Title:
Arens regularity of inverse semigroup algebras

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ARENS REGULARITY OF INVERSE SEMIGROUP ALGEBRAS

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Abstract. We present a characterization of Arens regular semigroup algebras $\ell^1(S)$, for a large class of semigroups. Mainly, we show that if the set of idempotents of an inverse semigroup $S$ is finite, then $\ell^1(S)$ is Arens regular if and only if $S$ is finite.

Keywords: Arens regularity, completely simple semigroup, inverse semigroup, left (right) group, weakly cancellative.

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1. Introduction

Let $\mathcal{A}$ be a Banach algebra. The second dual of $\mathcal{A}$ can be made into a Banach algebra in two different ways, as was first shown by Arens [1] and [2]. The algebra $\mathcal{A}$ is called Arens regular, if these two multiplications coincide. For a discrete semigroup $S$, Young [15] presented a criterion for Arens regularity of $\ell^1(S)$. Indeed, he showed that $\ell^1(S)$ is Arens regular if and only if there is no pair of sequences $(x_n)$ and $(y_m)$ in $S$ such that the sets $\{x_n y_m : m > n\}$ and $\{x_n y_m : m < n\}$ are disjoint. Another criterion has been provided in [3], which demonstrates that $\ell^1(S)$ is Arens regular if and only if for every pair of sequences of distinct elements of $S$, $(x_n), (y_m)$, there is a submatrix $(u_n v_m)$ of $(x_n y_m)$ of type C. Hosseini in a joint work with Duncan [8], asked for which semigroups $S$, the convolution Banach algebra $\ell^1(S)$ is Arens regular. Also in [8], they showed that if $S$ is a Brandt semigroup, then $\ell^1(S)$ is Arens regular if and only if $S$ is finite. We also refer to [6] as a helpful
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Furthermore, some authors have studied Arens regularity of $\ell^1(S, \omega)$, where $\omega : S \to (0, \infty)$ is a weight function on $S$. We just refer to some relevant works such as [3], [5] and [12].

The aim of the present work is to study Arens regularity of the convolution Banach algebra $\ell^1(S)$, for several classes of semigroups. In particular when $S$ is an inverse semigroup or a product of semigroups.

2. Preliminaries and definitions

In this section, we mention preliminaries and definitions which will be used throughout the paper. All the terminologies used in this work are from either [4] or [11].

Definition 2.1. Let $S$ be a discrete semigroup.

1. $S$ is called left (resp. right) zero semigroup if $xy = x$ (resp. $xy = y$), for all $x, y \in S$.

2. We say that $S$ is left (resp. right) cancellative if $xy = xz$ (resp. $yx = zx$) implies $y = z$. Moreover $S$ is called weakly cancellative if the sets $x^{-1}y = \{ t \in S : xt = y \}$ and $yx^{-1} = \{ t \in S : tx = y \}$ are finite, for all $x, y \in S$.

3. We say that $S$ is rectangular band if $aba = a$ for all $a, b \in S$. By [11, Theorem 1.1.3], $S$ is rectangular band if and only if it is isomorphic to the direct product of a left zero semigroup $E$ and a right zero semigroup $F$, with the multiplication defined by

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1, b_2).$$

4. $S$ is said to be left (resp. right) simple if it contains no proper left (resp. right) ideal. It is called simple if it has no proper two sided ideals. $S$ with a zero element 0 is called 0-simple if

(i) $\{0\}$ and $S$ are its only ideals,

(ii) $S^2 \neq \{0\}$.

5. $S$ is called left (resp. right) group if for each pair of elements $s, t$ in $S$ there exists a unique element $x \in S$ such that $xs = t$ (resp. $sx = t$). By [11, page 61], $S$ is left group if and only if $S$ is isomorphic to a direct product of a left zero semigroup $E$ and a group $G$. Also $S$ is right group if and only if $S$ is isomorphic to a direct product of a group $G$ and a right zero semigroup $F$. Moreover $S$ is left (resp. right) group if and only if it is left (resp. right) simple and right (resp. left) cancellative [4].
(6) $S$ is called regular if for every $a \in S$ there is $b \in S$ such that $a =aba$. Furthermore, $S$ is an inverse semigroup if for every $a \in S$ there is a unique $a^* \in S$ such that $aa^*a = a$ and $a^*aa^* = a^*$.

(7) An element $e \in S$ is called idempotent if $ee = e$. We denote by $E(S)$ the set of all idempotents of $S$. If $e, f \in E(S)$, we shall write $e \leq f$ if $ef = fe = e$. Then $\leq$ is a partial order relation on $E(S)$. An element $e \in E(S)$ is a primitive idempotent if for each $f \in E(S)$ such that $f \leq e$, then $f = e$. $S$ is called completely simple if it is simple and it has a primitive idempotent. If $S$ has a zero element $0$, then it will be called completely $0$-simple.

Now let $G$ be a group and $G^0 = G \cup \{0\}$ be the group with zero arising from $G$ by adjunction of a zero element. Let $I, \Lambda$ be non-empty sets and $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in $G^0$. Suppose that $P$ is regular in the sense that no row or column of $P$ consists entirely of zeros; formally for each $i \in I$ (resp. $\lambda \in \Lambda$), there exists $\lambda \in \Lambda$ (resp. $i \in I$) such that $p_{\lambda i} \neq 0$.

(1) Let $S = G \times I \times \Lambda$ and define a binary operation on $S$ by the rule that

$$(a, i, \lambda)(b, j, \mu) = (ap_{\lambda j}b, i, \mu).$$

Then $S$ will be denoted by $M[G; I, \Lambda; P]$ and will be called the $I \times \Lambda$ Rees matrix semigroup over the group $G$. By [11, Theorem 3.3.1] $S$ is completely simple semigroup. Conversely any completely simple semigroup is isomorphic to one constructed in this manner.

(2) Let $S = (G \times I \times \Lambda) \cup \{0\}$, and define a multiplication on $S$ by

$$(a, i, \lambda)(b, j, \mu) = \begin{cases} (ap_{\lambda j}b, i, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{if } p_{\lambda j} = 0 \end{cases}$$

and

$$(a, i, \lambda)0 = 0(a, i, \lambda) = 0.$$ 

The semigroup $S$ constructed in accordance with this recipe will be denoted by $M^0[G; I, \Lambda; P]$, and will be called the $I \times \Lambda$ Rees matrix semigroup over the $0$-group $G^0$ with the regular sandwich matrix $P$. Then by [11, Theorem 3.2.3], $S$ is a completely $0$-simple semigroup. Conversely, every completely $0$-simple semigroup is isomorphic to the one constructed in this way.

(3) Every completely $0$—simple inverse semigroup is called Brandt semigroup. By [4, Theorem 3.9] $S$ is Brandt semigroup if and
only if $S \cong M^0(G; E, E; \Delta)$, for some group $G$ and index set $E$ and the identity matrix $\Delta$.

3. Main results

Following [3], let $(x_n), (y_m)$ be sequences of distinct elements in $S$ such that the double matrix $(x_n y_m)$ satisfies one of the following conditions.

(i) There is $c \in S$ such that $x_n y_m = c$, for all $n, m \in \mathbb{N}$, i.e. every row and column of the matrix is constant.

(ii) There is a sequence $(c_m)$ of distinct elements in $S$ such that $x_n y_m = c_m$, for all $n, m \in \mathbb{N}$, i.e. every row of the matrix is constant.

(iii) There is a sequence $(d_n)$ of distinct elements in $S$ such that $x_n y_m = d_n$, for all $n, m \in \mathbb{N}$, i.e. every column of the matrix is constant.

Then the matrix $(x_n y_m)$ is called of type $C$. As we pointed out in Section 1, a criterion for Arens regularity of $\ell^1(S)$ has been provided in [3]. Indeed, $\ell^1(S)$ is Arens regular if and only if for every pair of sequences of distinct elements of $S$, $(x_n), (y_m)$, there is a submatrix $(u_n v_m)$ of $(x_n y_m)$ of type $C$. This criterion plays an essential role in the present work, to obtain the desired results.

We commence with the following theorem.

Theorem 3.1. Let $S$ be a discrete semigroup.

(i) If $S$ is a left or right zero semigroup, then $\ell^1(S)$ is Arens regular.

(ii) If $S = E \times G$ [resp. $G \times F$] is a left [resp. right] group, then $\ell^1(S)$ is Arens regular if and only if $G$ is finite.

(iii) If $S = E \times F$ is a rectangular band, then $\ell^1(S)$ is Arens regular if and only if $E$ or $F$ is finite.

(iv) If $S = M(G; E, F; P)$ [resp. $M^0(G; E, F; P)$] is a completely simple [resp. 0-simple] semigroup, then $\ell^1(S)$ is Arens regular if and only if $E \times G$ or $G \times F$ is finite.

Proof. (i). Let $S$ be a left (resp. right) zero semigroup and $(x_n), (y_m)$ be two sequences of distinct elements in $S$. Since $x_n y_m = x_n$ (resp. $x_n y_m = y_m$), for all $n, m \in \mathbb{N}$, it follows that every column (resp. row) of this matrix is constant. Thus $(x_n y_m)$ is a matrix of type $C$, which implies that $\ell^1(S)$ is Arens regular [3].

(ii). First let $\ell^1(S)$ be Arens regular. Since $\ell^1(E \times G) \cong \ell^1(E) \hat{\otimes} \ell^1(G)$, thus $\ell^1(G)$ is a closed subalgebra of $\ell^1(E \times G)$. It follows that $\ell^1(G)$ is
Arens regular [6, Corollary 3.15], which implies the finiteness of $G$ [14]. For the converse, let $G$ be finite and let $(x_n), (y_m)$ be two sequences with distinct elements in $S$. Thus $x_n = (a_n, b_n)$ and $y_m = (c_m, d_m)$, where $a_n, c_m \in E$ and $b_n, d_m \in G$, for all $m, n \in \mathbb{N}$. Hence $x_ny_m = (a_n, b_n, d_m)$. Since $\ell^1(G)$ is Arens regular, it follows that there is a submatrix of $(b_n d_m)$ of type $C$ [3]. By the definition of a matrix of type $C$, the only case that may occur is that $(b_n d_m)$ has a constant submatrix. Indeed, there is $g \in G$ and subsequences $(b_{n_k})$ and $(d_{m_l})$ of $(b_n)$ and $(d_m)$, respectively such that $b_{n_k} d_{m_l} = g$. Thus $x_{n_k} y_{m_l} = (a_{n_k}, g)$. It follows that $(x_{n_k} y_{m_l})$ is a submatrix of $(x_m y_n)$ of type $C$. Thus $\ell^1(S)$ is Arens regular. The same arguments can be used for the case where $S$ is a right group.

(iii). Let $E$ and $F$ be infinite and choose two sequences $(a_n)$ and $(d_m)$ of distinct elements in $E$ and $F$, respectively. For the arbitrary sequences $(b_n)$ and $(c_m)$ in $E$ and $F$, respectively, set $x_n = (a_n, b_n)$ and $y_m = (c_m, d_m)$. Thus $(x_n y_m) = (a_n, d_m)$, which has no submatrix of type $C$. Therefore $\ell^1(S)$ is not Arens regular [3]. Conversely, suppose that $E$ is finite and $x_n = (a_n, b_n)$ and $y_m = (c_m, d_m)$ be sequences with the distinct elements in $E \times F$. Thus $x_n y_m = (a_n, d_m)$, for all $m, n \in \mathbb{N}$. Hence there is an element $a \in E$ and a subsequence $(x_{n_k})$ of $(x_n)$ such that $x_{n_k} y_{m_l} = (a, y_m)$. It follows that $(x_{n_k} y_{m_l})$ is a submatrix of $(x_m y_n)$ of type $C$. Consequently $\ell^1(S)$ is Arens regular, by [3]. Similar arguments can be applied for the case where $F$ is finite.

(iv). Let $E \times G$ and $G \times F$ be infinite. If $G$ is infinite, then one can choose a sequence $(a_n)$ with distinct elements in $G$. Since $P = (p_{\lambda i})$ is regular, there are $i \in E$ and $\lambda \in F$ such that $p_{\lambda i} \neq 0$. Let $x_n = (a_n, i, \lambda)$. Then $x_n x_m = (a_n p_{\lambda i} a_m, i, \lambda)$ and for all $n_1, n_2, m_1, m_2 \in \mathbb{N}$ with $n_1 \neq n_2$ and $m_1 \neq m_2$, we have $x_{n_1} x_{m_1} \neq x_{n_2} x_{m_1}$ and $x_{n_1} x_{m_1} \neq x_{n_1} x_{m_2}$. Thus $(x_n x_m)$ has no submatrix of type $C$. Now suppose that $G$ is finite but $E$ and $F$ are both infinite. Then there are sequences $(i_n)$ and $(\lambda_m)$ with the distinct elements in $E$ and $F$, respectively. Let $e$ be the identity element of $G$ and $i_0 \in E$ and $\lambda_0 \in F$ be fixed. Also let $x_n = (e, i_n, \lambda_0)$ and $y_m = (e, i_0, \lambda_m)$. Then $x_n y_m = (p_{\lambda_0 i_0}, i_n, \lambda_m)$ and so $(x_n y_m)$ has no submatrix of type $C$. It follows that $\ell^1(S)$ is not Arens regular in any cases. Conversely, let $E \times G$ be finite, and let $(x_n)$ and $(y_m)$ be two sequences with distinct elements in $S$. Suppose that $x_n = (a_n, i_n, \lambda_n)$ and $y_m = (b_m, j_m, \mu_m)$. Then

$$
x_n y_m = \begin{cases} (a_n p_{\lambda_n j_m} b_m, i_n, \mu_m) & \text{if } p_{\lambda_n j_m} \neq 0 \\
0 & \text{if } p_{\lambda_n j_m} = 0. 
\end{cases}
$$
Moreover, there are \((i, g) \in E \times G\) and subsequences \((x_{n_k})\) and \((y_{m_l})\) of \((x_n)\) and \((y_m)\), respectively, such that \(x_{n_k}y_{m_l} = (g, i, \mu_{m_l})\). Thus \((x_{n_k}y_{m_l})\) is a submatrix of \((x_my_m)\) of type \(C\). Thus \(\ell^1(S)\) is regular. Some similar arguments will be used in the case where \(G \times F\) is finite. \(\square\)

The following result is immediately obtained from Theorem 3.1 part (iv).

**Corollary 3.2.** Let \(S\) be a Brandt semigroup. Then \(\ell^1(S)\) is Arens regular if and only if \(S\) is finite.

**Remark 3.3.** Suppose that \(S\) and \(T\) are two discrete semigroups such that \(\ell^1(S \times T)\) is Arens regular. Then \(\ell^1(S)\) and \(\ell^1(T)\) are Arens regular, by [6, Corollary 3.15]. The converse of this statement is not necessarily valid. For example, let \(S\) [resp. \(T\)] be an infinite left [resp. right] zero semigroup. Then Theorem 3.1 part (i) implies that \(\ell^1(S)\) and \(\ell^1(T)\) are Arens regular whereas \(\ell^1(S \times T)\) is not Arens regular, by Theorem 3.1 part (iii).

**Proposition 3.4.** Let \(S\) be a discrete semigroup.

(i) If \(S\) is weakly cancellative then \(\ell^1(S)\) is Arens regular if and only if \(S\) is finite.

(ii) If \(S\) is a simple [resp. 0-simple] inverse semigroup, then \(\ell^1(S)\) is Arens regular if and only if \(S\) is finite.

**Proof.** (i). Suppose that \(\ell^1(S)\) is Arens regular. If \(S\) is infinite, then there is a sequence \((x_n)\) with distinct elements in \(S\). Thus \((x_nx_m)\) has a submatrix \((x_{n_k}x_{m_l})\) of type \(C\) by [3]. Consequently one of the following cases occurs.

- (a) There is \(c \in S\) such that \(x_{n_k}x_{m_l} = c\), for all \(k, l \in \mathbb{N}\). Thus for fixed \(m_l\), \(\{x_{n_1}, x_{n_2}, x_{n_3}, \ldots\} \subseteq cx_{m_l}^{-1}\).
- (b) There is a sequence \((c_{m_l})\) of distinct elements in \(S\) such that \(x_{n_k}x_{m_l} = c_{m_l}\), for all \(k, l \in \mathbb{N}\). It follows that for fixed \(m_l\), \(\{x_{n_1}, x_{n_2}, x_{n_3}, \ldots\} \subseteq c_{m_l}x_{m_l}^{-1}\).
- (c) There is a sequence \((d_{n_k})\) of distinct elements in \(S\) such that \(x_{n_k}x_{m_l} = d_{n_k}\), for all \(k, l \in \mathbb{N}\). Then for fixed \(n_k\), \(\{x_{m_1}, x_{m_2}, x_{m_3}, \ldots\} \subseteq x_{n_k}^{-1}d_{n_k}\).

By assumption, the sets \(x^{-1}y = \{z \in S : xz = y\}\) and \(xy^{-1}\) are finite, for all \(x, y \in S\). But this is in contradiction to all the statements (a), (b) and (c). Therefore \(S\) is finite. The converse is clear.
(ii). Let $S$ be a simple [resp. 0-simple] inverse semigroup and let $\ell^1(S)$ be Arens regular. Then one can readily show, by using the Young’s criterion [15], that every chain of idempotents has finite length. It follows by Zorn’s lemma that $S$ has a primitive idempotent and so $S$ is completely simple [resp. 0-simple]. But only completely simple inverse semigroups are groups [11, page 151]. Moreover only completely 0-simple inverse semigroups are Brandt semigroups. Therefore, the arguments of [14] and also Corollary 3.2 imply that $\ell^1(S)$ is Arens regular if and only if $S$ is finite. \hfill $\Box$

Example 3.5. Given a non-empty set $X$, we define $I_X$ to consist of all partial one-one maps of $X$; i.e., one-one maps from subsets of $X$ to subsets of $X$. Then $I_X$ is a semigroup under the operation $\circ$ of composition of relations. Indeed, for $\rho, \sigma \in I_X$, $(x, y) \in \rho \circ \sigma$ if and only if there exists $z \in X$ such that $(x, z) \in \rho$ and $(z, y) \in \sigma$. By [11, Theorem 5.1.5] $I_X$ is an inverse semigroup, so called symmetric inverse semigroup on the set $X$. It is worth noting that every inverse semigroup can be embedded in a symmetric inverse semigroup, by Vagner-Preston Theorem [11, Theorem 5.1.7].

Now we show that $\ell_1(I_X)$ is Arens regular if and only if $X$ is finite. To prove, suppose that $X$ is infinite. Then there is a sequence $\{x_n\}$ with distinct elements in $X$. Let $I_n$ denote the one-one partial map on $\{x_1, x_2, \ldots, x_n\}$ defined by $I_n(x_i) = x_i$, for each $i$. Then we have

$$\{I_{2n+1} \circ I_{2m} : n < m\} \cap \{I_{2n+1} \circ I_{2m} : n > m\} = \emptyset.$$  
Thus $\ell_1(I_X)$ is not Arens regular by Young’s criterion [15]. The converse is clear.

Let $I$ be a proper ideal of $S$ and define the relation $\sim$ on $S$ as the following  

$$x \sim y \quad if \quad and \quad only \quad if \quad x = y \quad or \quad x, y \in I.$$  
The quotient semigroup $S/I$ is 

$$\{I\} \cup \{(x) : x \in S \setminus I\}.$$  
In $S/I$, the product of two elements in $S \setminus I$ is the same as their product in $S$ if this lies in $S \setminus I$; otherwise the product is $I$. Note that $I$ is the zero element of the semigroup $S/I$. See [11, page 33] for more information.

Proposition 3.6. Let $S$ be an inverse semigroup and $I$ be a maximal ideal in $S$. Then the following statements hold.
(i) $\ell^1(S/I)$ is Arens regular if and only if $S/I$ is finite.

(ii) $\ell^1(S)$ is Arens regular if and only if $\ell^1(S/I)$ and $\ell^1(I)$ are Arens regular.

Proof. (i). It is readily verified that $S/I$ is an inverse semigroup such that $(S/I)^2 \neq I$. By [11, Proposition 3.1.5], $S/I$ is a 0—simple inverse semigroup. Thus Proposition 3.4 implies that $\ell^1(S/I)$ is Arens regular if and only if $S/I$ is finite.

(ii). Let $\ell^1(S/I)$ and $\ell^1(I)$ be Arens regular. By part (i), $S/I$ is a 0—simple inverse semigroup and Arens regularity of $\ell^1(S/I)$ implies that $S/I$ is finite. It follows that $S\setminus I$ is a finite set. To that end, suppose that $(x_n)$ and $(y_m)$ are sequences of distinct elements in $S$. Since $S\setminus I$ is finite, there is $N > 0$ such that for each $n \geq N$, $(x_n)$ and $(y_m)$ can be considered as two sequences in $I$. Since $\ell^1(I)$ is Arens regular, there is a submatrix $\left(u_{n_k}v_{m_l}\right)$ of $(x_ny_m)_{n,m \geq N}$ of type $C$. Consequently $\ell^1(S)$ is Arens regular. The converse is obvious, by [6, Corollary 3.15].

Recall from [11] that $S = S_1 \supsetneq S_2 \supsetneq \cdots \supsetneq S_m$ is a principal series of $S$ if

(i) each $S_i$ is a two sided ideal of $S$;

(ii) there is no ideal of $S$ strictly between $S_i$ and $S_{i+1}$, for each $1 \leq i \leq m - 1$;

(iii) $S_m = K(S)$, where $K(S)$ is the kernel of the semigroup $S$; i.e. the minimal two-sided ideal of $S$. It is in fact the intersection of all two-sided ideals of $S$.

**Proposition 3.7.** Let $S$ be an inverse semigroup with a principal series. Then the following assertions are equivalent.

(i) $\ell^1(S)$ is Arens regular.

(ii) $\ell^1(K(S))$ and $\ell^1(S/K(S))$ are Arens regular.

(iii) $S \setminus K(S)$ is finite and $\ell^1(K(S))$ is Arens regular.

Proof. (i) $\Rightarrow$ (ii). It is clear.

(ii) $\Rightarrow$ (iii). Let $S = S_1 \supsetneq S_2 \supsetneq \cdots \supsetneq S_m = K(S)$ be a principal series for $S$. Since $\ell^1(S/K(S))$ is Arens regular and $S/K(S) \supset S_{m-1}/S_m$, it follows that $\ell^1(S_{m-1}/S_m)$ is Arens regular [6, Corollary 3.15]. Since $S_{m-1}/S_m$ is a 0—simple inverse semigroup [11, Proposition 3.1.5], then $S_{m-1}/S_m$ is finite by Proposition 3.4 part (ii). Inductively, for each
2 ≤ i ≤ m − 1, the semigroup $S_{i-1}/S_i$ is finite. Since

$$S \setminus K(S) = \bigcup_{i=2}^{m} S_{i-1} \setminus S_i,$$

it follows that $S \setminus K(S)$ is finite and so the result follows.

$(iii) \Rightarrow (i)$. It is proved exactly by the same arguments given in the proof of part $(ii)$ in Proposition 3.6. □

We now are in a position to prove the main theorem of the present work.

**Theorem 3.8.** Let $S$ be an inverse semigroup such that the set of idempotents of $S$ is finite. Then $\ell^1(S)$ is Arens regular if and only if $S$ is finite.

**Proof.** Suppose that $\ell^1(S)$ is Arens regular. Since $E(S)$ is finite, thus there exists a principal series as $S = S_0 \supsetneq S_1 \supsetneq \cdots \supsetneq S_m = G$, for some group $G$, such that $K(S) = G$ and also for each $0 \leq i < m$

$$S_i/S_{i+1} \cong (S_i/G)/(S_{i+1}/G),$$

as semigroup isomorphisms. Moreover natural maps from $S_i$ onto $S_i/G$ are homomorphisms and also $S_i/S_{i+1}$ is a completely 0-simple inverse semigroup, for each $0 \leq i < m$; see [7, Theorem 3.12] and also [9], for more details. By Proposition 3.7, $\ell^1(S)$ is Arens regular if and only if $S \setminus G$ is finite and $\ell^1(G)$ is Arens regular. But by [14], Arens regularity of $\ell^1(G)$ is equivalent to the finiteness of $G$. Thus the result is obtained. □

It should be noted that some other results related to regular and inverse semigroups have been provided in [10, Theorem 5.1].

We end this work with the following examples which are interesting in their own right.

**Examples 3.9.** Suppose that $T = \mathbb{N} \cup \{0\}$ with the multiplication defined by

$$n.m = \begin{cases} n & \text{if } n = m \\ 0 & \text{otherwise}. \end{cases}$$

(1) It is readily verified that $T$ is an inverse semigroup such that $E(T) = T$, which is infinite. But $\ell^1(T)$ is Arens regular. In fact, let $(x_n)$ and $(y_m)$ be two sequences with the distinct elements in $T$. We may find subsequences $(x_{n_k})$ and $(y_{m_l})$ of $(x_n)$ and $(y_m)$
respectively, such that \( \{x_{n_1}, x_{n_2}, x_{n_3}, \ldots \} \cap \{y_{m_1}, y_{m_2}, y_{m_3}, \ldots \} = \emptyset. \) So \((x_n, y_m)\) is the matrix 0 and consequently it is of type C. Then \( \ell^1(T) \) is Arens regular by [3]. This example shows that the condition of the finiteness of \( E(S) \) is necessary in Theorem 3.8.

(2) Let \( S = T \times T \). Then \( S \) is an inverse semigroup. Moreover, for each \((n, m) \in S, (n, m)(n, m) = (n, m) \) and so \( E(S) = S \). We show that \( \ell^1(S) \) is not Arens regular. First let us recall some spaces from [3]. We denote by \( \ell^\infty(S) \) the set of all functions \( f : S \to \mathbb{R} \) such that

\[
\|f\|_\infty = \sup\{|f(x)| : x \in S\}
\]

is finite. We say that \( f \in \ell^\infty(S) \) is weakly almost periodic if the set \( \{f_x : x \in S\} \) is relatively weakly compact in \( \ell^\infty(S) \), where \( f_x \) is the left translation of a function \( f \) by \( x \). The set of weakly almost periodic functions on \( S \) will be denoted by \( \text{wap}(S) \). By [3, Theorem 3.2], \( \ell^1(S) \) is Arens regular if and only if \( \text{wap}(S) = \ell^\infty(S) \). We use Ruppert criterion in [13], to show that \( \text{wap}(S) \neq \ell^\infty(S) \). This criterion mentions that for \( A \subseteq S, \chi_A \in \text{wap}(S) \) if and only if every infinite set \( B \) in \( S \) contains a finite subset \( F \) such that the sets

\[
\bigcap_{b \in F} Ab^{-1} \setminus \bigcap_{b \in B \setminus F} Ab^{-1}
\]

and also

\[
\bigcap_{b \in F} b^{-1} A \setminus \bigcap_{b \in B \setminus F} b^{-1} A
\]

are finite. Now let \( X = \{(k, 0) : k \in T\}, Y = \{(0, k) : k \in T\} \) and \( Z = X \cup Y \). To that end, we show that \( \chi(z) \notin \text{wap}(S) \), for each \( z \in Z \). Let \( B = \{(k, n) : k, n \in T\} \). Then for each \((k, n) \in B\)

\[
(k, n)^{-1}(k, 0) = \{(k, m) : m \neq n\} = B \setminus \{(k, n)\}.
\]

Thus for all finite subsets \( F \) of \( B \),

\[
\bigcap_{(k, n) \in F} (k, n)^{-1}(k, 0) \setminus \bigcap_{(k, n) \in B \setminus F} (k, n)^{-1}(k, 0) = \bigcap_{(k, n) \in F} (k, 0)(k, n)^{-1} \setminus \bigcap_{(k, n) \in B \setminus F} (k, n)^{-1}(k, 0) = \ (B \setminus F) \setminus F = B \setminus F.
\]

Since \( B \setminus F \) is infinite, [13] follows that \( \chi_{(k, 0)} \notin \text{wap}(S) \). Similarly \( \chi_{(0, k)} \notin \text{wap}(S) \). Therefore \( \ell^1(S) \) is not Arens regular.
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