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# A NOTE ON THE REMAINDERS OF RECTIFIABLE SPACES

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Abstract. In this paper, we mainly investigate how the generalized metrizability properties of the remainders affect the metrizability of rectifiable spaces, and how the character of the remainders affects the character and the size of a rectifiable space. Some results in [A. V. Arhangel'skiĭ and J. Van Mill, On topological groups with a first-countable remainder, *Topology Proc.* 42 (2013) 157–163.] and [F. C. Lin, C. Liu, S. Lin, A note on rectifiable spaces, *Topology Appl.* 159 (2012), no. 8, 2090–2101.] are improved, respectively.
**Keywords:** Rectifiable space, symmetrizable space, character.
**MSC(2010):** Primary: 54A25; Secondary: 54B05, 54E35.

## 1. Introduction

By a space we mean a Tychonoff topological space. A remainder of a space $X$ is the subspace $bX \setminus X$ of $bX$, where $bX$ is a Hausdorff compactification of $X$. The closure of a subset $A$ in the space $X$ is denoted by $\overline{A}^X$, and $\overline{A}$ stands for the closure of $A$ in $bX$. In this paper, $\tau$ is an infinite cardinal.

Remainders of a space $X$ have many interesting properties and have been studied extensively in literature. A famous classical result in this study is the following theorem of Henriksen and Isbell [12]:

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**Theorem 1.1.** [12, Theorem 3.6] *A space $X$ is of countable type if and only if the remiander in any (in some) compactification of $X$ is Lindelöf.*

Recall that a space $X$ is of *countable type* [10] if every compact subspace $P$ of $X$ is contained in a compact subspace $F \subseteq X$ that has a countable base of open neighborhoods in $X$.

A *rectification* on a space $X$ is a homeomorphism $\varphi : X \times X \to X \times X$ with the following two properties:

(R1) $\varphi(\{x\} \times X) = \{x\} \times X$ for each $x \in X$;

(R2) there exists an element $e \in X$ such that $\varphi(x, x) = (x, e)$ for every $x \in X$.

The point $e \in X$ is called the *neutral element* of the space $X$. A space with a rectification is called a *rectifiable space*. Every rectifiable space is homogeneous (see [8, 9]).

The following result is due to Čhoban.

**Theorem 1.2.** [9] *A topological space $G$ is a rectifiable space if and only if there exist $e \in G$ and two continuous maps $p : G \times G \to G$, $q : G \times G \to G$ such that for any $x \in G,\ y \in G$ the next identities hold:*

$$p(x, q(x, y)) = q(x, p(x, y)) = y \text{ and } q(x, x) = e.$$

In recent years, there have been many interesting and new results on rectifiable spaces and their remainders. In 2010, Arhangel'skiĭ and Čhoban [3] showed that for any Hausdorff compactification $bG$ of an arbitrary rectifiable space $G$, the remainder $bG \setminus G$ is either pseudocompact or Lindelöf. They also proved that the remainder $Y$ of a paracompact rectifiable space $G$ has a $G_\delta$-diagonal if and only if $Y, G$, and $bG$ are separable metrizable spaces [3]. In 2012, Fucai Lin, Chuan Liu and Shou Lin [14] investigated how the generalized metrizability properties of the remainder affect the metrizability of a rectifiable space. Some other results on a rectifiable space and its remainder can be found in [13, 15].

A space $X$ is said to have the property $(L)$ if $X$ satisfies one of the following conditions:

- $(L_1)$ if the cardinality of $X$ is Ulam non-measurable, then $X$ is weakly HN-complete[1];
- $(L_2)$ every Lindelöf $p$-subspace[2] of $X$ is metrizable;

[1] A space $X$ is *weakly HN-complete* if the remainder $Z$ of $X$ in the Čech-Stone compactification $\beta X$ of $X$ is a space of point-countable type.

[2] A space $X$ is a Lindelöf $p$-space if and only if it is the inverse image of a separable metric space by a perfect map.

($L_3$) every countably compact subset of $X$ is metrizable;
($L_4$) every compact subset of $X$ is a $G_\delta$-set in $X$.

Since every countably compact metrizable space is compact, the conditions ($L_3$) and ($L_4$) in ($L$) can be replaced by the following condition ($L_5$): every countably compact subset of $X$ is a metrizable $G_\delta$-set.

**Remark 1.3.** *A paracompact space has the property ($L_1$), since a paracompact space with Ulam non-measurable cardinality is HN-complete* [10]*, and hence it is weakly HN-complete.*

It was proved by Arhangel'skiĭ and Van Mill [6] that for every non-locally compact topological group $G$ with a first-countable remainder, the character of $G$ does not exceed $\omega_1$ and the cardinality of $G$ does not exceed $2^{\omega_1}$. Moreover, A.V. Arhangel'skiĭ and Van Mill [6] showed that there exists a non-metrizable non-locally compact topological group $G$ with a first-countable remainder. This fact shows that first-countability of some remainder of a topological group does not imply the metrizability of the group itself.

In section 2, we investigate how the generalized metrizability properties of the remainders affect the metrizability of rectifiable spaces. The following results are obtained:

(1) Let $G$ be a non-locally compact rectifiable space with property ($L_1$). If the remainder $Y = bG \setminus G$ has locally a property ($L_5$), then $G$ is separable metrizable and $Y$ is a first-countable, Lindelöf $p$-space. This result generalizes some known results on rectifiable spaces and their remainders.

(2) Let $G$ be a non-locally compact rectifiable space. Then $bG$ is separable metrizable if the remainder $Y = bG \setminus G$ of $G$ has a locally point-countable $p$-meta-base with $\pi\chi(Y) \leq \omega$.

(3) Let $G$ be a paracompact and non-locally compact rectifiable space and $Y = bG \setminus G$ be locally symmetrizable. Then $bG$ is separable and metrizable if each singleton of $Y$ is a $G_\delta$-set in $Y$.

In section 3, we study how the character of the remainders affect the character and the size of a rectifiable space. We generalize a result of A.V. Arhangel'skii and J. Van Mill's in [6]. We mainly show that: (1) If $G$ is a non-locally compact rectifiable space with a remainder $Y$ such that $\chi(Y) \leq \tau$, then $\chi(G) \leq \tau^+$; (2) If $G$ is a non-locally compact rectifiable space with a reminder $Y$ satisfying $\chi(Y) \leq \tau$, then $|G| \leq 2^{\tau^+}$.

## 2. **On some generalized metrizability properties**

In this section, we investigate how the generalized metrizability properties of the remainders affect the metrizability of rectifiable spaces. The following theorem was proved in [1].

**Lemma 2.1.** [1, Theorem 2.1] *If $X$ is a Lindelöf $p$-space, then every remainder of $X$ is a Lindelöf $p$-space.*

**Remark 2.2.** *If $G$ is a non-locally compact rectifiable space, then $G$ is nowhere locally compact since $G$ is homogeneous. Hence $Y = bG \setminus G$ is dense in $bG$, i.e., $bG$ is also a compactification of $Y$. By Theorem 1.1 and Lemma 2.1, the following statements hold:*
(1) *$Y$ is of countable type $\Leftrightarrow$ $G$ is Lindelöf;*
(2) *$Y$ is a Lindelöf $p$-space $\Leftrightarrow$ $G$ is a Lindelöf $p$-space.*

Recall that a space $X$ has *locally a property* $\Phi$ if for each point $x \in X$ there exists an open neighborhood $U(x)$ of $x$ such that $U(x)$ has property $\Phi$. Firstly, we give a lemma, which plays an important role in the proofs of our main results and is interesting itself as well.

**Lemma 2.3.** [3, Lemma 2.3] *Suppose that $B = X \cup Y$, where $B$ is a compact space, and $X, Y$ are dense nowhere locally compact subspaces of $B$. Suppose that $Y$ is of subcountable type* [3]. *Then each locally finite (in $X$) family of non-empty open subsets of $X$ is countable.*

**Lemma 2.4.** *Let $Y$ be a remainder of a paracompact and non-locally compact rectifiable space $G$. Then $Y$ is of countable type if and only if $Y$ is of subcountable type.*

*Proof. Necessity.* It is trivial.

*Sufficiency.* Suppose that $Y$ is a remainder of a paracompact and non-locally compact rectifiable space $G$ and that $Y$ is of subcountable type. We know that $Y$ is either pseudocompact or Lindelöf.

Case 1. $Y$ is pseudocompact.

Take an arbitrary compact subset $F$ of $Y$. Since $Y$ is of subcountable type, $F$ is contained in a compact $G_\delta$-set $L$ of $Y$. By the pseudocompactness of $Y$ it follows that the compact $G_\delta$-set $L$ has a countable base of open neighborhoods in $Y$, and hence $Y$ is of countable type.

Case 2. $Y$ is Lindelöf.

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[3] A space $X$ is of subcountable type [3] if every compact subset of $X$ is contained in a compact $G_\delta$-set of $X$.

Since $Y$ is of subcountable type, by Lemma 2.3 it follows that each locally finite (in $G$) family of non-empty open subsets of $G$ is countable. Thus $G$ is Lindelöf by the paracompactness of $G$. Therefore, $Y$ is of countable type by Remark 2.2. □

Recall that a $\pi$-network ($\pi$-base) of a space $X$ at a point $x \in X$ is a family $\xi$ of non-empty subsets (open subsets, respectively) of $X$ such that every open neighborhood of $x$ contains a member of $\xi$. The $\pi$-*character* of $x$ in $X$ is defined by $\pi\chi(x, X) = \omega + \min\{|\xi| : \xi$ is a local $\pi-$ base at $x$ *in* $X\}$. The $\pi$-*character* of $X$ is defined by $\pi\chi(X) = \sup \{\pi\chi(x, X) : x \in X\}$.

Before giving one of our main results we recall another result which was proved in [14].

**Lemma 2.5.** [14, Lemma 7.1] *Let $G$ be a non-locally compact rectifiable space. Then $G$ is metrizable and locally separable, if the remainder $Y = bG \setminus G$ has locally a property $(L_3)$ and $\pi$-character of $Y$ is countable.*

**Proposition 2.6.** *Let $G$ be a non-locally compact rectifiable space. If the remainder $Y = bG \setminus G$ with $\pi\chi(Y) \le \omega$ has locally a property $(L_5)$, then $G$ is separable metrizable and $Y$ is a first-countable, Lindelöf $p$-space.*

*Proof.* Claim. Every compact subset $F$ of $Y$ is a metrizable $G_\delta$-set in $Y$.

Suppose that $Y = bG \setminus G$ has locally a property $(L_5)$. Consider the open cover $\mathscr{U} = \{U(y) : y \in F\}$, where $U(y)$ is an open neighborhood of $y$ in $Y$ such that every countably compact subset of $U(y)$ is a metrizable $G_\delta$-set of $U(y)$ for each point $y \in F$. There exists a finite subfamily $\mathscr{U}'$ of $\mathscr{U}$ such that $\mathscr{U}'$ covers $F$ because $F$ is compact. For each $U \in \mathscr{U}'$ and each $z_U \in U \cap F$, take an open neighborhood $V(z_U)$ of $z_U$ in $Y$ such that $\overline{V(z_U)}^Y \subset U$. Clearly, $\overline{V(z_U)}^Y \cap F$ is countably compact in $U$, so $\overline{V(z_U)}^Y \cap F$ is a metrizable $G_\delta$-set of $U$. Since $U$ is open in $Y$, $\overline{V(z_U)}^Y \cap F$ is a $G_\delta$-set of $Y$. Put $\mathscr{V} = \bigcup\{\mathscr{V}_U : U \in \mathscr{U}'\}$, where $\mathscr{V}_U = \{V(z_U) : z_U \in U \cap F\}$. Then $\mathscr{V}$ is an open cover of $F$. There is a finite subfamily $\mathscr{V}'$ of $\mathscr{V}$ such that $\mathscr{V}'$ covers $F$. Clearly $F = \bigcup\{F \cap \overline{V}^Y : V \in \mathscr{V}'\}$. Since each $F \cap \overline{V}^Y$ is a metrizable $G_\delta$-set of $Y$, it is easy to show that $F$ is a metrizable $G_\delta$-set of $Y$.

By Lemma 2.5 it follows that $G$ is metrizable. Thus, according to Lemma 2.1 and Lemma 2.4 one can easily obtain that $G$ is separable metrizable and $Y$ is a Lindelöf $p$-space, since the claim above implies that $Y$ is of subcountable type. Then $Y$ is first-countable since $Y$ is a $p$-space with a countable pseudocharacter. □

**Theorem 2.7.** *Let $G$ be a non-locally compact rectifiable space with property $(L_1)$. If the remainder $Y = bG \setminus G$ has locally a property $(L_5)$, then $G$ is separable metrizable and $Y$ is a first-countable, Lindelöf $p$-space.*

*Proof.* According to Proposition 2.6 it is enough to show that $Y$ has countable $\pi$-character. We have seen above that $Y$ is either pseudocompact or Lindelöf. Thus it is enough to consider the following two cases.

Case 1. The space $Y$ is pseudocompact.

Since $Y$ has locally a property $(L_5)$, each singleton of $Y$ is a $G_\delta$-set. Thus $Y$ is first-countable by the pseudocompactness of $Y$.

Case 2. The space $Y$ is Lindelöf.

Since $Y$ is a space of countable pseudocharacter, it follows that the cardinality of $Y$ is Ulam non-measurable [5]. Since $G$ is a non-locally compact rectifiable space, the cardinality of $G$ is also Ulam non-measurable [5]. Then $G$ is weakly HN-complete by Remark 1.3. By [2, Theorem 4], each $G_\delta$-point of $Y$ is a point of bisequentiality of $Y$, it follows that $\pi\chi(Y) \leq \omega$. □

According to Remark 1.3 and Theorem 2.7 one can easily obtain the following result.

**Corollary 2.8.** *Let $G$ be a paracompact and non-locally compact rectifiable space. If the remainder $Y = bG \setminus G$ has locally a property $(L_5)$, then $G$ is separable metrizable and $Y$ is a first-countable, Lindelöf $p$-space.*

**Corollary 2.9.** [14, Proposition 7.2] *Let $G$ be a non-locally compact rectifiable space with property $(L_1)$. If the remainder $Y = bG \setminus G$ has locally a properties $(L_2)$ and $(L_5)$ , then $bG$ is separable metrizable.*

*Proof.* From Theorem 2.7 it follows that $G$ is separable metrizable, and $Y$ is a first-countable, Lindelöf $p$-space. Thus $Y$ is locally metrizable, since $Y$ has locally a property $(L_2)$ and the property Lindelöf $p$-space is hereditary with respect to closed subspaces. Then $Y$ is separable metrizable by [10, 5.4.A], since $Y$ is Lindelöf. Therefore, both $Y$ and $G$ have countable networks, which implies that $bG$ has a countable network as well. By the compactness of $bG$, one can easily obtain that $bG$ is separable metrizable. □

**Corollary 2.10.** [14, Proposition 7.5] *Let $G$ be a non-locally compact rectifiable space. If the remainder $Y = bG \setminus G$ with $\pi\chi(Y) \leq \omega$ has locally a property $(L_2)$ and $(L_5)$, then $bG$ is separable metrizable.*

*Proof.* From Lemma 2.5 it follows that $G$ is metrizable. Thus $G$ has property $(L_1)$ by Remark 1.3. It follows that $bG$ is separable metrizable from Corollary 2.9. □

We refer the reader to [16] for the definition of $p$-meta-base. The following result improves Corollary 7.6 in [14].

**Corollary 2.11.** *Let $G$ be a non-locally compact rectifiable space. Then $bG$ is separable metrizable if the remainder $Y = bG \setminus G$ of $G$ has locally a point-countable $p$-meta-base with $\pi\chi(Y) \leq \omega$.*

*Proof.* The property point-countable $p$-meta-base is hereditary with respect to subspaces. Thus, by [16, Theorem 3.1.8] and [7, Proposition 2.1], a space with a point-countable $p$-meta-base satisfies the properties $(L_2)$ and $(L_5)$. It follows that $bG$ is separable metrizable from Corollary 2.10. □

Let $X$ be a set and all non-negative real numbers be denoted by $\mathbb{R}^+$. A function $d : X \times X \to \mathbb{R}^+$ is *symmetric* on the set $X$ if, for each $x, y \in X$, (i) $d(x, y) = 0$ if and only if $x = y$; (ii) $d(x, y) = d(y, x)$. A space $X$ is said to be *symmetrizable* if there is a symmetric $d$ on $X$ satisfying the following condition: $U \subseteq X$ is open if and only if for each $x \in U$ there exists $\varepsilon > 0$ with $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\} \subset U$.

**Lemma 2.12.** [11, Lemma 9.12] *Every $\omega_1$-compact* [4] *symmetric space is hereditary Lindelöf.*

**Lemma 2.13.** [4, Corollary 2.8] *Let $G$ be a rectifiable space. Then $G$ is of countable type if and only if there exists a non-empty compact subset $F$ of $G$ such that $F$ has a countable base of open neighborhoods in $G$.*

The following result was proved in [17]. For completeness we give its proof.

**Lemma 2.14.** *Every countably compact subset of a symmetric space is compact and metrizable.*

[4] A space $X$ is $\omega_1$-*compact* if every closed discrete subset of $X$ has cardinality less than $\omega_1$.

*Proof.* Let $A$ be a countably compact subset of a symmetric space $X$. If $\{x_n\}$ is a sequence in $A$ which converges to a point $x$ in $X$, then the sequence $\{x_n\}$ has an accumulation point $a$ in $A$ by the countable compactness of $A$. Thus $a$ is also an accumulation point of $\{x_n\}$ in $X$, hence $x = a \in A$, i.e., $A$ is a sequentially closed subset in $X$. Since $X$ is symmetrizable, $X$ is a sequential space [11, p.481], i.e., every sequentially closed subset is closed in $X$. Therefore, $A$ is closed in $X$, which implies that $A$ is symmetrizable, thus $A$ is compact and metrizable by [11, Theorem 9.13]. □

**Theorem 2.15.** *Let $G$ be a paracompact and non-locally compact rectifiable space, and $Y = bG \setminus G$ be locally symmetrizable. Then $bG$ is separable and metrizable if each singleton of $Y$ is a $G_\delta$-set in $Y$.*

*Proof.* Case 1. $G$ is of countable type.

By Theorem 1.1, $Y$ is Lindelöf, thus $Y$ is $\omega_1$-compact. Then $Y$ is locally a hereditarily Lindelöf space by Lemma 2.12. Thus, from Lemma 2.14 it follows that $Y$ has locally a property $(L_5)$, since $Y$ is locally symmetrizable. Therefore, $G$ is separable metrizable and $Y$ is a Lindelöf $p$-space by Theorem 2.7. Since every symmetrizable, Lindelöf $p$-space is metrizable [11, Theorem 9.13], $Y$ is locally metrizable. Then $Y$ is separable metrizable by [10, 5.4.A], since $Y$ is Lindelöf. Therefore, both $Y$ and $G$ have a countable network, which implies that $bG$ has a countable network as well. By the compactness of $bG$, one can easily obtain that $bG$ is separable metrizable.

Case 2. Each singleton of $Y$ is a $G_\delta$-set in $bG$.

$Y$ is first-countable by the conditions $G_\delta$-subset and the compactness of $bG$. Then $G$ is metrizable and locally separable by Lemma 2.5, which implies that $G$ is of countable type. Then $bG$ is separable and metrizable by Case 1.

Case 3. There exists a point $y \in Y$ such that $\{y\}$ is not a $G_\delta$-set in $bG$.

There exists a $G_\delta$-set $P$ in $bG$ such that $\{y\} = P \cap Y$ and $P \cap G \neq \emptyset$. Take a sequence $\{U_n\}$ of open subsets in $bG$ with $P = \bigcap_{n\in\omega} U_n$. Fix a point $g \in P \setminus \{y\}$. There is an open subset $V_n$ in $bG$ such that $y \notin \overline{V_n}$, and $g \in V_{n+1} \subset \overline{V_{n+1}} \subseteq V_n \cap U_{n+1}$ for each $n \in \omega$. Put $F = \bigcap_{n\in\omega} V_n$. Clearly, $F$ is a non-empty closed $G_\delta$-set in $bG$ with $F \subseteq G$. One can easily obtain that $F$ has a countable base of open neighborhoods in $bG$ by the compactness of $bG$. Therefore $F$ has a countable base of open neighborhoods in $G$ as well. It is obvious that $F$ is a compact subset of

$G$. Then $G$ is of countable type by Lemma 2.13. Thus $bG$ is separable and metrizable by Case 1. □

**Corollary 2.16.** *Let $G$ be a paracompact and non-locally compact rectifiable space, and $Y = bG \setminus G$ be locally symmetrizable. Then $bG$ is separable and metrizable if $Y$ satisfies one of the following conditions.*
(1) *$Y$ is locally perfect;*
(2) *$Y$ is locally Lindelöf;*
(3) *$Y$ is locally $\omega_1$-compact.*

**Corollary 2.17.** *Let $G$ be a non-locally compact rectifiable space, and $Y = bG \setminus G$ be locally symmetrizable. Then $bG$ is separable and metrizable if $\pi$-character of $Y$ is countable.*

*Proof.* $G$ is metrizable and locally separable by Lemmas 2.5 and 2.14, then $Y$ is Lindelöf by Theorem 1.1. Thus the statement follows from Corollary 2.16. □

## 3. On the character of rectifiable spaces

In this section, we study how the character of the remainders affect the character and the size of a rectifiable space. The following proposition is similar to the result [6, Proposition 2.2].

Recall that the tightness of a space $X$ is the minimal cardinal $\tau \geq \omega$ with the property that for every point $x \in X$ and every set $P \subset X$ with $x \in \overline{P}$, there is a subset $Q$ of $P$ such that $|Q| \leq \tau$ and $x \in \overline{Q}$. The tightness of $X$ is denoted by $t(X)$.

The definition of *complete accumulation point* can be found in [10].

**Proposition 3.1.** *Suppose that $Y$ is a space of $\tau$-tightness satisfying the following condition: (c) for any subset $A$ of $Y$ with $|A| \leq \tau^+$, $\overline{A}^Y$ is compact. Then $Y$ is compact.*

*Proof.* Assume the contrary. We can regard $Y$ as a non-closed subspace of some Hausdorff compactification $X$ of the space $Y$. Pick $x \in \overline{Y} \setminus Y$.

Claim 1: For any $G_\tau$-subset $P = \{P_\alpha\}_{\alpha\in\tau}$ of $X$ with $x \in P$, we can conclude that $P \cap Y \neq \emptyset$.

Since $X$ is a Tychonoff space, there exists an open set $V_\alpha$ containing $x$ such that $x \in V_\alpha \subset \overline{V_\alpha} \subset P_\alpha$ for each $\alpha \in \tau$. Next we shall prove that $(\bigcap_{\alpha\in\tau} \overline{V_\alpha}) \cap Y \neq \emptyset$. Put $\mathscr{F} = \{\overline{V_\alpha}\}_{\alpha\in\tau}$ and $\mathscr{F}' = \{\bigcap \mathscr{F}'' : \mathscr{F}'' \subset \mathscr{F}$ and $|\mathscr{F}''| < \omega\} = \{K_\alpha\}_{\alpha\in\tau}$. Since $x \in \overline{Y} \setminus Y$, there is $x_\alpha \in K_\alpha \cap Y$

for each $\alpha \in \tau$. Put $A = \{x_\alpha\}_{\alpha\in\tau} \subset Y$. Then $|A| \leq \tau$. Thus $\overline{A}^Y$ is compact and $\{\overline{V_\alpha} \cap \overline{A}^Y\}_{\alpha\in\tau}$ is a family of non-empty closed subsets of $\overline{A}^Y$ which has finite intersection property. Therefore, $\emptyset \neq \bigcap_{\alpha\in\tau}(\overline{V_\alpha} \cap \overline{A}^Y) \subset (\bigcap_{\alpha\in\tau} \overline{V_\alpha}) \cap Y \subset P \cap Y$, which completes the proof of the Claim 1.

By Claim 1, we define a point $y_\alpha \in Y$ and a closed $G_\tau$- subset $P_\alpha$ of $X$ containing $x$ for each $\alpha < \tau^+$. Let $y_0$ be any element of $Y$ and $P_0 = X$. Assume that $\alpha \in \tau^+$, and that the points $y_\beta \in Y$ and the closed $G_\tau$-subsets $P_\beta$ have been defined for each $\beta < \alpha$. Let $F_\alpha = \overline{\{y_\beta : \beta < \alpha\}}$. Thus, $F_\alpha \subset Y$ and $x \notin F_\alpha$. Since $F_\alpha$ is closed in $X$, there exists a closed $G_\delta$-subset $V_\alpha$ of $x$ in $X$ such that $x \in V_\alpha$ and $V_\alpha \cap F_\alpha = \emptyset$. Let $P_\alpha = V_\alpha \cap \bigcap_{\beta<\alpha} P_\beta$. It is obvious that $P_\alpha$ is a closed $G_\tau$-subset of $X$ and $x \in P_\alpha$. We can conclude that $P_\alpha \cap Y \neq \emptyset$ by Claim 1. Pick a point $y_\alpha \in P_\alpha \cap Y$. Then the sequences $\{y_\alpha : \alpha \in \tau^+\}$ and $\{P_\alpha : \alpha \in \tau^+\}$ are constructed. It is clear that the following statements hold for any $\alpha \in \tau^+$.

Claim 2: $F_\alpha \cap P_\alpha = \emptyset$.

Claim 3: $\overline{\{y_\beta : \alpha \leq \beta < \tau^+\}} \subset P_\alpha$.

Claim 4: $F_\alpha \cap \overline{\{y_\beta : \alpha \leq \beta < \tau^+\}} = \emptyset$.

Let $\eta = \{y_\alpha : \alpha \in \tau^+\}$. It is obvious that $\eta \subset Y$ and $|\eta| \leq \tau^+$. Some point $z$ of $Y$ is a complete accumulation point for $\eta$. Since $t(Y) \leq \tau$, it follows from Claim 4 that no point of $Y$ is a complete accumulation point for $\eta$. □

**Proposition 3.2.** *Suppose that $X$ is a nowhere locally compact space with a remainder $Y$ such that $t(Y) \leq \tau$ and $\pi\chi(Y) \leq \tau^+$. Then the $\pi$-character of the space $X$ does not exceed $\tau^+$ at some point of $X$.*

*Proof.* Assume that $bX$ is a compactification of the space $X$ such that $Y = bX \setminus X$. Since $X$ is nowhere locally compact, $Y$ is not closed in $bX$, that is, $Y$ is not compact. It follows from Proposition 3.1 that $Y$ does not satisfy the condition $(c)$. Therefore, there exists a subset $A$ of $Y$ such that $|A| \leq \tau^+$ and $\overline{A}^Y$ is not compact. Then there is $x \in \overline{A} \setminus Y$. Clearly, $\overline{Y} = bX$ by the fact that $X$ is nowhere locally compact. Since $\pi\chi(Y) \leq \tau$, $\pi\chi(y, bX) \leq \tau$ for each $y \in Y$. Thus, we can fix a local $\pi$-base $\xi_y$ of $bX$ at $y$ for every $y \in Y$ such that $|\xi_y| \leq \tau$. Let $\gamma = \bigcup_{y\in A} \xi_y$ and $\mathscr{P} = \{W \cap X : W \in \gamma\}$. Since $X$ is dense in $bX$ and $x \in \overline{A}$, the family $\mathscr{P}$ is a $\pi$-base of $X$ at $x$. Clearly, $|\mathscr{P}| \leq \tau^+$. □

The following theorems generalize A.V. Arhangel'skiĭ and J. Van Mill's results [6, Theorem 2.1 and Theoem 2.4].

**Theorem 3.3.** *Suppose that $G$ is a non-locally compact rectifiable space with a remainder $Y$ such that $\chi(Y) \leq \tau$. Then $\chi(G) \leq \tau^+$.*

*Proof.* It follows from Proposition 3.2 that there exists a $\pi$-base $\mathscr{P}$ of $G$ at the neutral element $e$ of $G$ such that $|\mathscr{P}| \leq \tau^+$. Therefore, the family $\mathscr{B} = \{q(P, P) : P \in \mathscr{P}\}$ is a base of $G$ at $e$ such that $|\mathscr{B}| \leq \tau^+$. □

**Theorem 3.4.** *If $G$ is a non-locally compact rectifiable space with a remainder $Y$ satisfying $\chi(Y) \leq \tau$ , then $|G| \leq 2^{\tau^+}$.*

*Proof.* Let $bG$ be a compactification of the space $G$ such that the remainder $Y = bG \setminus G$ satisfies $\chi(Y) \leq \tau$. Then $\chi(G) \leq \tau^+$ by Theorem 3.3. Since $\chi(Y) \leq \tau$ and $\overline{Y} = \overline{G} = bG$, $\chi(bG) \leq \tau^+$, it follows from compactness of $bG$ that $|bG| \leq 2^{\tau^+}$. Therefore, $|G| \leq 2^{\tau^+}$. □

Since it is consistent with ZFC that $2^\tau = 2^{\tau^+}$, it follows that the next statement holds.

**Corollary 3.5.** *It is consistent with ZFC that if $G$ is any non-locally compact rectifiable space with a remainder $Y$ satisfying $\chi(Y) \leq \tau$ , then $|G| \leq 2^{\tau^+}$.*

**Theorem 3.6.** *Suppose that $G$ is a non-locally compact rectifiable space with a remainder $Y$ such that the tightness of $Y$ is $\tau$ and $\pi\chi(Y) \leq \tau^+$. Then $\chi(G) \leq \tau^+$.*

*Proof.* Since $\overline{Y} = bG$, $\pi\chi(y, Y) = \pi\chi(y, bG)$ for each $y \in Y$. Therefore, $\chi(G) \leq \tau^+$ by Proposition 3.2 and Theorem 3.3. □

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