ISSN: 1017-060X (Print)



ISSN: 1735-8515 (Online)

# **Bulletin of the**

# Iranian Mathematical Society

Vol. 40 (2014), No. 6, pp. 1527-1538

Title:

Arens regularity of inverse semigroup algebras

Author(s):

F. Abtahi, B. Khodsiani and A. Rejali

Published by Iranian Mathematical Society http://bims.ims.ir

Bull. Iranian Math. Soc. Vol. 40 (2014), No. 6, pp. 1527–1538 Online ISSN: 1735-8515

# ARENS REGULARITY OF INVERSE SEMIGROUP ALGEBRAS

F. ABTAHI\*, B. KHODSIANI AND A. REJALI

(Communicated by Gholam Hossein Esslamzadeh)

ABSTRACT. We present a characterization of Arens regular semigroup algebras  $\ell^1(S)$ , for a large class of semigroups. Mainly, we show that if the set of idempotents of an inverse semigroup S is finite, then  $\ell^1(S)$  is Arens regular if and only if S is finite. **Keywords:** Arens regularity, completely simple semigroup, inverse semigroup, left (right) group, weakly cancellative. **MSC(2010):** Primary: 43A20; Secondary: 46H05.

### 1. Introduction

Let  $\mathcal{A}$  be a Banach algebra. The second dual of  $\mathcal{A}$  can be made into a Banach algebra in two different ways, as was first shown by Arens [1] and [2]. The algebra  $\mathcal{A}$  is called Arens regular, if these two multiplications coincide. For a discrete semigroup S, Young [15] presented a criterion for Arens regularity of  $\ell^1(S)$ . Indeed, he showed that  $\ell^1(S)$  is Arens regular if and only if there is no pair of sequences  $(x_n)$  and  $(y_m)$  in Ssuch that the sets  $\{x_n y_m : m > n\}$  and  $\{x_n y_m : m < n\}$  are disjoint. Another criterion has been provided in [3], which demonstrates that  $\ell^1(S)$  is Arens regular if and only if for every pair of sequences of distinct elements of S,  $(x_n), (y_m)$ , there is a submatrix  $(u_n v_m)$  of  $(x_n y_m)$  of type C. Hosseiniun in a joint work with Duncan [8], asked for which semigroups S, the convolution Banach algebra  $\ell^1(S)$  is Arens regular. Also in [8], they showed that if S is a Brandt semigroup, then  $\ell^1(S)$  is Arens regular if and only if S is finite. We also refer to [6] as a helpful

O2014 Iranian Mathematical Society

Article electronically published on December 11, 2014.

Received: 11 April 2013, Accepted: 18 November 2013.

<sup>\*</sup>Corresponding author.

<sup>1527</sup> 

reference. Furthermore, some authors have studied Arens regularity of  $\ell^1(S,\omega)$ , where  $\omega: S \to (0,\infty)$  is a weight function on S. We just refer to some relevant works such as [3], [5] and [12].

The aim of the present work is to study Arens regularity of the convolution Banach algebra  $\ell^1(S)$ , for several classes of semigroups. In particular when S is an inverse semigroup or a product of semigroups.

## 2. Preliminaries and definitions

In this section, we mention preliminaries and definitions which will be used throughout the paper. All the terminologies used in this work are from either [4] or [11].

**Definition 2.1.** Let S be a discrete semigroup.

- (1) S is called left (resp. right) zero semigroup if xy = x (resp. xy = y), for all  $x, y \in S$ .
- (2) We say that S is left (resp. right) cancellative if xy = xz (resp. yx = zx) implies y = z. Moreover S is called weakly cancellative if the sets  $x^{-1}y = \{t \in S : xt = y\}$  and  $yx^{-1} = \{t \in S : tx = y\}$  are finite, for all  $x, y \in S$ .
- (3) We say that S is rectangular band if aba = a for all  $a, b \in S$ . By [11, Theorem 1.1.3], S is rectangular band if and only if it is isomorphic to the direct product of a left zero semigroup E and a right zero semigroup F, with the multiplication defined by

$$(a_1, b_1).(a_2, b_2) = (a_1, b_2).$$

- (4) S is said to be left (resp. right) simple if it contains no proper left (resp. right) ideal. It is called simple if it has no proper two sided ideals. S with a zero element 0 is called 0-simple if

  (i) {0} and S are its only ideals,
  - (ii)  $S^2 \neq \{0\}.$
- (5) S is called left (resp. right) group if for each pair of elements s,t in S there exists a unique element x ∈ S such that xs = t (resp. sx = t). By [11, page 61], S is left group if and only if S is isomorphic to a direct product of a left zero semigroup E and a group G. Also S is right group if and only if S is isomorphic to a direct product of a group G and a right zero semigroup F. Moreover S is left (resp. right) group if and only if it is left (resp. right) simple and right (resp. left) cancellative [4].

- (6) S is called regular if for every a ∈ S there is b ∈ S such that a = aba. Furthermore, S is an inverse semigroup if for every a ∈ S there is a unique a\* ∈ S such that aa\*a = a and a\*aa\* = a\*.
- (7) An element e ∈ S is called idempotent if ee = e. We denote by E(S) the set of all idempotents of S. If e, f ∈ E(S), we shall write e ≤ f if ef = fe = e. Then ≤ is a partial order relation on E(S). An element e ∈ E(S) is a primitive idempotent if for each f ∈ E(S) such that f ≤ e, then f = e. S is called completely simple if it is simple and it has a primitive idempotent. If S has a zero element 0, then it will be called completely 0-simple if it is 0-simple and contains a nonzero primitive idempotent.

Now let G be a group and  $G^0 = G \cup \{0\}$  be the group with zero arising from G by adjunction of a zero element. Let I,  $\Lambda$  be non-empty sets and  $P = (p_{\lambda i})$  be a  $\Lambda \times I$  matrix with entries in  $G^0$ . Suppose that P is regular in the sense that no row or column of P consists entirely of zeros; formally for each  $i \in I$  (resp.  $\lambda \in \Lambda$ ), there exists  $\lambda \in \Lambda$  (resp.  $i \in I$ ) such that  $p_{\lambda i} \neq 0$ .

(1) Let  $S = G \times I \times \Lambda$  and define a binary operation on S by the rule that

$$(a, i, \lambda)(b, j, \mu) = (ap_{\lambda j}b, i, \mu).$$

Then S will be denoted by  $M[G; I, \Lambda; P]$  and will be called the  $I \times \Lambda$  Rees matrix semigroup over the group G. By [11, Theorem 3.3.1] S is completely simple semigroup. Conversely any completely simple semigroup is isomorphic to one constructed in this manner.

(2) Let  $S = (G \times I \times \Lambda) \cup \{0\}$ , and define a multiplication on S by

$$(a, i, \lambda)(b, j, \mu) = \begin{cases} (ap_{\lambda j}b, i, \mu) & \text{if } p_{\lambda j} \neq 0\\ 0 & \text{if } p_{\lambda j} = 0 \end{cases}$$

and

$$(a, i, \lambda)0 = 0(a, i, \lambda) = 0.$$

The semigroup S constructed in accordance with this recipe will be denoted by  $M^0[G; I, \Lambda; P]$ , and will be called the  $I \times \Lambda$  Rees matrix semigroup over the 0-group  $G^0$  with the regular sandwich matrix P. Then by [11, Theorem 3.2.3], S is a completely 0simple semigroup. Conversely, every completely 0-simple semigroup is isomorphic to the one constructed in this way.

(3) Every completely 0-simple inverse semigroup is called Brandt semigroup. By [4, Theorem 3.9] S is Brandt semigroup if and

1529

only if  $S \cong M^0(G; E, E; \Delta)$ , for some group G and index set E and the identity matrix  $\Delta$ .

### 3. Main results

Following [3], let  $(x_n), (y_m)$  be sequences of distinct elements in S such that the double matrix  $(x_n y_m)$  satisfies one of the following conditions.

- (i) There is  $c \in S$  such that  $x_n y_m = c$ , for all  $n, m \in \mathbb{N}$ , i.e. every row and column of the matrix is constant.
- (ii) There is a sequence  $(c_m)$  of distinct elements in S such that  $x_n y_m = c_m$ , for all  $n, m \in \mathbb{N}$ , i.e. every row of the matrix is constant.
- (iii) There is a sequence  $(d_n)$  of distinct elements in S such that  $x_n y_m = d_n$ , for all  $n, m \in \mathbb{N}$ , i.e. every column of the matrix is constant.

Then the matrix  $(x_n y_m)$  is called of type C. As we pointed out in Section 1, a criterion for Arens regularity of  $\ell^1(S)$  has been provided in [3]. Indeed,  $\ell^1(S)$  is Arens regular if and only if for every pair of sequences of distinct elements of S,  $(x_n), (y_m)$ , there is a submatrix  $(u_n v_m)$  of  $(x_n y_m)$  of type C. This criterion plays an essential role in the present work, to obtain the desired results.

We commence with the following theorem.

**Theorem 3.1.** Let S be a discrete semigroup.

- (i) If S is a left or right zero semigroup, then  $\ell^1(S)$  is Arens regular.
- (ii) If  $S = E \times G$  [resp.  $G \times F$ ] is a left [resp. right] group, then  $\ell^1(S)$  is Arens regular if and only if G is finite.
- (iii) If  $S = E \times F$  is a rectangular band, then  $\ell^1(S)$  is Arens regular if and only if E or F is finite.
- (iv) If S = M(G; E, F; P) [resp.  $M^0(G; E, F; P)$ ] is a completely simple [resp.0-simple] semigroup, then  $\ell^1(S)$  is Arens regular if and only if  $E \times G$  or  $G \times F$  is finite.

*Proof.* (i). Let S be a left (resp. right) zero semigroup and  $(x_n), (y_m)$  be two sequences of distinct elements in S. Since  $x_n y_m = x_n$  (resp.  $x_n y_m = y_m$ ), for all  $n, m \in \mathbb{N}$ , it follows that every column (resp. row) of this matrix is constant. Thus  $(x_n y_m)$  is a matrix of type C, which implies that  $\ell^1(S)$  is Arens regular [3].

(ii). First let  $\ell^1(S)$  be Arens regular. Since  $\ell^1(E \times G) \cong \ell^1(E) \hat{\otimes} \ell^1(G)$ , thus  $\ell^1(G)$  is a closed subalgebra of  $\ell^1(E \times G)$ . It follows that  $\ell^1(G)$  is Abtahi, Khodsiani and Rejali

Arens regular [6, Corollary 3.15], which implies the finiteness of G [14]. For the converse, let G be finite and let  $(x_n), (y_m)$  be two sequences with distinct elements in S. Thus  $x_n = (a_n, b_n)$  and  $y_m = (c_m, d_m)$ , where  $a_n, c_m \in E$  and  $b_n, d_m \in G$ , for all  $m, n \in \mathbb{N}$ . Hence  $x_n y_m = (a_n, b_n d_m)$ . Since  $\ell^1(G)$  is Arens regular, it follows that there is a submatrix of  $(b_n d_m)$  of type C [3]. By the definition of a matrix of type C, the only case that may occur is that  $(b_n d_m)$  has a constant submatrix. Indeed there is  $g \in G$  and subsequences  $(b_{n_k})$  and  $(d_{m_l})$  of  $(b_n)$  and  $(d_m)$ , respectively such that  $b_{n_k} d_{m_l} = g$ . Thus  $x_{n_k} y_{m_l} = (a_{n_k}, g)$ . It follows that  $(x_{n_k} y_{m_l})$  is a submatrix of  $(x_m y_m)$  of type C. Thus  $\ell^1(S)$  is Arens regular. The same arguments can be used for the case where S is a right group.

(iii). Let E and F be infinite and choose two sequences  $(a_n)$  and  $(d_m)$  of distinct elements in E and F, respectively. For the arbitrary sequences  $(b_n)$  and  $(c_m)$  in E and F, respectively, set  $x_n = (a_n, b_n)$  and  $y_m = (c_m, d_m)$ . Thus  $(x_n y_m) = (a_n, d_m)$ , which has no submatrix of type C. Therefore  $\ell^1(S)$  is not Arens regular [3]. Conversely, suppose that E is finite and  $x_n = (a_n, b_n)$  and  $y_m = (c_m, d_m)$  be sequences with the distinct elements in  $E \times F$ . Thus  $x_n y_m = (a_n, d_m)$ , for all  $m, n \in \mathbb{N}$ . Hence there is an element  $a \in E$  and a subsequence  $(x_{n_k})$  of  $(x_n)$  such that  $x_{n_k} y_m = (a, y_m)$ . It follows that  $(x_{n_k} y_m)$  is a submatrix of  $(x_m y_m)$  of type C. Consequently  $\ell^1(S)$  is Arens regular, by [3]. Similar arguments can be applied for the case where F is finite.

(iv). Let  $E \times G$  and  $G \times F$  be infinite. If G is infinite, then one can choose a sequence  $(a_n)$  with distinct elements in G. Since  $P = (p_{\lambda i})$  is regular, there are  $i \in E$  and  $\lambda \in F$  such that  $p_{\lambda i} \neq 0$ . Let  $x_n = (a_n, i, \lambda)$ . Then  $x_n x_m = (a_n p_{\lambda i} a_m, i, \lambda)$  and for all  $n_1, n_2, m_1, m_2 \in \mathbb{N}$  with  $n_1 \neq n_2$ and  $m_1 \neq m_2$ , we have  $x_{n_1} x_{m_1} \neq x_{n_2} x_{m_1}$  and  $x_{n_1} x_{m_1} \neq x_{n_1} x_{m_2}$ . Thus  $(x_n x_m)$  has no submatrix of type C. Now suppose that G is finite but E and F are both infinite. Then there are sequences  $(i_n)$  and  $(\lambda_m)$  with the distinct elements in E and F, respectively. Let e be the identity element of G and  $i_0 \in E$  and  $\lambda_0 \in F$  be fixed. Also let  $x_n = (e, i_n, \lambda_0)$ and  $y_m = (e, i_0, \lambda_m)$ . Then  $x_n y_m = (p_{\lambda_0 i_0}, i_n, \lambda_m)$  and so  $(x_n y_m)$  has no submatrix of type C. It follows that  $\ell^1(S)$  is not Arens regular in any cases. Conversely, let  $E \times G$  be finite, and let  $(x_n)$  and  $(y_m)$  be two sequences with distinct elements in S. Suppose that  $x_n = (a_n, i_n, \lambda_n)$ and  $y_m = (b_m, j_m, \mu_m)$ . Then

$$x_n y_m = \begin{cases} (a_n p_{\lambda_n j_m} b_m, i_n, \mu_m) & \text{if } p_{\lambda_n j_m} \neq 0\\ 0 & \text{if } p_{\lambda_n j_m} = 0. \end{cases}$$

Moreover, there are  $(i,g) \in E \times G$  and subsequences  $(x_{n_k})$  and  $(y_{m_l})$ of  $(x_n)$  and  $(y_m)$ , respectively such that  $x_{n_k}y_{m_l} = (g, i, \mu_{m_l})$ . Thus  $(x_{n_k}y_{m_l})$  is a submatrix of  $(x_my_m)$  of type C. Thus  $\ell^1(S)$  is regular. Some similar arguments will be used in the case where  $G \times F$  is finite.  $\Box$ 

The following result is immediately obtained from Theorem 3.1 part (iv).

**Corollary 3.2.** Let S be a Brandt semigroup. Then  $\ell^1(S)$  is Arens regular if and only if S is finite.

**Remark 3.3.** Suppose that S and T are two discrete semigroups such that  $\ell^1(S \times T)$  is Arens regular. Then  $\ell^1(S)$  and  $\ell^1(T)$  are Arens regular, by [6, Corollary 3.15]. The converse of this statement is not necessarily valid. For example, let S [resp. T] be an infinite left [resp. right] zero semigroup. Then Theorem 3.1 part (i) implies that  $\ell^1(S)$  and  $\ell^1(T)$  are Arens regular whereas  $\ell^1(S \times T)$  is not Arens regular, by Theorem 3.1 part (iii).

**Proposition 3.4.** Let S be a discrete semigroup.

- (i) If S is weakly cancellative then l<sup>1</sup>(S) is Arens regular if and only if S is finite.
- (ii) If S is a simple [resp. 0-simple] inverse semigroup, then  $\ell^1(S)$  is Arens regular if and only if S is finite.

*Proof.* (i). Suppose that  $\ell^1(S)$  is Arens regular. If S is infinite, then there is a sequence  $(x_n)$  with distinct elements in S. Thus  $(x_n x_m)$  has a submatrix  $(x_{n_k} x_{m_l})$  of type C by [3]. Consequently one of the following cases occurs.

- (a) There is c ∈ S such that x<sub>nk</sub>x<sub>ml</sub> = c, for all k, l ∈ N. Thus for fixed m<sub>l</sub>, {x<sub>n1</sub>, x<sub>n2</sub>, x<sub>n3</sub>, ···} ⊆ cx<sup>-1</sup><sub>ml</sub>.
  (b) There is a sequence (c<sub>ml</sub>) of distinct elements in S such that
- (b) There is a sequence  $(c_{m_l})$  of distinct elements in S such that  $x_{n_k}x_{m_l} = c_{m_l}$ , for all  $k, l \in \mathbb{N}$ . It follows that for fixed  $m_l$ ,  $\{x_{n_1}, x_{n_2}, x_{n_3}, \dots\} \subseteq c_{m_l} x_{m_l}^{-1}$ .
- (c) There is a sequence  $(d_{n_k})$  of distinct elements in S such that  $x_{n_k}x_{m_l} = d_{n_k}$ , for all  $k, l \in \mathbb{N}$ . Then for fixed  $n_k$ ,

$$\{x_{m_1}, x_{m_2}, x_{m_3}, \cdots\} \subseteq x_{n_k}^{-1} d_{n_k}.$$

By assumption, the sets  $x^{-1}y = \{z \in S : xz = y\}$  and  $xy^{-1}$  are finite, for all  $x, y \in S$ . But this is in contradiction to all the statements (a), (b) and (c). Therefore S is finite. The converse is clear.

(ii). Let S be a simple [resp. 0-simple] inverse semigroup and let  $\ell^1(S)$  be Arens regular. Then one can readily show, by using the Young's criterion [15], that every chain of idempotents has finite length. It follows by Zorn's lemma that S has a primitive idempotent and so S is completely simple [resp. 0-simple]. But only completely simple inverse semigroups are groups [11, page 151]. Moreover only completely 0-simple inverse semigroups are Brandt semigroups. Therefore, the arguments of [14] and also Corollary 3.2 imply that  $\ell^1(S)$  is Arens regular if and only if S is finite.

**Example 3.5.** Given a non-empty set X, we define  $\mathcal{I}_X$  to consist of all partial one-one maps of X; i.e., one-one maps from subsets of X to subsets of X. Then  $\mathcal{I}_X$  is a semigroup under the operation  $\circ$  of composition of relations. Indeed, for  $\rho, \sigma \in \mathcal{I}_X$ ,  $(x, y) \in \rho \circ \sigma$  if and only if there exists  $z \in X$  such that  $(x, z) \in \rho$  and  $(z, y) \in \sigma$ . By [11, Theorem 5.1.5]  $\mathcal{I}_X$  is an inverse semigroup, so called symmetric inverse semigroup on the set X. It is worth noting that every inverse semigroup can be embedded in a symmetric inverse semigroup, by Vagner-Preston Theorem [11, Theorem 5.1.7].

Now we show that  $\ell_1(\mathcal{I}_X)$  is Arens regular if and only if X is finite. To prove, suppose that X is infinite. Then there is a sequence  $\{x_n\}$  with distinct elements in X. Let  $I_n$  denote the one-one partial map on  $\{x_1, x_2, \dots, x_n\}$  defined by  $I_n(x_i) = x_i$ , for each i. Then we have

$$\{I_{2n+1} \circ I_{2m} : n < m\} \cap \{I_{2n+1} \circ I_{2m} : n > m\} = \emptyset.$$

Thus  $\ell_1(\mathcal{I}_X)$  is not Arens regular by Young's criterion [15]. The converse is clear.

Let I be a proper ideal of S and define the relation  $\sim$  on S as the following

$$x \sim y$$
 if and only if  $x = y$  or  $x, y \in I$ .

The quotient semigroup S/I is

$$\{I\} \cup \{\{x\} : x \in S \setminus I\}\}.$$

In S/I, the product of two elements in  $S \setminus I$  is the same as their product in S if this lies in  $S \setminus I$ ; otherwise the product is I. Note that I is the zero element of the semigroup S/I. See [11, page 33] for more information.

**Proposition 3.6.** Let S be an inverse semigroup and I be a maximal ideal in S. Then the following statements hold.

1533

- (i)  $\ell^1(S/I)$  is Arens regular if and only if S/I is finite.
- (ii)  $\ell^1(S)$  is Arens regular if and only if  $\ell^1(S/I)$  and  $\ell^1(I)$  are Arens regular.

*Proof.* (i). It is readily verified that S/I is an inverse semigroup such that  $(S/I)^2 \neq I$ . By [11, Proposition 3.1.5], S/I is a 0-simple inverse semigroup. Thus Proposition 3.4 implies that  $\ell^1(S/I)$  is Arens regular if and only if S/I is finite.

(*ii*). Let  $\ell^1(S/I)$  and  $\ell^1(I)$  be Arens regular. By part (*i*), S/I is a 0-simple inverse semigroup and Arens regularity of  $\ell^1(S/I)$  implies that S/I is finite. It follows that  $S\backslash I$  is a finite set. To that end, suppose that  $(x_n), (y_m)$  are sequences of distinct elements in S. Since  $S\backslash I$  is finite, there is N > 0 such that for each  $n \ge N$ ,  $(x_n)$  and  $(y_m)$  can be considered as two sequences in I. Since  $\ell^1(I)$  is Arens regular, there is a submatrix  $(u_{n_k}v_{m_l})$  of  $(x_ny_m)_{n,m\ge N}$  of type C. Consequently  $\ell^1(S)$  is Arens regular. The converse is obvious, by [6, Corollary 3.15].

Recall from [11] that  $S = S_1 \supseteq a S_2 \supseteq \cdots a S_m$  is a principal series of S if

- (i) each  $S_i$  is a two sided ideal of S;
- (ii) there is no ideal of S strictly between  $S_i$  and  $S_{i+1}$ , for each  $1 \le i \le m-1$ ;
- (iii)  $S_m = K(S)$ , where K(S) is the kernel of the semigroup S; i.e. the minimal two-sided ideal of S. It is in fact the intersection of all two-sided ideals of S.

**Proposition 3.7.** Let S be an inverse semigroup with a principal series. Then the following assertions are equivalent.

- (i)  $\ell^1(S)$  is Arens regular.
- (ii)  $\ell^1(K(S))$  and  $\ell^1(S/K(S))$  are Arens regular.
- (iii)  $S \setminus K(S)$  is finite and  $\ell^1(K(S))$  is Arens regular.

*Proof.*  $(i) \Rightarrow (ii)$ . It is clear.

 $(ii) \Rightarrow (iii)$ . Let  $S = S_1 \supseteq S_2 \supseteq \cdots \supseteq S_m = K(S)$  be a principal series for S. Since  $\ell^1(S/K(S))$  is Arens regular and  $S/K(S) \supseteq S_{m-1}/S_m$ , it follows that  $\ell^1(S_{m-1}/S_m)$  is Arens regular [6, Corollary 3.15]. Since  $S_{m-1}/S_m$  is a 0-simple inverse semigroup [11, Proposition 3.1.5], then  $S_{m-1}/S_m$  is finite by Proposition 3.4 part (*ii*). Inductively, for each

 $2 \leq i \leq m-1$ , the semigroup  $S_{i-1}/S_i$  is finite. Since

$$S \setminus K(S) = \bigcup_{i=2}^{m} S_{i-1} \setminus S_i,$$

it follows that  $S \setminus K(S)$  is finite and so the result follows.

 $(iii) \Rightarrow (i)$ . It is proved exactly by the same arguments given in the proof of part (ii) in Proposition 3.6.

We now are in a position to prove the main theorem of the present work.

**Theorem 3.8.** Let S be an inverse semigroup such that the set of idempotents of S is finite. Then  $\ell^1(S)$  is Arens regular if and only if S is finite.

*Proof.* Suppose that  $\ell^1(S)$  is Arens regular. Since E(S) is finite, thus there exists a principal series as  $S = S_0 \supseteq S_1 \supseteq \cdots \supseteq S_m = G$ , for some group G, such that K(S) = G and also for each  $0 \leq i < m$ 

$$S_i/S_{i+1} \cong (S_i/G)/(S_{i+1}/G),$$

as semigroup isomorphisms. Moreover natural maps from  $S_i$  onto  $S_i/G$  are homomorphisms and also  $S_i/S_{i+1}$  is a completely 0-simple inverse semigroup, for each  $0 \leq i < m$ ; see [7, Theorem 3.12] and also [9], for more details. By Proposition 3.7,  $\ell^1(S)$  is Arens regular if and only if  $S \setminus G$  is finite and  $\ell^1(G)$  is Arens regular. But by [14], Arens regularity of  $\ell^1(G)$  is equivalent to the finiteness of G. Thus the result is obtained.  $\Box$ 

It should be noted that some other results related to regular and inverse semigroups have been provided in [10, Theorem 5.1].

We end this work with the following examples which are interesting in their own right.

**Examples 3.9.** Suppose that  $T = \mathbb{N} \cup \{0\}$  with the multiplication defined by

$$n.m = \left\{ \begin{array}{ll} n & if \quad n = m \\ 0 & otherwise. \end{array} \right.$$

(1) It is readily verified that T is an inverse semigroup such that E(T) = T, which is infinite. But  $\ell^1(T)$  is Arens regular. In fact, let  $(x_n)$  and  $(y_m)$  be two sequences with the distinct elements in T. We may find subsequences  $(x_{n_k})$  and  $(y_{m_l})$  of  $(x_n)$  and  $(y_m)$ 

1535

respectively, such that  $\{x_{n_1}, x_{n_2}, x_{n_3}, \dots\} \cap \{y_{m_1}, y_{m_2}, y_{m_3}, \dots\} = \emptyset$ . So  $(x_{n_k}y_{m_l})$  is the matrix 0 and consequently it is of type C. Then  $\ell^1(T)$  is Arens regular by [3]. This example shows that the condition of the finiteness of E(S) is necessary in Theorem 3.8.

(2) Let S = T × T. Then S is an inverse semigroup. Moreover, for each (n,m) ∈ S, (n,m).(n,m) = (n,m) and so E(S) = S. We show that l<sup>1</sup>(S) is not Arens regular. First let us recall some spaces from [3]. We denote by l<sup>∞</sup>(S) the set of all functions f : S → ℝ such that

$$||f||_{\infty} = \sup\{|f(x)| : x \in S\}$$

is finite. We say that  $f \in \ell^{\infty}(S)$  is weakly almost periodic if the set  $\{f_x : x \in S\}$  is relatively weakly compact in  $\ell^{\infty}(S)$ , where  $f_x$ is the left translation of a function f by x. The set of weakly almost periodic functions on S will be denoted by wap(S). By [3, Theorem 3.2],  $\ell^1(S)$  is Arens regular if and only if wap(S) = $\ell^{\infty}(S)$ . We use Ruppert criterion in [13], to show that wap $(S) \neq$  $\ell^{\infty}(S)$ . This criterion mentions that for  $A \subseteq S$ ,  $\chi_A \in wap(S)$  if and only if every infinite set B in S contains a finite subset Fsuch that the sets

$$\bigcap_{b \in F} Ab^{-1} \setminus \bigcap_{b \in B \setminus F} Ab^{-1}$$

and also

$$\bigcap_{b\in F} b^{-1}A \setminus \bigcap_{b\in B\setminus F} b^{-1}A$$

are finite. Now let  $X = \{(k,0) : k \in T\}$ ,  $Y = \{(0,k) : k \in T\}$ and  $Z = X \cup Y$ . To that end, we show that  $\chi_{\{z\}} \notin wap(S)$ , for each  $z \in Z$ . Let  $B = \{(k,n) : k, n \in T\}$ . Then for each  $(k,n) \in B$ 

$$(k,n)^{-1}(k,0) = \{(k,m) : m \neq n\} = B \setminus \{(k,n)\}.$$

Thus for all finite subsets F of B,

$$\bigcap_{(k,n)\in F} (k,n)^{-1}(k,0) \setminus \bigcap_{(k,n)\in B\setminus F} (k,n)^{-1}(k,0) = \bigcap_{(k,n)\in F} (k,0)(k,n)^{-1} \setminus \bigcap_{(k,n)\in B\setminus F} (k,n)^{-1}(k,0)$$
$$= (B\setminus F)\setminus F = B\setminus F.$$

Since  $B \setminus F$  is infinite, [13] follows that  $\chi_{(k,0)} \notin wap(S)$ . Similarly  $\chi_{(0,k)} \notin wap(S)$ . Therefore  $\ell^1(S)$  is not Arens regular.

### Acknowledgments

The authors would like to thank the referee of the paper for his/her invaluable comments. This research was partially supported by the Banach algebra Center of Excellence for Mathematics, University of Isfahan.

#### References

- [1] R. Arens, Operations induced in function classes, Monatsh. Math. 55 (1951) 1–19.
- [2] R. Arens, The adjoint of a bilinear operation, Proc. Amer. Math. Soc. 2 (1951) 839–848.
- [3] J. W. Baker and A. Rejali, On the Arens regularity of weighted convolution algebras, J. London Math. Soc. (2) 40 (1989), no. 3, 535–546.
- [4] A. H. Clifford and G. B. Preston, The Algebraic Theory of Semigroups, Vol I, Amer. Math. Soc., Providence, 1961.
- [5] I. G. Craw and N. J. Young, Regularity of multiplications in weighted group and semigroup algebras, Quart. J. Math. Oxford ser. (2) 25 (1974) 351–358.
- [6] H. G. Dales and A. T. M. Lau, The second duals of Beurling algebras, Mem. Amer. Math. Soc. 177 (2005), no. 836, 191 pages.
- [7] H. G. Dales, A. T. M. Lau and D. Strauss, Banach algebras on semigroups and their compactifications, *Mem. Amer. Math. Soc.* **205** (2010), no. 966, 165 pages.
- [8] J. Duncan and S. A. R. Hosseiniun, The second dual of a Banach algebra, Proc. Roy. Soc. Edinburgh Sect. A. 84 (1979), no. 3-4, 309–325.
- [9] J. Duncan and I. Namioka, Amenability of inverse semigroups and their semigroup algebras, Proc. Roy. Soc. Edinburgh Sect. A. 80 (1978), no. 3-4, 309–321.
- [10] G. H. Esslamzadeh, Duals and topological center of a class of Matrix algebras with applications, *Proc. Amer. Math. Soc.* **128** (2000), no. 12, 3493–3503.
- [11] J. M. Howie, Fundamentals of semigroup theory, London Mathematical Society Monographs, Oxford Science Publications, Oxford University Press, New York, 1995.
- [12] A. Rejali, The Arens regularity of weighted convolution algebras on semitopological semigroups, 227–244, Proceedings of the 21st. Annual Iranian Math. Conf., Isfahan, 1990.
- [13] W. A. F. Ruppert, On weakly almost periodic sets, Semigroup Forum 32 (1985), no. 3, 267–281.
- [14] N. J. Young, The irregularity of multiplication in group algebras, Quart J. Math. Oxford Ser. (2) 24 (1973) 59–62.
- [15] N. J. Young, Semigroup algebras having regular multiplication, Studia Math. 47 (1973) 191–196.

(Fatemeh Abtahi) Department of Mathematics, University of Isfahan, P.O. Box 81746-73441, Isfahan, Iran

E-mail address: f.abtahi@sci.ui.ac.ir

(Bahram Khodsiani) Department of Mathematics, University of Isfahan, P.O. Box 81746-73441,

*E-mail address:* b\_khodsiani@sci.ui.ac.ir

(Ali Rejali) Department of Mathematics, University of Isfahan, P.O. Box 81746-73441, Isfahan, Iran

*E-mail address*: rejali@sci.ui.ac.ir