implies that $x^{r+1} = xx^r \in x(N : M) \subseteq (N : M)$. By induction, we get $x^t \in (N : M) \subseteq R$ for each $t \geq r$. However, the seminormality of $R$ then implies that $x \in R$, a contradiction. □

**Theorem 3.7.** *Let $N$ be a proper strongly primary submodule of an $R$-module $M$, and let $R'$ be an overing of $R$. Then either $(N : M)R' = R'$ or $(N : M)R' = (N : M)$.*

*Proof.* Assume that $(N : M)R' \neq R'$ and pick $x \in R' - R$. If $x^{-n} \in (N : M)$ for some $n \geq 1$, then since $(N : M)R' \neq R'$, $x^{-n}$ is a non-unit of $R'$, a contradiction. Hence $x^{-1} \in E(N : M)$, and we have $x(N : M) \subseteq (N : M)$ by Lemma 3.5. Thus $(N : M)R' = (N : M)$. □

**Corollary 3.8.** *Let $R'$ be an integral closure of the domain $R$ and $N$ be a proper strongly primary submodule of an $R$- module $M$, then $(N : M)R' = (N : M)$. Moreover, $(N : M)^3$ is powerful in both $R$ and $R'$.*

*Proof.* The first conclusion follows from Theorem 3.7 and the lying over property of integral extensions. Since $(N : M)$ is automatically strongly primary in $M$, $N$ is powerful in $M$ by Theorem 3.6. It follows that $(N : M)^3$ is powerful in $R$ and $R'$ by Corollary 2.29. □

**Corollary 3.9.** *If $N$ is a proper strongly primary submodule of an $R$-module $M$, then $\bigcap(N : M)^n$ is a strongly prime ideal of $R$.*

*Proof.* This follows from Theorem 2.16 and the fact that $(N : M)^3$ is powerful. □

**Corollary 3.10.** *If $N$ is a strongly primary submodule of an $R$-module $M$, then $(N : M)$ is comparable to every radical ideal of $R$. Moreover, the prime submodules of $M$ which are properly contained in $N$ are strongly prime and linearly ordered.*

*Proof.* Let $J$ be a radical ideal of $R$, and suppose that $(N : M) \nsubseteq J$. Choose $a \in (N : M) - J$, and $b \in J$. Since $(\frac{a^2}{b})(\frac{b}{a}) = a \in (N : M)$ and $\frac{a^2}{b} \in E(R) \subseteq E(N : M)$, we have $\frac{b}{a} \in (N : M)$. Hence $J \in (N : M)$, as desired.
If $Q$ is a prime submodule which is properly contained in $N$, then, since $(N : M)^3$ is powerful and $(Q : M) \subseteq (N : M)^3$, $Q$ is also powerful. Then $Q$ is strongly prime. □

**Corollary 3.11.** *If $N$ is a prime submodule of $M$ which is strongly primary but not strongly prime, then $N$ is the only prime with this property.*

**Corollary 3.12.** *Let $R$ be an integral domain, $M$ be an $R$-module, also $R$-module $M' \neq S^{-1}M$ $(S = R - 0)$ be an overing of $M$, that is $M \subseteq M'$ and $N$ be a strongly primary submodule of $M$. Then we have the following cases:*

1. *If $(N : M)M' \neq M'$, then $(N : M)M' = N$ is a common strongly primary submodule.*
2. *If $(N : M)M' = M'$, then $M'$ is strongly primary, and for each maximal submodule $N'$ of $M'$, $N \cap M$ is a common strongly prime submodule of $M$ and $M'$.*

**Proposition 3.13.** *Let $N$ be a strongly primary submodule of $M$. Then:*

1. *$(N : M) \subseteq xR$ for every $x \in R \setminus \sqrt{(N : M)}$.*
2. *if $(N : M)$ is finitely generated, then $R$ is quasilocal with maximal ideal $\sqrt{(N : M)}$.*

*Proof.* Let $x \in R \setminus \sqrt{(N : M)}$. Then $x \in E(N : M)$ and so (by Lemma 3.5) $x^{-1}(N : M) \subseteq (N : M)$. Hence $(N : M) \subseteq x(N : M) \subseteq xR$, proving (1).

(2) The relation $x^{-1}(N : M) \subseteq (N : M)$ shows that $x^{-1}$ is integral over $R$. Since $x \in R$, we have $x^{-1} \in R$. It follows that $R$ is quasilocal with maximal ideal $\sqrt{(N : M)}$. □

## References

[1] D. D. Anderson and D. F. Anderson, Multiplicatively closed subset of fields, *Houston J. Math.* **13** (1987), no. 1, 1–11.

[2] A. Badawi, A visit to valuation and pseudo-valuation domains, Zero-dimensional commutative rings (Knoxville, TN, 1994), 155–161, Lecture Notes in Pure and Appl. Math., 171, Dekker, New York, 1995.

[3] R. Gilmer, Multiplicative Ideal Theory, New York, Marcel Dekker, Inc., New York, 1972.

[4] J. Hedstrom and E. Houston, Pseudo-Valuation domains, *Pacific J. Math.* **75** (1978), no. 1, 137–147.

[5] J. Hedstrom, E. Houston, Pseudo-Valuation domains, II, *Houston J. Math.* **4** (1978), no. 2, 199–207.

[6] I. Kaplansky, Commutative Rings, The University of Chicago Press, Chicago-London, 1974.

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