ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 40 (2014), No. 6, pp. 1527–1538

Title:

Arens regularity of inverse semigroup algebras

Author(s):

F. Abtahi, B. Khodsiani and A. Rejali

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 40 (2014), No. 6, pp. 1527–1538
Online ISSN: 1735-8515

ARENS REGULARITY OF INVERSE SEMIGROUP ALGEBRAS

F. ABTAHI*, B. KHODSIANI AND A. REJALI

(Communicated by Gholam Hossein Esslamzadeh)

Abstract. We present a characterization of Arens regular semigroup algebras $\ell^1(S)$, for a large class of semigroups. Mainly, we show that if the set of idempotents of an inverse semigroup S is finite, then $\ell^1(S)$ is Arens regular if and only if S is finite.
Keywords: Arens regularity, completely simple semigroup, inverse semigroup, left (right) group, weakly cancellative.
MSC(2010): Primary: 43A20; Secondary: 46H05.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra. The second dual of $\mathcal{A}$ can be made into a Banach algebra in two different ways, as was first shown by Arens [1] and [2]. The algebra $\mathcal{A}$ is called Arens regular, if these two multiplications coincide. For a discrete semigroup S, Young [15] presented a criterion for Arens regularity of $\ell^1(S)$. Indeed, he showed that $\ell^1(S)$ is Arens regular if and only if there is no pair of sequences (x_n) and (y_m) in S such that the sets $\{x_ny_m : m > n\}$ and $\{x_ny_m : m < n\}$ are disjoint. Another criterion has been provided in [3], which demonstrates that $\ell^1(S)$ is Arens regular if and only if for every pair of sequences of distinct elements of S, $(x_n), (y_m)$, there is a submatrix (u_nv_m) of (x_ny_m) of type C. Hosseiniun in a joint work with Duncan [8], asked for which semigroups S, the convolution Banach algebra $\ell^1(S)$ is Arens regular. Also in [8], they showed that if S is a Brandt semigroup, then $\ell^1(S)$ is Arens regular if and only if S is finite. We also refer to [6] as a helpful

Article electronically published on December 11, 2014.
Received: 11 April 2013, Accepted: 18 November 2013.
*Corresponding author.

reference. Furthermore, some authors have studied Arens regularity of $\ell^1(S,\omega)$, where $\omega : S \to (0,\infty)$ is a weight function on S. We just refer to some relevant works such as [3], [5] and [12].

The aim of the present work is to study Arens regularity of the convolution Banach algebra $\ell^1(S)$, for several classes of semigroups. In particular when S is an inverse semigroup or a product of semigroups.

2. **Preliminaries and definitions**

In this section, we mention preliminaries and definitions which will be used throughout the paper. All the terminologies used in this work are from either [4] or [11].

Definition 2.1. *Let S be a discrete semigroup.*

(1) *S is called left (resp. right) zero semigroup if $xy = x$ (resp. $xy = y$), for all $x, y \in S$.*
(2) *We say that S is left (resp. right) cancellative if $xy = xz$ (resp. $yx = zx$) implies $y = z$. Moreover S is called weakly cancellative if the sets $x^{-1}y = \{t \in S : xt = y\}$ and $yx^{-1} = \{t \in S : tx = y\}$ are finite, for all $x, y \in S$.*
(3) *We say that S is rectangular band if $aba = a$ for all $a, b \in S$. By* [11, Theorem 1.1.3]*, S is rectangular band if and only if it is isomorphic to the direct product of a left zero semigroup E and a right zero semigroup F, with the multiplication defined by*

$$(a_1, b_1).(a_2, b_2) = (a_1, b_2).$$

(4) *S is said to be left (resp. right) simple if it contains no proper left (resp. right) ideal. It is called simple if it has no proper two sided ideals. S with a zero element 0 is called 0-simple if*
   (i) *$\{0\}$ and S are its only ideals,*
   (ii) *$S^2 \neq \{0\}$.*
(5) *S is called left (resp. right) group if for each pair of elements s, t in S there exists a unique element $x \in S$ such that $xs = t$ (resp. $sx = t$). By* [11, page 61]*, S is left group if and only if S is isomorphic to a direct product of a left zero semigroup E and a group G. Also S is right group if and only if S is isomorphic to a direct product of a group G and a right zero semigroup F. Moreover S is left (resp. right) group if and only if it is left (resp. right) simple and right (resp. left) cancellative* [4].

(6) *S is called regular if for every $a \in S$ there is $b \in S$ such that $a = aba$. Furthermore, S is an inverse semigroup if for every $a \in S$ there is a unique $a^* \in S$ such that $aa^*a = a$ and $a^*aa^* = a^*$.*

(7) *An element $e \in S$ is called idempotent if $ee = e$. We denote by $E(S)$ the set of all idempotents of S. If $e, f \in E(S)$, we shall write $e \leq f$ if $ef = fe = e$. Then $\leq$ is a partial order relation on $E(S)$. An element $e \in E(S)$ is a primitive idempotent if for each $f \in E(S)$ such that $f \leq e$, then $f = e$. S is called completely simple if it is simple and it has a primitive idempotent. If S has a zero element 0, then it will be called completely 0-simple if it is 0-simple and contains a nonzero primitive idempotent.*

Now let G be a group and $G^0 = G \cup \{0\}$ be the group with zero arising from G by adjunction of a zero element. Let I, Λ be non-empty sets and $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in G^0. Suppose that P is regular in the sense that no row or column of P consists entirely of zeros; formally for each $i \in I$ (resp. $\lambda \in \Lambda$), there exists $\lambda \in \Lambda$ (resp. $i \in I$) such that $p_{\lambda i} \neq 0$.

(1) *Let $S = G \times I \times \Lambda$ and define a binary operation on S by the rule that*

$$(a, i, \lambda)(b, j, \mu) = (ap_{\lambda j}b, i, \mu).$$

Then S will be denoted by $M[G; I, \Lambda; P]$ and will be called the $I \times \Lambda$ Rees matrix semigroup over the group G. By [11, Theorem 3.3.1] *S is completely simple semigroup. Conversely any completely simple semigroup is isomorphic to one constructed in this manner.*

(2) *Let $S = (G \times I \times \Lambda) \cup \{0\}$, and define a multiplication on S by*

$$(a, i, \lambda)(b, j, \mu) = \begin{cases} (ap_{\lambda j}b, i, \mu) & \text{if} \quad p_{\lambda j} \neq 0 \\ 0 & \text{if} \quad p_{\lambda j} = 0 \end{cases}$$

and

$$(a, i, \lambda)0 = 0(a, i, \lambda) = 0.$$

The semigroup S constructed in accordance with this recipe will be denoted by $M^0[G; I, \Lambda; P]$, and will be called the $I \times \Lambda$ Rees matrix semigroup over the 0-group G^0 with the regular sandwich matrix P. Then by [11, Theorem 3.2.3]*, S is a completely 0-simple semigroup. Conversely, every completely 0-simple semigroup is isomorphic to the one constructed in this way.*

(3) *Every completely $0-$simple inverse semigroup is called Brandt semigroup. By* [4, Theorem 3.9] *S is Brandt semigroup if and*

only if $S \cong M^0(G; E, E; \Delta)$*, for some group* G *and index set* E *and the identity matrix* Δ.

3. **Main results**

Following [3], let $(x_n), (y_m)$ be sequences of distinct elements in S such that the double matrix $(x_n y_m)$ satisfies one of the following conditions.

(i) There is $c \in S$ such that $x_n y_m = c$, for all $n, m \in \mathbb{N}$, i.e. every row and column of the matrix is constant.

(ii) There is a sequence (c_m) of distinct elements in S such that $x_n y_m = c_m$, for all $n, m \in \mathbb{N}$, i.e. every row of the matrix is constant.

(iii) There is a sequence (d_n) of distinct elements in S such that $x_n y_m = d_n$, for all $n, m \in \mathbb{N}$, i.e. every column of the matrix is constant.

Then the matrix $(x_n y_m)$ is called of type C. As we pointed out in Section 1, a criterion for Arens regularity of $\ell^1(S)$ has been provided in [3]. Indeed, $\ell^1(S)$ is Arens regular if and only if for every pair of sequences of distinct elements of S, $(x_n), (y_m)$, there is a submatrix $(u_n v_m)$ of $(x_n y_m)$ of type C. This criterion plays an essential role in the present work, to obtain the desired results.

We commence with the following theorem.

Theorem 3.1. *Let* S *be a discrete semigroup.*

(i) *If* S *is a left or right zero semigroup, then* $\ell^1(S)$ *is Arens regular.*

(ii) *If* $S = E \times G$ *[resp.* $G \times F$*] is a left [resp. right] group, then* $\ell^1(S)$ *is Arens regular if and only if* G *is finite.*

(iii) *If* $S = E \times F$ *is a rectangular band, then* $\ell^1(S)$ *is Arens regular if and only if* E *or* F *is finite.*

(iv) *If* $S = M(G; E, F; P)$ *[resp.* $M^0(G; E, F; P)$*] is a completely simple [resp.0-simple] semigroup, then* $\ell^1(S)$ *is Arens regular if and only if* $E \times G$ *or* $G \times F$ *is finite.*

Proof. (i). Let S be a left (resp. right) zero semigroup and $(x_n), (y_m)$ be two sequences of distinct elements in S. Since $x_n y_m = x_n$ (resp. $x_n y_m = y_m$), for all $n, m \in \mathbb{N}$, it follows that every column (resp. row) of this matrix is constant. Thus $(x_n y_m)$ is a matrix of type C, which implies that $\ell^1(S)$ is Arens regular [3].

(ii). First let $\ell^1(S)$ be Arens regular. Since $\ell^1(E \times G) \cong \ell^1(E) \hat{\otimes} \ell^1(G)$, thus $\ell^1(G)$ is a closed subalgebra of $\ell^1(E \times G)$. It follows that $\ell^1(G)$ is

Arens regular [6, Corollary 3.15], which implies the finiteness of G [14]. For the converse, let G be finite and let $(x_n), (y_m)$ be two sequences with distinct elements in S. Thus $x_n = (a_n, b_n)$ and $y_m = (c_m, d_m)$, where $a_n, c_m \in E$ and $b_n, d_m \in G$, for all $m, n \in \mathbb{N}$. Hence $x_n y_m = (a_n, b_n d_m)$. Since $\ell^1(G)$ is Arens regular, it follows that there is a submatrix of $(b_n d_m)$ of type C [3]. By the definition of a matrix of type C, the only case that may occur is that $(b_n d_m)$ has a constant submatrix. Indeed there is $g \in G$ and subsequences (b_{n_k}) and (d_{m_l}) of (b_n) and (d_m), respectively such that $b_{n_k} d_{m_l} = g$. Thus $x_{n_k} y_{m_l} = (a_{n_k}, g)$. It follows that $(x_{n_k} y_{m_l})$ is a submatrix of $(x_m y_m)$ of type C. Thus $\ell^1(S)$ is Arens regular. The same arguments can be used for the case where S is a right group.

(iii). Let E and F be infinite and choose two sequences (a_n) and (d_m) of distinct elements in E and F, respectively. For the arbitrary sequences (b_n) and (c_m) in E and F, respectively, set $x_n = (a_n, b_n)$ and $y_m = (c_m, d_m)$. Thus $(x_n y_m) = (a_n, d_m)$, which has no submatrix of type C. Therefore $\ell^1(S)$ is not Arens regular [3]. Conversely, suppose that E is finite and $x_n = (a_n, b_n)$ and $y_m = (c_m, d_m)$ be sequences with the distinct elements in $E \times F$. Thus $x_n y_m = (a_n, d_m)$, for all $m, n \in \mathbb{N}$. Hence there is an element $a \in E$ and a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} y_m = (a, y_m)$. It follows that $(x_{n_k} y_m)$ is a submatrix of $(x_m y_m)$ of type C. Consequently $\ell^1(S)$ is Arens regular, by [3]. Similar arguments can be applied for the case where F is finite.

(iv). Let $E \times G$ and $G \times F$ be infinite. If G is infinite, then one can choose a sequence (a_n) with distinct elements in G. Since $P = (p_{\lambda i})$ is regular, there are $i \in E$ and $\lambda \in F$ such that $p_{\lambda i} \neq 0$. Let $x_n = (a_n, i, \lambda)$. Then $x_n x_m = (a_n p_{\lambda i} a_m, i, \lambda)$ and for all $n_1, n_2, m_1, m_2 \in \mathbb{N}$ with $n_1 \neq n_2$ and $m_1 \neq m_2$, we have $x_{n_1} x_{m_1} \neq x_{n_2} x_{m_1}$ and $x_{n_1} x_{m_1} \neq x_{n_1} x_{m_2}$. Thus $(x_n x_m)$ has no submatrix of type C. Now suppose that G is finite but E and F are both infinite. Then there are sequences (i_n) and (λ_m) with the distinct elements in E and F, respectively. Let e be the identity element of G and $i_0 \in E$ and $\lambda_0 \in F$ be fixed. Also let $x_n = (e, i_n, \lambda_0)$ and $y_m = (e, i_0, \lambda_m)$. Then $x_n y_m = (p_{\lambda_0 i_0}, i_n, \lambda_m)$ and so $(x_n y_m)$ has no submatrix of type C. It follows that $\ell^1(S)$ is not Arens regular in any cases. Conversely, let $E \times G$ be finite, and let (x_n) and (y_m) be two sequences with distinct elements in S. Suppose that $x_n = (a_n, i_n, \lambda_n)$ and $y_m = (b_m, j_m, \mu_m)$. Then

$$x_n y_m = \begin{cases} (a_n p_{\lambda_n j_m} b_m, i_n, \mu_m) & \text{if} \quad p_{\lambda_n j_m} \neq 0 \\ 0 & \text{if} \quad p_{\lambda_n j_m} = 0. \end{cases}$$

Moreover, there are $(i,g) \in E \times G$ and subsequences (x_{n_k}) and (y_{m_l}) of (x_n) and (y_m), respectively such that $x_{n_k} y_{m_l} = (g, i, \mu_{m_l})$. Thus $(x_{n_k} y_{m_l})$ is a submatrix of $(x_m y_m)$ of type C. Thus $\ell^1(S)$ is regular. Some similar arguments will be used in the case where $G \times F$ is finite. $\square$

The following result is immediately obtained from Theorem 3.1 part (iv).

Corollary 3.2. *Let S be a Brandt semigroup. Then $\ell^1(S)$ is Arens regular if and only if S is finite.*

Remark 3.3. *Suppose that S and T are two discrete semigroups such that $\ell^1(S \times T)$ is Arens regular. Then $\ell^1(S)$ and $\ell^1(T)$ are Arens regular, by* [6, Corollary 3.15]*. The converse of this statement is not necessarily valid. For example, let S [resp. T] be an infinite left [resp. right] zero semigroup. Then Theorem 3.1 part (i) implies that $\ell^1(S)$ and $\ell^1(T)$ are Arens regular whereas $\ell^1(S \times T)$ is not Arens regular, by Theorem 3.1 part (iii).*

Proposition 3.4. *Let S be a discrete semigroup.*

(i) *If S is weakly cancellative then $\ell^1(S)$ is Arens regular if and only if S is finite.*
(ii) *If S is a simple [resp. 0-simple] inverse semigroup, then $\ell^1(S)$ is Arens regular if and only if S is finite.*

Proof. (i). Suppose that $\ell^1(S)$ is Arens regular. If S is infinite, then there is a sequence (x_n) with distinct elements in S. Thus $(x_n x_m)$ has a submatrix $(x_{n_k} x_{m_l})$ of type C by [3]. Consequently one of the following cases occurs.

(a) There is $c \in S$ such that $x_{n_k} x_{m_l} = c$, for all $k, l \in \mathbb{N}$. Thus for fixed m_l, $\{x_{n_1}, x_{n_2}, x_{n_3}, \cdots\} \subseteq c x_{m_l}^{-1}$.
(b) There is a sequence (c_{m_l}) of distinct elements in S such that $x_{n_k} x_{m_l} = c_{m_l}$, for all $k, l \in \mathbb{N}$. It follows that for fixed m_l, $\{x_{n_1}, x_{n_2}, x_{n_3}, \cdots\} \subseteq c_{m_l} x_{m_l}^{-1}$.
(c) There is a sequence (d_{n_k}) of distinct elements in S such that $x_{n_k} x_{m_l} = d_{n_k}$, for all $k, l \in \mathbb{N}$. Then for fixed n_k,

$$\{x_{m_1}, x_{m_2}, x_{m_3}, \cdots\} \subseteq x_{n_k}^{-1} d_{n_k}.$$

By assumption, the sets $x^{-1}y = \{z \in S : xz = y\}$ and xy^{-1} are finite, for all $x, y \in S$. But this is in contradiction to all the statements (a), (b) and (c). Therefore S is finite. The converse is clear.

(ii). Let S be a simple [resp. 0-simple] inverse semigroup and let $\ell^1(S)$ be Arens regular. Then one can readily show, by using the Young's criterion [15], that every chain of idempotents has finite length. It follows by Zorn's lemma that S has a primitive idempotent and so S is completely simple [resp. 0-simple]. But only completely simple inverse semigroups are groups [11, page 151]. Moreover only completely 0-simple inverse semigroups are Brandt semigroups. Therefore, the arguments of [14] and also Corollary 3.2 imply that $\ell^1(S)$ is Arens regular if and only if S is finite. □

Example 3.5. *Given a non-empty set X, we define $\mathcal{I}_X$ to consist of all partial one-one maps of X; i.e., one-one maps from subsets of X to subsets of X. Then $\mathcal{I}_X$ is a semigroup under the operation $\circ$ of composition of relations. Indeed, for $\rho, \sigma \in \mathcal{I}_X$, $(x, y) \in \rho \circ \sigma$ if and only if there exists $z \in X$ such that $(x, z) \in \rho$ and $(z, y) \in \sigma$. By* [11, Theorem 5.1.5] *$\mathcal{I}_X$ is an inverse semigroup, so called symmetric inverse semigroup on the set X. It is worth noting that every inverse semigroup can be embedded in a symmetric inverse semigroup, by Vagner-Preston Theorem* [11, Theorem 5.1.7].

Now we show that $\ell_1(\mathcal{I}_X)$ is Arens regular if and only if X is finite. To prove, suppose that X is infinite. Then there is a sequence $\{x_n\}$ with distinct elements in X. Let I_n denote the one-one partial map on $\{x_1, x_2, \cdots, x_n\}$ defined by $I_n(x_i) = x_i$, for each i. Then we have

$$\{I_{2n+1} \circ I_{2m} : n < m\} \cap \{I_{2n+1} \circ I_{2m} : n > m\} = \emptyset.$$

Thus $\ell_1(\mathcal{I}_X)$ is not Arens regular by Young's criterion [15]. *The converse is clear.*

Let I be a proper ideal of S and define the relation $\sim$ on S as the following

$$x \sim y \ \ if\ and\ only\ if \ \ x = y \ \ or \ \ x, y \in I.$$

The quotient semigroup S/I is

$$\{I\} \cup \{\{x\} : x \in S \setminus I\}\}.$$

In S/I, the product of two elements in $S \setminus I$ is the same as their product in S if this lies in $S \setminus I$; otherwise the product is I. Note that I is the zero element of the semigroup S/I. See [11, page 33] for more information.

Proposition 3.6. *Let S be an inverse semigroup and I be a maximal ideal in S. Then the following statements hold.*

(i) *$\ell^1(S/I)$ is Arens regular if and only if S/I is finite.*
(ii) *$\ell^1(S)$ is Arens regular if and only if $\ell^1(S/I)$ and $\ell^1(I)$ are Arens regular.*

Proof. (i). It is readily verified that S/I is an inverse semigroup such that $(S/I)^2 \neq I$. By [11, Proposition 3.1.5], S/I is a $0-$simple inverse semigroup. Thus Proposition 3.4 implies that $\ell^1(S/I)$ is Arens regular if and only if S/I is finite.

(ii). Let $\ell^1(S/I)$ and $\ell^1(I)$ be Arens regular. By part (i), S/I is a $0-$simple inverse semigroup and Arens regularity of $\ell^1(S/I)$ implies that S/I is finite. It follows that $S\backslash I$ is a finite set. To that end, suppose that $(x_n), (y_m)$ are sequences of distinct elements in S. Since $S\backslash I$ is finite, there is $N > 0$ such that for each $n \geq N$, (x_n) and (y_m) can be considered as two sequences in I. Since $\ell^1(I)$ is Arens regular, there is a submatrix $(u_{n_k} v_{m_l})$ of $(x_n y_m)_{n,m\geq N}$ of type C. Consequently $\ell^1(S)$ is Arens regular. The converse is obvious, by [6, Corollary 3.15]. □

Recall from [11] that $S = S_1 \supsetneqq S_2 \supsetneqq \cdots \supsetneqq S_m$ is a principal series of S if

(i) each S_i is a two sided ideal of S;
(ii) there is no ideal of S strictly between S_i and S_{i+1}, for each $1 \leq i \leq m-1$;
(iii) $S_m = K(S)$, where $K(S)$ is the kernel of the semigroup S; i.e. the minimal two-sided ideal of S. It is in fact the intersection of all two-sided ideals of S.

Proposition 3.7. *Let S be an inverse semigroup with a principal series. Then the following assertions are equivalent.*

(i) *$\ell^1(S)$ is Arens regular.*
(ii) *$\ell^1(K(S))$ and $\ell^1(S/K(S))$ are Arens regular.*
(iii) *$S \setminus K(S)$ is finite and $\ell^1(K(S))$ is Arens regular.*

Proof. $(i) \Rightarrow (ii)$. It is clear.

$(ii) \Rightarrow (iii)$. Let $S = S_1 \supsetneqq S_2 \supsetneqq \cdots \supsetneqq S_m = K(S)$ be a principal series for S. Since $\ell^1(S/K(S))$ is Arens regular and $S/K(S) \supseteq S_{m-1}/S_m$, it follows that $\ell^1(S_{m-1}/S_m)$ is Arens regular [6, Corollary 3.15]. Since S_{m-1}/S_m is a $0-$simple inverse semigroup [11, Proposition 3.1.5], then S_{m-1}/S_m is finite by Proposition 3.4 part (ii). Inductively, for each

$2 \leq i \leq m-1$, the semigroup S_{i-1}/S_i is finite. Since

$$S \setminus K(S) = \bigcup_{i=2}^{m} S_{i-1} \setminus S_i,$$

it follows that $S \setminus K(S)$ is finite and so the result follows.

$(iii) \Rightarrow (i)$. It is proved exactly by the same arguments given in the proof of part (ii) in Proposition 3.6. $\square$

We now are in a position to prove the main theorem of the present work.

Theorem 3.8. *Let S be an inverse semigroup such that the set of idempotents of S is finite. Then $\ell^1(S)$ is Arens regular if and only if S is finite.*

Proof. Suppose that $\ell^1(S)$ is Arens regular. Since $E(S)$ is finite, thus there exists a principal series as $S = S_0 \supsetneqq S_1 \supsetneqq \cdots \supsetneqq S_m = G$, for some group G, such that $K(S) = G$ and also for each $0 \leq i < m$

$$S_i/S_{i+1} \cong (S_i/G)/(S_{i+1}/G),$$

as semigroup isomorphisms. Moreover natural maps from S_i onto S_i/G are homomorphisms and also S_i/S_{i+1} is a completely 0-simple inverse semigroup, for each $0 \leq i < m$; see [7, Theorem 3.12] and also [9], for more details. By Proposition 3.7, $\ell^1(S)$ is Arens regular if and only if $S \setminus G$ is finite and $\ell^1(G)$ is Arens regular. But by [14], Arens regularity of $\ell^1(G)$ is equivalent to the finiteness of G. Thus the result is obtained. $\square$

It should be noted that some other results related to regular and inverse semigroups have been provided in [10, Theorem 5.1].

We end this work with the following examples which are interesting in their own right.

Examples 3.9. *Suppose that $T = \mathbb{N} \cup \{0\}$ with the multiplication defined by*

$$n.m = \begin{cases} n & \text{if} \quad n = m \\ 0 & \text{otherwise.} \end{cases}$$

(1) *It is readily verified that T is an inverse semigroup such that $E(T) = T$, which is infinite. But $\ell^1(T)$ is Arens regular. In fact, let (x_n) and (y_m) be two sequences with the distinct elements in T. We may find subsequences (x_{n_k}) and (y_{m_l}) of (x_n) and (y_m)*

respectively, such that $\{x_{n_1}, x_{n_2}, x_{n_3}, \cdots\} \cap \{y_{m_1}, y_{m_2}, y_{m_3}, \cdots\} = \emptyset$. *So* $(x_{n_k} y_{m_l})$ *is the matrix* 0 *and consequently it is of type* C. *Then* $\ell^1(T)$ *is Arens regular by* [3]. *This example shows that the condition of the finiteness of* $E(S)$ *is necessary in Theorem 3.8.*

(2) *Let* $S = T \times T$. *Then* S *is an inverse semigroup. Moreover, for each* $(n, m) \in S$, $(n, m).(n, m) = (n, m)$ *and so* $E(S) = S$. *We show that* $\ell^1(S)$ *is not Arens regular. First let us recall some spaces from* [3]. *We denote by* $\ell^\infty(S)$ *the set of all functions* $f : S \to \mathbb{R}$ *such that*

$$\|f\|_\infty = \sup\{|f(x)| : x \in S\}$$

is finite. We say that $f \in \ell^\infty(S)$ *is weakly almost periodic if the set* $\{f_x : x \in S\}$ *is relatively weakly compact in* $\ell^\infty(S)$, *where* f_x *is the left translation of a function* f *by* x. *The set of weakly almost periodic functions on* S *will be denoted by* $wap(S)$. *By* [3, Theorem 3.2], $\ell^1(S)$ *is Arens regular if and only if* $wap(S) = \ell^\infty(S)$. *We use Ruppert criterion in* [13], *to show that* $wap(S) \neq \ell^\infty(S)$. *This criterion mentions that for* $A \subseteq S$, $\chi_A \in wap(S)$ *if and only if every infinite set* B *in* S *contains a finite subset* F *such that the sets*

$$\bigcap_{b \in F} Ab^{-1} \setminus \bigcap_{b \in B \setminus F} Ab^{-1}$$

and also

$$\bigcap_{b \in F} b^{-1}A \setminus \bigcap_{b \in B \setminus F} b^{-1}A$$

are finite. Now let $X = \{(k, 0) : k \in T\}$, $Y = \{(0, k) : k \in T\}$ *and* $Z = X \cup Y$. *To that end, we show that* $\chi_{\{z\}} \notin wap(S)$, *for each* $z \in Z$. *Let* $B = \{(k, n) : k, n \in T\}$. *Then for each* $(k, n) \in B$

$$(k, n)^{-1}(k, 0) = \{(k, m) : m \neq n\} = B \setminus \{(k, n)\}.$$

Thus for all finite subsets F *of* B,

$$\begin{aligned} \bigcap_{(k,n) \in F} (k, n)^{-1}(k, 0) \setminus \bigcap_{(k,n) \in B \setminus F} (k, n)^{-1}(k, 0) &= \bigcap_{(k,n) \in F} (k, 0)(k, n)^{-1} \setminus \bigcap_{(k,n) \in B \setminus F} (k, n)^{-1}(k, 0) \\ &= (B \setminus F) \setminus F = B \setminus F. \end{aligned}$$

Since $B \setminus F$ *is infinite,* [13] *follows that* $\chi_{(k,0)} \notin wap(S)$. *Similarly* $\chi_{(0,k)} \notin wap(S)$. *Therefore* $\ell^1(S)$ *is not Arens regular.*

Acknowledgments

The authors would like to thank the referee of the paper for his/her invaluable comments. This research was partially supported by the Banach algebra Center of Excellence for Mathematics, University of Isfahan.

References

[1] R. Arens, Operations induced in function classes, *Monatsh. Math.* **55** (1951) 1–19.

[2] R. Arens, The adjoint of a bilinear operation, *Proc. Amer. Math. Soc.* **2** (1951) 839–848.

[3] J. W. Baker and A. Rejali, On the Arens regularity of weighted convolution algebras, *J. London Math. Soc. (2)* **40** (1989), no. 3, 535–546.

[4] A. H. Clifford and G. B. Preston, The Algebraic Theory of Semigroups, Vol I, Amer. Math. Soc., Providence, 1961.

[5] I. G. Craw and N. J. Young, Regularity of multiplications in weighted group and semigroup algebras, *Quart. J. Math. Oxford ser. (2)* **25** (1974) 351–358.

[6] H. G. Dales and A. T. M. Lau, The second duals of Beurling algebras, *Mem. Amer. Math. Soc.* **177** (2005), no. 836, 191 pages.

[7] H. G. Dales, A. T. M. Lau and D. Strauss, Banach algebras on semigroups and their compactifications, *Mem. Amer. Math. Soc.* **205** (2010), no. 966, 165 pages.

[8] J. Duncan and S. A. R. Hosseiniun, The second dual of a Banach algebra, *Proc. Roy. Soc. Edinburgh Sect. A.* **84** (1979), no. 3-4, 309–325.

[9] J. Duncan and I. Namioka, Amenability of inverse semigroups and their semigroup algebras, *Proc. Roy. Soc. Edinburgh Sect. A.* **80** (1978), no. 3-4, 309–321.

[10] G. H. Esslamzadeh, Duals and topological center of a class of Matrix algebras with applications, *Proc. Amer. Math. Soc.* **128** (2000), no. 12, 3493–3503.

[11] J. M. Howie, Fundamentals of semigroup theory, London Mathematical Society Monographs, Oxford Science Publications, Oxford University Press, New York, 1995.

[12] A. Rejali, The Arens regularity of weighted convolution algebras on semitopological semigroups, 227–244, Proceedings of the 21st. Annual Iranian Math. Conf., Isfahan, 1990.

[13] W. A. F. Ruppert, On weakly almost periodic sets, *Semigroup Forum* **32** (1985), no. 3, 267–281.

[14] N. J. Young, The irregularity of multiplication in group algebras, *Quart J. Math. Oxford Ser. (2)* **24** (1973) 59–62.

[15] N. J. Young, Semigroup algebras having regular multiplication, *Studia Math.* **47** (1973) 191–196.

(Fatemeh Abtahi) Department of Mathematics, University of Isfahan, P.O. Box 81746-73441, Isfahan, Iran
E-mail address: f.abtahi@sci.ui.ac.ir

(Bahram Khodsiani) Department of Mathematics, University of Isfahan, P.O. Box 81746-73441,
E-mail address: b_khodsiani@sci.ui.ac.ir

(Ali Rejali) Department of Mathematics, University of Isfahan, P.O. Box 81746-73441, Isfahan, Iran
E-mail address: rejali@sci.ui.ac.ir