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GENERALIZED MULTIVALUED F-CONTRACTIONS ON COMPLETE METRIC SPACES

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Abstract. In the present paper, we introduce the concept of generalized multivalued F-contraction mappings and give a fixed point result, which is a proper generalization of some multivalued fixed point theorems including Nadler's.
Keywords: Fixed point, multivalued map, generalized F-contraction.
MSC(2010): Primary: 54H25; Secondary: 47H10.

1. Introduction and preliminaries

The beginning of metric fixed point theory is related to the Banach's Contraction Principle, published in 1922. Let (X, d) be a metric space and $T : X \to X$ be a selfmap of X. Then, T is said to be a contraction mapping if there exists a constant $L \in [0, 1)$ (called the contraction factor of T), such that

$$d(Tx, Ty) \leq Ld(x, y) \text{ for all } x, y \in X. \tag{1.1}$$

Banach's Contraction Principle says that, whenever (X, d) is complete, then any contraction selfmap of X has a unique fixed point. This fixed point result is one of the most powerful tools for many existence and uniqueness problems arising in mathematics. Because of its importance, Banach Contraction Principle has been extended and generalized in many ways; see, for instance [2–6, 11, 13, 16, 20, 23, 24, 27]. Among all these, an interesting generalization was given by Wardowski [26]. For

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the sake of completeness, we will discuss its basic lines. Let $\mathcal{F}$ be the set of all functions $F : (0,\infty) \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing, i.e., for all $\alpha, \beta \in (0,\infty)$ such that $\alpha < \beta, F(\alpha) < F(\beta)$,

(F2) For each sequence $\{\alpha_n\}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$

(F3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 1.1 ([26]). *Let (X,d) be a metric space and $T : X \to X$ be a mapping. Given $F \in \mathcal{F}$, we say that T is F-contraction, if there exists $\tau > 0$ such that*

$$x, y \in X,\ d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \le F(d(x,y)). \tag{1.2}$$

Taking in (1.2) different functions $F \in \mathcal{F}$, one gets a variety of F-contractions, some of them being already known in the literature. The following examples will certify this assertion:

Example 1 ([26]). Let $F_1 : (0,\infty) \to \mathbb{R}$ be given by the formula $F_1(\alpha) = \ln\alpha$. It is clear that $F_1 \in \mathcal{F}$. Then each self mapping T on a metric space (X,d) satisfying (1.2) is an F_1-contraction such that

$$d(Tx,Ty) \le e^{-\tau} d(x,y),\ \text{for all } x,y \in X, Tx \ne Ty. \tag{1.3}$$

It is clear that for $x, y \in X$ such that $Tx = Ty$ the inequality $d(Tx,Ty) \le e^{-\tau} d(x,y)$ also holds. Therefore T satisfies (1.1) with $L = e^{-\tau}$, thus T is a contraction.

Example 2 ([26]). Let $F_2 : (0,\infty) \to \mathbb{R}$ be given by the formula $F_2(\alpha) = \alpha + \ln\alpha$. It is clear that $F_2 \in \mathcal{F}$. Then each self mapping T on a metric space (X,d) satisfying (1.2) is an F_2-contraction such that

$$\frac{d(Tx,Ty)}{d(x,y)} e^{d(Tx,Ty)-d(x,y)} \le e^{-\tau},\ \text{for all } x,y \in X, Tx \ne Ty. \tag{1.4}$$

We can find some different examples for the function F belonging to $\mathcal{F}$ in [26]. In addition, Wardowski concluded that every F-contraction T is a contractive mapping, i.e.,

$$d(Tx,Ty) < d(x,y),\ \text{for all } x,y \in X, Tx \ne Ty.$$

Thus, every F-contraction is a continuous mapping.

Also, Wardowski concluded that if $F_1, F_2 \in \mathcal{F}$ with $F_1(\alpha) \le F_2(\alpha)$ for all $\alpha > 0$ and $G = F_2 - F_1$ is nondecreasing, then every F_1-contraction T is an F_2-contraction.

He noted that for the mappings $F_1(\alpha) = \ln \alpha$ and $F_2(\alpha) = \alpha + \ln \alpha$, $F_1 < F_2$, the mapping $F_2 - F_1$ is strictly increasing. Hence, it was obtained that every Banach contraction (1.3) satisfies the contractive condition (1.4). On the other side, Example 2.5 in [26] shows that the mapping T is not an F_1-contraction (Banach contraction), but still is an F_2-contraction. Thus, the following theorem, which was given by Wardowski, is a proper generalization of Banach Contraction Principle.

Theorem 1.2 ([26])**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point in X.*

Following Wardowski, Mınak et al [17] introduced the concept of a Ćirić type generalized F-contraction. Let (X, d) be a metric space and $T : X \to X$ be a mapping. Given $F \in \mathcal{F}$, we say that T is a Ćirić type generalized F-contraction if there exists $\tau > 0$ such that:

$$x, y \in X,\ d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(m(x, y)), \tag{1.5}$$

where

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}.$$

Then the following theorem was given:

Theorem 1.3. *Let (X, d) be a complete metric space and $T : X \to X$ be a Ćirić type generalized F-contraction. If T or F is continuous, then T has a unique fixed point in X.*

Concerning the multivalued versions of the preceding results, note that, in Altun et al [1], an extension of this type is considered for Theorem 1.2. It is our main aim in this work to establish a multivalued version of Theorem 1.3 as well. First we recall some useful properties of multivalued mappings. Using the concept of the Hausdorff metric, Nadler [19] introduced the notion of multivalued contraction mapping and proved a multivalued version of the well known Banach contraction principle. Let (X, d) be a metric space. Denote by $P(X)$ the family of all nonempty subsets of X, $CB(X)$ the family of all nonempty, closed and bounded subsets of X and $K(X)$ the family of all nonempty compact subsets of X. It is well known that, $H : CB(X) \times CB(X) \to \mathbb{R}$ is defined by, for every $A, B \in CB(X)$,

$$H(A, B) = \max\left\{\sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A)\right\}$$

is a metric on $CB(X)$, which is called Hausdorff metric induced by d, where $D(x, B) = \inf\{d(x, y) : y \in B\}$. Let $T : X \to CB(X)$ be a map, then T is called multivalued contraction if for all $x, y \in X$ there exists $L \in [0, 1)$ such that

$$H(Tx, Ty) \leq Ld(x, y).$$

Then Nadler [19] proved that every multivalued contraction mapping on a complete metric space has a fixed point.

Inspired by his result, various fixed point results concerning multivalued contractions appeared in the last decades; see, for instance, [7–9, 12, 14, 15]. Concerning these, the following problem was formulated by Reich [22]. Let (X, d) be a complete metric space. Suppose that $T : X \to CB(X)$ satisfies

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y),$$

for all $x, y \in X$, $x \neq y$, where $\alpha : (0, \infty) \to [0, 1)$ fulfills

$$\limsup_{s \to t^+} \alpha(s) < 1, \ \forall t \in (0, \infty).$$

Does T have a fixed point? A first partial affirmative answer to this was already given by Reich [21], in the case of $T : X \to K(X)$. A second partial answer to the same was obtained by Mizoguchi and Takahashi [18], for functions $\alpha : (0, \infty) \to [0, 1)$ taken so as

$$\limsup_{s \to t^+} \alpha(s) < 1, \ \forall t \in [0, \infty).$$

For a simple proof of this, we refer to Suzuki [25]; in addition, he showed that the result in question is a real generalization of Nadler's in [25]. Further aspects may be found in Du [10].

Also multivalued F-contractions by combining the ideas of Wardowski and Nadler was introduced in [1] and a fixed point result for these type mappings on complete metric space was given as:

Definition 1.4. *Let (X, d) be a metric space and $T : X \to CB(X)$ be a mapping. Then T is said to be a multivalued F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that*

$$x, y \in X, \ H(Tx, Ty) > 0 \Rightarrow \tau + F(H(Tx, Ty)) \leq F(d(x, y)). \tag{1.6}$$

By considering $F(\alpha) = \ln \alpha$, we can say that every multivalued contraction is also a multivalued F-contraction.

Theorem 1.5. *Let (X, d) be a complete metric space and $T : X \to K(X)$ be a multivalued F-contraction, then T has a fixed point in X.*

Remark 1.6. *Note that in Theorem 1.5, Tx is compact for all $x \in X$. Thus, we can present the following problem: Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued F-contraction. Does T have a fixed point? By adding a condition on F, we can give a partial answer to this problem as follows:*

Theorem 1.7. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued F-contraction. Suppose that, F also satisfies*

(F4) $F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

Then T has a fixed point.

Remark 1.8. *Note that if F is continuous and satisfies (F1), then it satisfies (F4).*

2. Main result

Definition 2.1. *Let (X, d) be a metric space and $T : X \to CB(X)$ be a mapping. Then T is said to be a generalized multivalued F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that*

$$x, y \in X,\ H(Tx, Ty) > 0 \Rightarrow \tau + F(H(Tx, Ty)) \le F(M(x, y)), \tag{2.1}$$

where

$$M(x, y) = \max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{1}{2}\left[D(x, Ty) + D(y, Tx)\right]\right\}.$$

Our main result is as follows:

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to K(X)$ be a generalized multivalued F-contraction. If T or F is continuous, then T has a fixed point in X.*

Proof. Let $x_0 \in X$. As Tx is nonempty for all $x \in X$, we can choose $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T and so the proof is completed. Let $x_1 \notin Tx_1$. Then $D(x_1, Tx_1) > 0$ since Tx_1 is closed. On the other hand, from $D(x_1, Tx_1) \le H(Tx_0, Tx_1)$ and (F1), we obtain

$$F(D(x_1, Tx_1)) \le F(H(Tx_0, Tx_1)).$$

From (2.1), we can write that

$$\begin{aligned}
F(D(x_1,Tx_1)) &\leq F(H(Tx_0,Tx_1)) \leq F(M(x_0,x_1)) - \tau \\
&= F(\max\left\{\begin{array}{c} d(x_0,x_1), D(x_0,Tx_0), D(x_1,Tx_1), \\ \frac{1}{2}\left[D(x_0,Tx_1)+D(x_1,Tx_0)\right] \end{array}\right\}) - \tau \\
&\leq F(\max\left\{d(x_0,x_1), \frac{1}{2}D(x_0,Tx_1)\right\}) - \tau \\
&\leq F(\max\left\{d(x_0,x_1), \frac{1}{2}\left[d(x_0,x_1)+D(x_1,Tx_1)\right]\right\}) - \tau \\
&\leq F(\max\left\{d(x_0,x_1), D(x_1,Tx_1)\right\}) - \tau \\
&= F(d(x_0,x_1)) - \tau. \qquad (2.2)
\end{aligned}$$

Since Tx_1 is compact, we obtain that $x_2 \in Tx_1$ such that $d(x_1,x_2) = D(x_1,Tx_1)$. Then, from (2.2)

$$F(d(x_1,x_2)) \leq F(H(Tx_0,Tx_1)) \leq F(d(x_1,x_0)) - \tau.$$

If we continue recursively, then we obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and

$$F(d(x_n,x_{n+1})) \leq F(d(x_n,x_{n-1})) - \tau \qquad (2.3)$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} \in Tx_{n_0}$, then x_{n_0} is a fixed point of T and so the proof is completed. Thus, suppose that for every $n \in \mathbb{N}$, $x_n \notin Tx_n$. Denote $a_n = d(x_n,x_{n+1})$, for $n = 0,1,2,\cdots$. Then, $a_n > 0$ for all n and, using (2.3), the following holds:

$$F(a_n) \leq F(a_{n-1}) - \tau \leq F(a_{n-2}) - 2\tau \leq \cdots \leq F(a_0) - n\tau. \qquad (2.4)$$

From (2.4), we get $\lim_{n\to\infty} F(a_n) = -\infty$. Thus, from (F2), we have

$$\lim_{n\to\infty} a_n = 0.$$

From (F3) there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} a_n^k F(a_n) = 0.$$

By (2.4), the following holds for all $n \in \mathbb{N}$

$$a_n^k F(a_n) - a_n^k F(a_0) \leq -a_n^k n\tau \leq 0. \qquad (2.5)$$

Letting $n \to \infty$ in (2.5), we obtain that

$$\lim_{n\to\infty} n a_n^k = 0. \qquad (2.6)$$

From (2.6), there exits $n_1 \in \mathbb{N}$ such that $na_n^k \le 1$ for all $n \ge n_1$. So we have

$$a_n \le \frac{1}{n^{1/k}}. \tag{2.7}$$

for all $n \ge n_1$. In order to show that $\{x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \ge n_1$. Using the triangle inequality for the metric and from (2.7), we have

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= a_n + a_{n+1} + \cdots + a_{m-1} \\ &= \sum_{i=n}^{m-1} a_i \\ &\le \sum_{i=n}^{\infty} a_i \\ &\le \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

By the convergence of the series $\sum\limits_{i=1}^{\infty} \frac{1}{i^{1/k}}$, we get $d(x_n, x_m) \to 0$ as $n \to \infty$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is, $\lim_{n\to\infty} x_n = z$.

If T is continuous, then we have $Tx_n \to Tz$ and

$$D(x_n, Tz) \le H(Tx_n, Tz),$$

so $D(z, Tz) = 0$ and $z \in Tz$.

Now, suppose F is continuous. In this case, we claim that $z \in Tz$. Assume the contrary, that is, $z \notin Tz$. In this case, there exist an $n_0 \in \mathbb{N}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $D(x_{n_k+1}, Tz) > 0$ for all $n_k \ge n_0$. (Otherwise, there exists $n_1 \in \mathbb{N}$ such that $x_n \in Tz$ for all $n \ge n_1$, which implies that $z \in Tz$. This is a contradiction, since $z \notin Tz$). Since $D(x_{n_k+1}, Tz) > 0$ for all $n_k \ge n_0$, then we have

$$\begin{aligned} \tau + F(D(x_{n_k+1}, Tz)) &\le \tau + F(H(Tx_{n_k}, Tz)) \\ &\le F(M(x_{n_k}, z)) \\ &\le F(\max\{d(x_{n_k}, z), d(x_{n_k}, x_{n_k+1}), D(z, Tz), \\ &\quad \frac{1}{2}[D(x_{n_k}, Tz) + d(z, x_{n_k+1})]\}). \end{aligned}$$

Taking the limit as $k \to \infty$ and using the continuity of F we have $\tau + F(D(z, Tz)) \leq F(D(z, Tz))$, which is a contradiction. Thus, we get $z \in \overline{Tz} = Tz$. This completes the proof. □

In the light of the Example 2.5 of [26], we can give the following example. This example shows that T is a generalized multivalued F-contraction but it is not generalized multivalued contraction.

Example 3. Let $X = \{x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\}$ and $d(x, y) = |x - y|$, $x, y \in X$. Then (X, d) is a complete metric space. Define the mapping $T : X \to K(X)$ by the:

$$Tx = \begin{cases} \{x_1\} & , \quad x = x_1 \\ \\ \{x_1, x_2, \cdots, x_{n-1}\} & , \quad x = x_n \end{cases}.$$

We claim that T is a multivalued F-contraction with respect to $F(\alpha) = \alpha + \ln \alpha$ and $\tau = 1$. To see this, we consider the following cases.

First, observe that

$$m, n \in \mathbb{N},\ H(Tx_m, Tx_n) > 0 \Leftrightarrow (m > 2 \text{ and } n = 1) \text{ or } (m > n > 1).$$

Case 1. For $m > 2$ and $n = 1$, we have

$$\begin{aligned} \frac{H(Tx_m, Tx_1)}{M(x_m, x_1)} e^{H(Tx_m,Tx_1) - M(x_m,x_1)} &= \frac{x_{m-1} - x_1}{x_m - x_1} e^{x_{m-1} - x_m} \\ &= \frac{m^2 - m - 2}{m^2 + m - 2} e^{-m} < e^{-m} < e^{-1}. \end{aligned}$$

Case 2. For $m > n > 1$, we have

$$\begin{aligned} \frac{H(Tx_m, Tx_n)}{M(x_m, x_n)} e^{H(Tx_m,Tx_n) - M(x_m,x_n)} &= \frac{x_{m-1} - x_{n-1}}{x_m - x_n} e^{x_{m-1} - x_{n-1} - x_m + x_n} \\ &= \frac{m + n - 1}{m + n + 1} e^{n-m} < e^{n-m} \leq e^{-1}. \end{aligned}$$

This shows that T is a generalized multivalued F-contraction, therefore, all conditions of Theorem 2.2 are satisfied and so T has a fixed point in X.

On the other hand, since

$$\lim_{n \to \infty} \frac{H(Tx_n, Tx_1)}{M(x_n, x_1)} = \lim_{n \to \infty} \frac{x_{n-1} - 1}{x_n - 1} = 1,$$

then T is not a generalized multivalued contraction.

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