ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 40 (2014), No. 6, pp. 1553–1571

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 40 (2014), No. 6, pp. 1553–1571
Online ISSN: 1735-8515

DILATIONS, MODELS, SCATTERING AND SPECTRAL PROBLEMS OF 1D DISCRETE HAMILTONIAN SYSTEMS

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(Communicated by Hamid Reza Ebrahimi Vishki)

Abstract. In this paper, the maximal dissipative extensions of a symmetric singular 1D discrete Hamiltonian operator with maximal deficiency indices $(2,2)$ (in limit-circle cases at $\pm\infty$) and acting in the Hilbert space $\ell^2_\Omega(\mathbb{Z};\mathbb{C}^2)$ ($\mathbb{Z} := \{0, \pm1, \pm2, ...\}$) are considered. We deal with two classes of dissipative operators with separated boundary conditions both at $-\infty$ and ∞. For each of these cases, we establish a self-adjoint dilation of the dissipative operator and construct the incoming and outgoing spectral representations. Then, it becomes possible to determine the scattering function (matrix) of the dilation. Further, a functional model of the dissipative operator and its characteristic function in terms of the Weyl function of a self-adjoint operator are constructed. Finally, we show that the system of root vectors of the dissipative operators are complete in the Hilbert space $\ell^2_\Omega(\mathbb{Z};\mathbb{C}^2)$.
Keywords: Discrete Hamiltonian system, dissipative operator, self-adjoint dilation, characteristic function, completeness of the root vectors.
MSC(2010): Primary: 47B39; Secondary: 47B44, 47A20, 47A40, 47A45, 47B25, 47A75, 39A70.

1. Introduction

In the Hilbert space, one of the useful tools to investigate the abstract and applied theories is the functional model theory connected with dissipative or contractive operators. To construct the functional models for dissipative (contractive) operators, one can use the well known theory

Article electronically published on December 11, 2014.
Received: 28 June 2013, Accepted: 2 October 2013.

of Sz. Nagy-Foiaş [9] which is related to the spectral decompositions for self-adjoint (unitary) operators and Lax-Phillips scattering theory [8]. In this theory, characteristic function plays the main role by carrying the complete information on the spectral properties of the dissipative operator. The dissipative operator becomes the model in the incoming spectral representation of the dilation. The characteristic function can be factored, which gives some information about the completeness of the system of eigenvectors and associated (or root) vectors. This theory has been used for dissipative discrete singular Hamiltonian and Dirac operators in [1]-[4].

In this paper, the minimal symmetric one dimensional (1D) discrete Hamiltonian operator which acts in the Hilbert space $\ell^2_\Omega(\mathbb{Z};E)$, where $E:=\mathbb{C}^2$, is considered with maximal deficiency indices $(2,2)$. The deficiency indices $(2,2)$ is known as Weyl's limit-circle cases at $\pm\infty$. The boundary conditions at $\pm\infty$ will allow us to construct a space of boundary values of minimal symmetric operator and describe all maximal dissipative, maximal accumulative, self-adjoint and other extensions of such a symmetric operator. Two classes of maximal dissipative operators generated with separated boundary conditions, called 'dissipative at $-\infty$' and 'dissipative at ∞' are investigated. In each of these cases we construct a self-adjoint dilation and its incoming and outgoing spectral representations. With these representations we determine the scattering function (matrix) of a dilation according to the scheme of Lax and Phillips [8]. We also construct a functional model of the dissipative operator and construct its characteristic function in terms of the Weyl function of a self-adjoint operator with the help of the incoming spectral representation. Finally, based on the results obtained regarding the theory of the characteristic function we prove theorems on completeness of the system of eigenvectors and associated vectors of dissipative discrete Hamiltonian operators.

2. **Preliminaries**

We consider the one dimensional (1D) discrete Hamiltonian system on the whole line

(2.1)

$$(L_1u)_k := \begin{pmatrix} -a_k u^{(2)}_{k+1} + b_k u^{(2)}_k + p_k u^{(1)}_k = \lambda(c_k u^{(1)}_k + d_k u^{(2)}_k) \\ b_k u^{(1)}_k - a_{k-1} u^{(1)}_{k-1} + q_k u^{(2)}_k = \lambda(d_k u^{(1)}_k + r_k u^{(2)}_k) \end{pmatrix},$$

where λ is a complex spectral parameter, $u^{(1)} = \{u_k^{(1)}\}$ and $u^{(2)} = \{u_k^{(2)}\}$ $(k \in \mathbb{Z})$ are the sequences of complex numbers $u_k^{(1)}$ and $u_k^{(2)}$, $a_k \neq 0$, $b_k \neq 0$, $a_k, b_k, p_k, q_k, c_k, d_k, r_k \in \mathbb{R} := (-\infty, \infty)$ and $\Omega_k := \begin{pmatrix} c_k & d_k \\ d_k & r_k \end{pmatrix} > 0$ $(k \in \mathbb{Z})$.

Clearly (2.1) can be regarded as a discrete analog (for $a_k = \pm 1$, $k \in \mathbb{Z}$) of differential Hamiltonian (or Dirac-type) system given by

$$J\frac{du(t)}{dt} + Q(t)u(t) = \lambda\Omega(t)u(t),\ t \in \mathbb{R}, \tag{2.2}$$

where

$$J = \begin{pmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{pmatrix},\ u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix},$$

$$Q(t) = \begin{pmatrix} p(t) & k(t) \\ k(t) & q(t) \end{pmatrix},\ \Omega(t) = \begin{pmatrix} c(t) & d(t) \\ d(t) & r(t) \end{pmatrix},$$

$\Omega(t) > 0$ for almost all $t \in \mathbb{R}$, and the entries of the (2×2) matrices $Q(t)$ and $\Omega(t)$ are real-valued, Lebesgue measurable and locally integrable functions on $\mathbb{R}$ (see, for example, [5]).

Let $[u, v]$ denote the sequence of complex numbers with components $[u, v]_k$ defined by

$$[u, v]_k = a_k(u_k^{(1)}\bar{v}_{k+1}^{(2)} - u_{k+1}^{(2)}\bar{v}_k^{(1)})\ (k \in \mathbb{Z}) \tag{2.2}$$

for two arbitrary vector-valued sequences

$$u := \{u_k\} := \begin{Bmatrix} u_k^{(1)} \\ u_k^{(2)} \end{Bmatrix} \text{ and } v := \{v_k\} := \begin{Bmatrix} v_k^{(1)} \\ v_k^{(2)} \end{Bmatrix}\ (k \in \mathbb{Z})$$

To introduce the Hilbert space $H := \ell^2_\Omega(\mathbb{Z}; E)$ $(\Omega := \{\Omega_k\},\ k \in \mathbb{Z})$ consisting of all the vector-valued sequences $u = \{u_k\}$ $(k \in \mathbb{Z})$ with $\sum_{k=-\infty}^{\infty}(\Omega_k u_k, u_k)_E < +\infty$, we consider the inner product $(u, v) = \sum_{k=-\infty}^{\infty}(\Omega_k u_k, v_k)_E$. Denote by $L_1 u$ (respectively Lu) the vector-valued sequence with components $(L_1 u)_k$ (respectively $(Lu)_k := \Omega_k^{-1}(L_1 u)_k$) $(k \in \mathbb{Z})$. We consider the set $\mathfrak{D}_{\max}$ consisting of all the vectors $u \in H$ such that $Lu \in H$. The *maximal operator* $T_{\max}$ on $\mathfrak{D}_{\max}$ is defined by the equality $T_{\max} u = Lu$.

For $m, k \in \mathbb{Z}$ and $k < m$, we have the Green's formula

$$\sum_{j=k}^{m}[((L_1 u)_j, v_j)_E - (u_j, (L_1 v)_j)_E] = [u, v]_m - [u, v]_{k-1}\,. \tag{2.3}$$

It is clear from the Green's formula (2.3) that for two arbitrary vectors $u, v \in \mathfrak{D}_{\max}$ the limits $[u,v]_{\infty} = \lim_{m\to\infty}[u,v]_m$ and $[u,v]_{-\infty} = \lim_{k\to-\infty}[u,v]_k$ exist and are finite. Therefore, passing to the limit as $k \to -\infty$ and $m \to \infty$, we get for all $u, v \in \mathfrak{D}_{\max}$ that

$$(2.4) \qquad (T_{\max}u, v) - (u, T_{\max}v) = [u,v]_{\infty} - [u,v]_{-\infty}\,.$$

Let $T_{\min}$ denote the closure of the symmetric operator $T'_{\min}$ defined by $T'_{\min}u = T_{\max}u$ on the linear set $\mathfrak{D}'_{\min}$ consisting of finite vectors $u = \{u_k\}$ $(k \in \mathbb{Z})$ (i.e., vectors $u = \{u_k\}$ $(k \in \mathbb{Z})$ having only finitely many nonzero component). The *minimal operator* $T_{\min}$ is symmetric, and $T^*_{\min} = T_{\max}$.

The deficiency indices of $T_{\min}$ can be calculated using the deficiency indices for the case of half-line. Indeed, $\ell^2_{\Omega}(\mathbb{Z};E)$ is the orthogonal sum of the space $\ell^2_{\Omega}(\mathbb{N}_-;E)$ $(\mathbb{N}_- = \{-1,-2,-3,...\})$ and $\ell^2_{\Omega}(\mathbb{N};E)$ $(\mathbb{N}_+ = \{0,1,2,...\})$, which are imbedded in the natural way in $\ell^2_{\Omega}(\mathbb{Z};E)$. Denote by $T^-_{\min}$ $(T^-_{\max})$ and $T^+_{\min}$ $(T^+_{\max})$ the minimal (maximal) operators generated by the expression L in the spaces $\ell^2_{\Omega}(\mathbb{N}_-;E)$ and $\ell^2_{\Omega}(\mathbb{N};E)$, respectively, and $\mathfrak{D}^{\mp}_{\min}$ $(\mathfrak{D}^{\mp}_{\max})$ is a domain of $T^{\mp}_{\min}$ $(T^{\mp}_{\max})$. Then it is not hard to see that the equality $defT_{\min} = defT^-_{\min} + defT^+_{\min}$ holds for the defect number $defT_{\min} := \dim\{(T_{\min} - \lambda I)\,\mathfrak{D}\,(T_{\min})\}^{\perp}$, $Im\lambda \neq 0$, of $T_{\min}$. This implies that the deficiency indices of $T_{\min}$ has the form (k,k), where $k = 0, 1$ or 2. For the deficiency indices $(0,0)$, the operator $T_{\min}$ is self-adjoint, that is, $T^*_{\min} = T_{\min} = T_{\max}$ (see [1-7, 13-17]).

We assume that the symmetric operator $T_{\min}$ has deficiency indices $(2,2)$. In other words, we assume that Weyl's limit-circle case holds at $\pm\infty$ for the expression L or the operator $T_{\min}$ (see [1-7, 13-17]). The domain $\mathfrak{D}_{\min}$ of $T_{\min}$ consists of precisely those vectors $u \in \mathfrak{D}_{\max}$ satisfying the condition

$$(2.5) \qquad [u,v]_{\infty} - [u,v]_{-\infty} = 0, \ \forall v \in \mathfrak{D}_{\max}.$$

Let us consider the solutions of the system (2.1) $\theta(\lambda) = \{\theta_k(\lambda)\}$ and $\chi(\lambda) = \{\chi_k(\lambda)\}(k \in \mathbb{Z})$ satisfying the initial conditions

$$\theta^{(1)}_{-1}(\lambda) = 1,\ \theta^{(2)}_0(\lambda) = 0,\ \chi^{(1)}_{-1}(\lambda) = 0,\ \chi^{(2)}_0(\lambda) = 1/a_{-1}.$$

The *Wronskian* of the two solutions $u = \{u_k\}$ and $v = \{v_k\}$ $(k \in \mathbb{Z})$ of (2.1) is defined as $\mathcal{W}_k(u,v) := a_k(u^{(1)}_k v^{(2)}_{k+1} - u^{(2)}_{k+1}v^{(1)}_k)$ and thus we get $\mathcal{W}_k(u,v) = [u,\bar{v}]_k$ $(k \in \mathbb{Z})$. Since the Wronskian of the two solutions of (2.1) does not depend on k and the two solutions of this system are linearly independent if and only if their Wronskian is nonzero, we get from the conditions (2.5) and the constancy of the Wronskian that $\mathcal{W}_k(\theta, \chi) = 1$ $(k \in \mathbb{Z})$. Consequently, $\theta(\lambda)$ and $\chi(\lambda)$ form a fundamental system of solutions of (2.1), and $\theta(\lambda), \chi(\lambda) \in H$ for all $\lambda \in \mathbb{C}$.

Setting $\sigma = \theta(0)$ and $\omega = \chi(0)$, we arrive at the following equality for arbitrary vectors $u, v \in \mathfrak{D}_{\max}$

$$[u,v]_k = [u,\sigma]_k [\bar{v},\omega]_k - [u,\omega]_k [\bar{v},\sigma]_k \quad (k \in \mathbb{Z} \cup \{-\infty, \infty\}). \tag{2.6}$$

We can conclude that the domain $\mathfrak{D}_{\min}$ of the operator $T_{\min}$ consists of the vectors $u \in \mathfrak{D}_{\max}$ satisfying the boundary conditions ([4])

$$[u,\sigma]_{-\infty} = [u,\omega]_{-\infty} = [u,\sigma]_{\infty} = [u,\omega]_{\infty} = 0. \tag{2.7}$$

It is better to recall that a linear operator $\mathcal{T}$ (with dense domain $\mathfrak{D}(\mathcal{T})$) acting in some Hilbert space $\mathcal{H}$ is called *dissipative* (*accumulative*) if $Im(\mathcal{T}f, f) \geq 0$ $(Im(\mathcal{T}f, f) \leq 0)$ for all $f \in \mathfrak{D}(\mathcal{T})$ and *maximal dissipative* (*maximal accumulative*) if it does not have a proper dissipative (accumulative) extension.

The following linear maps of $\mathfrak{D}_{\max}$ into E will allow us to construct the maximal dissipative (accumulative, self-adjoint) operators:

$$\Phi_1 u = \begin{pmatrix} [u,\omega]_{-\infty} \\ [u,\sigma]_{\infty} \end{pmatrix}, \; \Phi_2 u = \begin{pmatrix} [u,\sigma]_{-\infty} \\ [u,\omega]_{\infty} \end{pmatrix} \; (u \in \mathfrak{D}_{\max}). \tag{2.8}$$

Then we have the following conclusion ([4]).

Theorem 2.1. *For any contraction S in E, the restriction of the operator T to the set of vectors $u \in \mathfrak{D}_{\max}$ satisfying the boundary condition*

$$(S - I)\,\Phi_1 u + i\,(S + I)\,\Phi_2 u = 0 \tag{2.9}$$

or

$$(S - I)\,\Phi_1 u - i\,(S + I)\,\Phi_2 u = 0 \tag{2.10}$$

is, respectively, a maximal dissipative or a maximal accumulative extension of the operator $T_{\min}$. Conversely, every maximal dissipative (maximal accumulative) extension of $T_{\min}$ is the restriction of $T_{\max}$ to the set of vectors $u \in \mathfrak{D}_{\max}$ satisfying (2.9) ((2.10)), and the contraction S is uniquely determined by the extension. These conditions give a self-adjoint extension if and only if S is unitary. In the latter case, (2.9) and (2.10) are equivalent to the condition $(\cos S)\,\Phi_1 u - (\sin S)\,\Phi_2 u = 0$, *where S is a self-adjoint operator (Hermitian matrix) in E.*

In particular, if S is a diagonal matrix, the boundary conditions

$$[u,\omega]_{-\infty} - \beta_1\,[u,\sigma]_{-\infty} = 0 \tag{2.11}$$

$$[u,\sigma]_{\infty}-\beta_2\,[u,\omega]_{\infty}=0 \tag{2.12}$$

with $Im\beta_1 \geq 0$ *or* $\beta_1 = \infty$, *and* $Im\beta_2 \geq 0$ *or* $\beta_2 = \infty$ *(*$Im\beta_1 \leq 0$ *or* $\beta_1 = \infty$, *and* $Im\beta_2 \leq 0$ *or* $\beta_2 = \infty$*) describe all maximal dissipative (maximal accumulative) extensions of* $T_{\min}$ *with separated boundary conditions. The self-adjoint extensions of* $T_{\min}$ *are obtained precisely when* $Im\beta_1 = 0$ *or* $\beta_1 = \infty$, *and* $Im\beta_2 = 0$ *or* $\beta_2 = \infty$. *Here for* $\beta_1 = \infty$ $(\beta_2 = \infty)$, *the condition (2.11) ((2.12)) should be replaced by* $[u,\sigma]_{-\infty} = 0$ $([u,\omega]_{\infty} = 0)$.

3. **Self-adjoint dilations of the maximal dissipative operators**

Now consider the dissipative operators $T^{-}_{\beta_1\beta_2}$ and $T^{+}_{\beta_1\beta_2}$generated by (2.1) and the boundary conditions (2.11) and (2.12) which can be regarded as 'dissipative at $-\infty$', i.e. when either $Im\beta_1 > 0$ and $Im\beta_2 = 0$ *or* $\beta_2 = \infty$; and 'dissipative at ∞', when $Im\beta_1 = 0$ *or* $\beta_1 = \infty$ and $Im\beta_2 > 0$.

To reach our main aim, we shall construct a self-adjoint dilation of the maximal dissipative operator $T^{-}_{\beta_1\beta_2}$ in the case of 'dissipative at $-\infty$ $(Im\beta_1 > 0$ and $Im\beta_2 = 0$ or $\beta_2 = \infty)$. For this purpose, we adjoint the spaces $\mathcal{L}^2(-\infty,0)$ and $\mathcal{L}^2(0,\infty)$ to H. Hence we have formed the orthogonal sum $\mathbf{H} = \mathcal{L}^2(-\infty,0) \oplus H \oplus \mathcal{L}^2(0,\infty)$ called the *main Hilbert space of the dilation*. In the space $\mathbf{H}$, we consider the operator $\mathbf{T}^{-}_{\beta_1\beta_2}$ generated by the expression

$$\mathbf{T}\langle\phi_-,u,\phi_+\rangle = \langle i\frac{d\phi_-}{d\xi}, L(u), i\frac{d\phi_+}{d\varsigma}\rangle \tag{3.1}$$

on the set $\mathfrak{D}(\mathbf{T}^{-}_{\beta_1\beta_2})$ consisting of all vectors $\langle\phi_-,u,\phi_+\rangle$ satisfying the conditions $\phi_- \in \mathcal{W}^1_2(-\infty,0)$, $\phi_+ \in \mathcal{W}^1_2(0,\infty)$, $u \in \mathfrak{D}_{\max}$,

$$[u,\omega]_{-\infty}-\beta_1\,[u,\sigma]_{-\infty}=\gamma\phi_-(0), [u,\omega]_{-\infty}-\overline{\beta}_1\,[u,\sigma]_{-\infty}=\gamma\phi_+(0),$$

$$[u,\sigma]_{\infty}-\beta_2\,[u,\omega]_{\infty}=0, \tag{3.2}$$

where $\mathcal{W}^1_2$ is the Sobolev space, and $\gamma^2 := 2Im\beta_1$, $\gamma > 0$. Then the following result holds.

Theorem 3.1. *The operator* $\mathbf{T}^{-}_{\beta_1\beta_2}$ *is self-adjoint in* $\mathbf{H}$ *and is a self-adjoint dilation of the maximal dissipative operator* $T^{-}_{\beta_1\beta_2}$.

Proof. Consider the vectors $\mathbf{f},\mathbf{g}\in\mathfrak{D}(\mathbf{T}^{-}_{\beta_1\beta_2})$, $\mathbf{f}=\langle\phi_-,u,\phi_+\rangle$ and $\mathbf{g}=\langle\psi_-,v,\psi_+\rangle$. Using (3.1) and integrating by parts one obtains

$$(\mathbf{T}^{-}_{\beta_1\beta_2}\mathbf{f},\mathbf{g})_{\mathbf{H}}=\int_{-\infty}^{0} i\phi'_-\bar{\psi}_-d\xi+(T_{\max}u,v)_H+\int_0^{\infty} i\phi'_+\bar{\psi}_+d\varsigma=i\phi_-(0)\overline{\psi_-(0)}$$

$$-i\phi_+(0)\overline{\psi_+(0)}+[u,v]_\infty-[u,v]_{-\infty}+(\mathbf{f},\mathbf{T}^{-}_{\beta_1\beta_2})_{\mathbf{H}}. \tag{3.3}$$

Taking the boundary conditions (3.2) into account for the components of the vectors $\mathbf{f}$, $\mathbf{g}$ and using relation (2.6), with a direct calculation, we see that $i\phi_-(0)\bar{\psi}_-(0)-i\phi_+(0)\bar{\psi}_+(0)+[u,v]_\infty-[u,v]_{-\infty}=0$. This shows that $\mathbf{T}^{-}_{\beta_1\beta_2}$ is symmetric.

For proving the self-adjointness one should prove that $(\mathbf{T}^{-}_{\beta_1\beta_2})^*\subseteq\mathbf{T}^{-}_{\beta_1\beta_2}$. Consider the vector $\mathbf{g}=\langle\psi_-,v,\psi_+\rangle\in\mathfrak{D}((\mathbf{T}^{-}_{\beta_1\beta_2})^*)$ so that $(\mathbf{T}^{-}_{\beta_1\beta_2})^*\mathbf{g}=\mathbf{g}^*=\langle\psi^*_-,v^*,\psi^*_+\rangle\in\mathbf{H}$ and

$$(\mathbf{T}^{-}_{\beta_1\beta_2}\mathbf{f},\mathbf{g})_{\mathbf{H}}=(\mathbf{f},\mathbf{g}^*)_{\mathbf{H}},\ \forall\mathbf{f}\in\mathfrak{D}(\mathbf{T}^{-}_{\beta_1\beta_2}). \tag{3.4}$$

In (3.4), if we chose vectors with suitable components for $\mathbf{f}\in\mathfrak{D}(\mathbf{T}^{-}_{\beta_1\beta_2})$ we can show that $\psi_-\in\mathcal{W}^1_2(-\infty,0)$, $\psi_+\in\mathcal{W}^1_2(0,\infty)$, $v\in\mathfrak{D}_{\max}$ and $\mathbf{g}^*=\mathbf{T}\mathbf{g}$, where the operation $\mathbf{T}$ is defined by (3.1). Hence for arbitrary $\mathbf{f}\in\mathfrak{D}(\mathbf{T}^{-}_{\beta_1\beta_2})$, (3.4) takes the form $(\mathbf{T}\mathbf{f},\mathbf{g})_{\mathbf{H}}=(\mathbf{f},\mathbf{T}\mathbf{g})_{\mathbf{H}}$. Therefore for all $\mathbf{f}=\langle\phi_-,u,\phi_+\rangle\in\mathfrak{D}(\mathbf{T}^{-}_{\beta_1\beta_2})$ the sum of the integrated terms in the bilinear form $(\mathbf{T}\mathbf{f},\mathbf{g})_{\mathbf{H}}$ must be equal to zero:

$$i\phi_-(0)\overline{\psi_-(0)}-i\phi_+(0)\overline{\psi_+(0)}+[u,v]_\infty-[u,v]_{-\infty}=0. \tag{3.5}$$

On the other hand, using the boundary conditions (3.2) for $[u,\sigma]_{-\infty}$ and $[u,\omega]_{-\infty}$ we obtain that

$$[u,\sigma]_{-\infty}=\frac{i}{\gamma}(\phi_+(0)-\phi_-(0)),\ \ [u,\omega]_{-\infty}=\gamma\phi_-(0)+\frac{i\beta_1}{\gamma}(\phi_+(0)-\phi_-(0)). \tag{3.6}$$

Therefore, from (3.6), we find that (3.5) is equivalent to the equality

$$i\phi_-(0)\overline{\psi_-(0)}-i\phi_+(0)\overline{\psi_+(0)}=[u,v]_{-\infty}-[u,v]_\infty$$

$$=\frac{i}{\gamma}(\phi_+(0)-\phi_-(0))\,[\overline{v},\omega]_{-\infty}-[\gamma\phi_-(0)+\frac{i\beta_1}{\gamma^2}(\phi_+(0)-\phi_-(0))]\,[\overline{v},\sigma]_{-\infty}$$

$$-[u,\sigma]_\infty\,[\overline{v},\omega]_\infty+[u,\omega]_\infty\,[\overline{v},\sigma]_\infty=\frac{i}{\gamma}(\phi_+(0)-\phi_-(0))\,[\overline{v},\omega]_{-\infty}$$

$$-[\gamma\phi_-(0)+\frac{i\beta_1}{\gamma^2}(\phi_+(0)-\phi_-(0))]\,[\overline{v},\sigma]_{-\infty}-([\overline{v},\sigma]_\infty-\beta_2\,[\overline{v},\omega]_\infty)[u,\omega]_\infty.$$

Since the values $\phi_{\pm}(0)$ can be arbitrary complex numbers, a comparison of the coefficients of $\phi_{\pm}(0)$ on the left and right of the last equality gives us that the vector $\mathbf{g} = \langle\psi_-, v, \psi_+\rangle$ satisfies the boundary conditions $[v,\omega]_{-\infty} - \beta_1 [v,\sigma]_{-\infty} = \gamma\psi_-(0)$, $[v,\omega]_{-\infty} - \overline{\beta}_1 [v,\sigma]_{-\infty} = \gamma\psi_+(0)$, $[v,\sigma]_{\infty} - \beta_2 [v,\omega]_{\infty} = 0$. Consequently, establishing the inclusion $(\mathbf{T}^-_{\beta_1\beta_2})^* \subseteq \mathbf{T}^-_{\beta_1\beta_2}$ we have proved that $\mathbf{T}^-_{\beta_1\beta_2} = (\mathbf{T}^-_{\beta_1\beta_2})^*$.

It is well known that the self-adjoint operator $\mathbf{T}^-_{\beta_1\beta_2}$ generates a unitary group $\mathcal{V}^-(r) = \exp[i\mathbf{T}^-_{\beta_1\beta_2} r]$ $(r \in \mathbb{R})$ in $\mathbf{H}$. Let us denote by $\mathcal{P} : \mathbf{H} \to H$ and $\mathcal{P}_1 : H \to \mathbf{H}$ the mappings acting according to the formulae $\mathcal{P} : \langle\phi_-, u, \phi_+\rangle \to u$ and $\mathcal{P}_1 : u \to \langle 0, u, 0\rangle$. The family $\{\mathcal{Y}_r\}$ $(r \geq 0)$ of operators, where $\mathcal{Y}_r = \mathcal{P}\mathcal{V}^-(r)\mathcal{P}_1$ $(r \geq 0)$ is a strongly continuous semigroup of completely nonunitary contractions on H. Let us denote the generator of this semigroup by $A_{\beta_1\beta_2}$ such that $A_{\beta_1\beta_2} u = \lim_{r\to+0}[(ir)^{-1}(\mathcal{Y}_r u - u)]$. The domain of $A_{\beta_1\beta_2}$ consists of all vectors for which the limit exists. It is known that the operator $A_{\beta_1\beta_2}$ is maximal dissipative. Further the operator $\mathbf{T}^-_{\beta_1\beta_2}$ is called the self-adjoint dilation of $A_{\beta_1\beta_2}$ [9]. We shall show that $A_{\beta_1\beta_2} = T^-_{\beta_1\beta_2}$, and this will prove that $\mathbf{T}^-_{\beta_1\beta_2}$ is a self-adjoint dilation of $T_{\beta_1\beta_2}$. For the last purpose, we shall verify the equality [9, 11]

$$\mathcal{P}(\mathbf{T}^-_{\beta_1\beta_2} - \lambda I)^{-1}\mathcal{P}_1 u = (T^-_{\beta_1\beta_2} - \lambda I)^{-1} u, \ u \in H, \ Im\lambda < 0. \tag{3.7}$$

Let us set $(\mathbf{T}^-_{\beta_1\beta_2} - \lambda I)^{-1}\mathcal{P}_1 u = \mathbf{g} = \langle\psi_-, v, \psi_+\rangle$. This gives us $(\mathbf{T}^-_{\beta_1\beta_2} - \lambda I)\mathbf{g} = \mathcal{P}_1 u$, and consequently $T^-_{\beta_1\beta_2} v - \lambda v = u$, $\psi_-(\xi) = \psi_-(0)e^{-i\lambda\xi}$ and $\psi_+(\varsigma) = \psi_+(0)e^{-i\lambda\varsigma}$. Since the vector $\mathbf{g}$ belongs to $\mathfrak{D}(\mathbf{T}^-_{\beta_1\beta_2})$, $\psi_- \in \mathcal{L}^2(-\infty, 0)$ and hence $\psi_-(0) = 0$. Consequently, v satisfies the boundary condition $[v,\omega]_{-\infty} - \beta_1 [v,\sigma]_{-\infty} = 0$, $[v,\sigma]_{\infty} - \beta_2[v,\omega]_{\infty} = 0$. Therefore, $v \in \mathfrak{D}(T^-_{\beta_1\beta_2})$, and since a point λ with $Im\lambda < 0$ cannot be an eigenvalue of a dissipative operator, it follows that $v = (T^-_{\beta_1\beta_2} - \lambda I)^{-1}u$. It is better to remark that $\psi_+(0)$ is found from the formula $\psi_+(0) = \gamma^{-1}\left([v,\omega]_{-\infty} - \overline{\beta}_1 [v,\sigma]_{-\infty}\right)$. Then

$$(\mathbf{T}^-_{\beta_1\beta_2} - \lambda I)^{-1}\mathcal{P}_1 u$$

$$= \left\langle 0, (T^-_{\beta_1\beta_2} - \lambda I)^{-1}u, \gamma^{-1}\left([v,\omega]_{-\infty} - \overline{\beta}_1 [v,\sigma]_{-\infty}\right) e^{-i\lambda\varsigma}\right\rangle$$

for $u \in H$ and $Im\lambda < 0$. On applying the mapping $\mathcal{P}$, we obtain (3.7). On the other side, using (3.7) one can arrive at

$$\begin{aligned}(T^{-}_{\beta_1\beta_2} - \lambda I)^{-1} &= \mathcal{P}(\mathbf{T}^{-}_{\beta_1\beta_2} - \lambda I)^{-1}\mathcal{P}_1 = -i\mathcal{P}\int_0^\infty \mathcal{V}^{-}(r)e^{-i\lambda r}dr\mathcal{P}_1 \\ &= -i\int_0^\infty \mathcal{Y}_r e^{-i\lambda r}dr = (A_{\beta_1\beta_2} - \lambda I)^{-1},\ Im\lambda < 0,\end{aligned}$$

from which it is clear that $T^{-}_{\beta_1\beta_2} = A_{\beta_1\beta_2}$. Theorem 3.1. is proved. $\square$
For the case $Im\beta_1 = 0$ or $\beta_1 = 0$ and $Im\beta_2 > 0$, that is dissipative at ∞, we shall construct a self-adjoint dilation of the dissipative operator $T^{+}_{\beta_1\beta_2}$. So in the space $\mathbf{H}$ we consider the operator $\mathbf{T}^{+}_{\beta_1\beta_2}$ generated by the expression (3.1) on the set $\mathfrak{D}(\mathbf{T}^{+}_{\beta_1\beta_2})$ of vectors $\langle\phi_-, u, \phi_+\rangle$ satisfying the conditions: $\phi_- \in \mathcal{W}^1_2(-\infty, 0)$, $\phi_+ \in \mathcal{W}^1_2(0,\infty)$, $u \in \mathfrak{D}_{\max}$,

$$[u,\omega]_{-\infty} - \beta_1 [u,\sigma]_{-\infty} = 0, [u,\sigma]_\infty - \beta_2 [u,\omega]_\infty = \delta\phi_-(0),$$

$$[u,\sigma]_\infty - \overline{\beta}_2 [u,\omega]_\infty = \delta\phi_+(0), \tag{3.8}$$

where $\delta^2 := 2Im\beta_2$, $\delta > 0$.

We may prove the next result similar to Theorem 3.1.

Theorem 3.2. *The operator $\mathbf{T}^{+}_{\beta_1\beta_2}$ is self-adjoint in $\mathbf{H}$ and is a self-adjoint dilation on the maximal dissipative operator $T^{+}_{\beta_1\beta_2}$.*

4. Scattering theory of dilations, functional model and completeness of root vectors of dissipative operators

In order to apply the Lax-Phillips scattering theory [8] let us consider the unitary group $\mathcal{V}^{\pm}(r) = \exp[i\mathbf{T}^{\pm}_{\beta_1\beta_2}r]$ $(r \in \mathbb{R})$ with the incoming and outgoing subspaces $\mathbf{D}_{in} = \langle\mathcal{L}^2(-\infty,0), 0, 0\rangle$ and $\mathbf{D}_{out} = \langle 0, 0, \mathcal{L}^2(0,\infty)\rangle$ which satisfy the following properties

(**1**) $\mathcal{V}^{\pm}(r)\mathbf{D}_{in} \subset \mathbf{D}_{in}$, $r \le 0$ and $\mathcal{V}^{\pm}(r)\mathbf{D}_{out} \subset \mathbf{D}_{out}$, $r \ge 0$;

(**2**) $\bigcap_{r\le 0} \mathcal{V}^{\pm}(r)\mathbf{D}_{in} = \bigcap_{r\ge 0} \mathcal{V}^{\pm}(r)\mathbf{D}_{out} = \{0\}$;

(**3**) $\overline{\bigcup_{r\ge 0} \mathcal{V}^{\pm}(r)\mathbf{D}_{in}} = \overline{\bigcup_{r\le 0} \mathcal{V}^{\pm}(r)\mathbf{D}_{out}} = \mathbf{H}$;

(**4**) $\mathbf{D}_{in} \perp \mathbf{D}_{out}$.

Property (**4**) is obvious. To prove the property (**1**) for $\mathbf{D}_{out}$ (the proof for $\mathbf{D}_{in}$ is similar), we set $\mathcal{R}^{\pm}_\lambda = (\mathbf{T}^{\pm}_{\beta_1\beta_2} - \lambda I)^{-1}$, for all λ with $Im\lambda < 0$

and for any $\mathbf{f} = \langle 0, 0, \phi_+ \rangle \in \mathbf{D}_{out}$ we have

$$\mathcal{R}_\lambda^{\pm}\mathbf{f} = \langle 0, 0, -ie^{-i\lambda\varsigma} \int_0^{\varsigma} e^{-i\lambda s}\phi_+(s)ds.$$

This gives that $\mathcal{R}_\lambda^{\pm}\mathbf{f} \in \mathbf{D}_{out}$. Hence for $\mathbf{g} \perp \mathbf{D}_{out}$, we find

$$0 = \left(\mathcal{R}_\lambda^{\pm}\mathbf{f}, \mathbf{g}\right)_{\mathbf{H}} = -i\int_0^{\infty} e^{-i\lambda r}\left(\mathcal{V}^{\pm}(r)\mathbf{f}, \mathbf{g}\right)_{\mathbf{H}} d\lambda,\ Im\lambda < 0,$$

which implies that $(\mathcal{V}^{\pm}(r)\mathbf{f}, \mathbf{g})_{\mathbf{H}} = 0$ for all $r \geq 0$. Hence $\mathcal{V}^{\pm}(r)\mathbf{D}_{out} \subset \mathbf{D}_{out}$ for $r \geq 0$, and thus property (**1**) has been proved.

To prove the property (**2**) for $\mathbf{D}_{out}$ (the proof for $\mathbf{D}_{in}$ is similar), we consider the linear mappings $\mathcal{P}^+ : \mathbf{H} \to \mathcal{L}^2(0, \infty)$ and $\mathcal{P}_1^+ : \mathcal{L}^2(0, \infty) \to \mathbf{D}_{out}$ defined by $\mathcal{P}^+ : \langle \phi_-, u, \phi_+ \rangle \to \phi_+$ and $\mathcal{P}_1^+ : \phi \to \langle 0, 0, \phi \rangle$, respectively. It is better to note that the semigroup of isometries $\mathcal{Z}_r = \mathcal{P}^+\mathcal{V}^-(r)\mathcal{P}_1^+$, $r \geq 0$ is a one-sided shift in $\mathcal{L}^2(0, \infty)$. Indeed, the generator of the semigroup of the one-sided shift $\mathcal{X}_r$ in $\mathcal{L}^2(0, \infty)$ is the differential operator $i(\frac{d}{d\varsigma})$ with boundary condition $\phi(0) = 0$. Besides, the generator A of the semigroup of isometries $\mathcal{Z}_r$, $r \geq 0$, is the operator

$$A\phi = \mathcal{P}^+\mathbf{T}_{\beta_1\beta_2}^{-}\mathcal{P}_1^+\mathbf{f} = \mathcal{P}^+\mathbf{T}_{\beta_1\beta_2}^{-}\langle 0, 0, \phi \rangle = \mathcal{P}^+\langle 0, 0, i\frac{d\phi}{d\varsigma} \rangle = i\frac{d\phi}{d\varsigma},$$

where $\phi \in \mathcal{W}_2^1(0, \infty)$ and $\phi(0) = 0$. Since a semigroup is uniquely determined by its generator, it follows that $\mathcal{Z}_r = \mathcal{X}_r$, and hence

$$\bigcap_{r\geq 0} \mathcal{V}^-(r)\mathbf{D}_{out} = \langle 0, 0, \bigcap_{r\geq 0} \mathcal{X}_r\mathcal{L}^2(0, \infty) \rangle = \{0\},$$

which proves (the proof for $\mathcal{V}^+(r)$ is similar) the property (**2**).

We should remind that a maximal dissipative operator $\mathcal{S}$ (with domain $\mathfrak{D}(\mathcal{S})$) acting in a Hilbert space $\mathcal{H}$ is called *totally nonself-adjoint* (or *pure*) if there are no nonzero subspaces $\mathcal{M} \subseteq \mathfrak{D}(\mathcal{S})$ of $\mathcal{H}$ such that $\mathcal{S}$ induces a self-adjoint operator in $\mathcal{M}$.

Lemma 4.1. *The operator $T_{\beta_1\beta_2}^{\pm}$ is totally nonself-adjoint (pure).*

Proof. Let $H' \subset H$ be a nontrivial subspace in which $T_{\beta_1\beta_2}^{-}$ induces a self-adjoint operator T' with domain $\mathfrak{D}(T') = H' \cap \mathfrak{D}(T_{\beta_1\beta_2}^{-})$ in a $H' \subset H$ (the proof for $T_{\beta_1\beta_2}^{+}$ is similar). For $z \in \mathfrak{D}(T')$ one can conclude that $z \in \mathfrak{D}(T'^*)$ and $[z, \omega]_{-\infty} - \beta_1 [z, \sigma]_{-\infty} = 0$, $[z, \omega]_{-\infty} - \overline{\beta}_1 [z, \sigma]_{-\infty} = 0$, $[z, \sigma]_{\infty} - \beta_2 [z, \omega]_{\infty} = 0$. From this, we have $[z, \sigma]_{-\infty} = 0$, where $z(\lambda)$ is the eigenvector of the operator $T_{\beta_1\beta_2}^{-}$ that lies in H' and thus it is also an eigenvector of T'. From the boundary condition $[z,\omega]_{-\infty} - \beta_1 [z,\sigma]_{-\infty} = 0$, we obtain $[z,\omega]_{-\infty} = 0$ and $z(\lambda) = 0$. Since all solutions of (2.1) are from $\ell^2_\Omega(\mathbb{Z};E)$, it is obtained that the resolvent $\mathcal{R}_\lambda(T^-_{\beta_1\beta_2})$ of the operator $T_{\beta_1\beta_2}$ is a Hilbert-Schmidt operator and hence the spectrum of $T^-_{\beta_1\beta_2}$ is purely discrete. Hence, by the theorem on expansion in eigenvectors of the self-adjoint operator T', we have $H' = \{0\}$, i.e. the operator $T^-_{\beta_1\beta_2}$ is pure. This proves the lemma. □

According to the Lax-Phillips scattering theory, one can define the scattering matrix with the help of spectral representations. Using this construction, we will also prove property (**3**) of the incoming and outgoing subspaces.

Consider the spaces

$$\mathbf{H}^{\pm}_{-} = \overline{\bigcup_{r\geq 0} \mathcal{V}^{\pm}(r)\mathbf{D}_{in}},\ \ \mathbf{H}^{\pm}_{+} = \overline{\bigcup_{r\leq 0} \mathcal{V}^{\pm}(r)\mathbf{D}_{out}}.$$

Lemma 4.2. *The equality* $\mathbf{H}^{\pm}_{-} + \mathbf{H}^{\pm}_{+} = \mathbf{H}$ *holds.*

Proof. One can see that the subspace $\mathbf{H}'_{\pm} = \mathbf{H} \ominus \left(\mathbf{H}^{\pm}_{-} + \mathbf{H}^{\pm}_{+}\right)$ is invariant relative to the group $\{\mathcal{V}^{\pm}(r)\}$. Further we can consider the space $\mathbf{H}'_{\pm}$ as $\mathbf{H}'_{\pm} = \left\langle 0, H'_{\pm}, 0\right\rangle$, where $H'_{\pm}$ is a subspace in H. Hence, if the subspace $\mathbf{H}'_{\pm}$ (and hence also $H'_{\pm}$) were nontrivial, then the unitary group $\{\mathcal{V}^{\pm}(r)\}$, restricted to this subspace, would be a unitary part of the group $\{\mathcal{V}^{\pm}(r)\}$, and hence the restriction $T^{\pm\prime}_{\beta_1\beta_2}$ of $T^{\pm}_{\beta_1\beta_2}$ to $H'_{\pm}$ would be a self-adjoint operator in $H'_{\pm}$. We know that the operator $T^{\pm}_{\beta_1\beta_2}$ is completely nonself-adjoint. Hence it follows that $H'_{\pm} = \{0\}$, i.e. $\mathbf{H}'_{\pm} = \{0\}$. The lemma is proved. □

Consider the solutions $\varphi(\lambda)$ and $\psi(\lambda)$ of the system (2.1) satisfying the conditions

$$[\varphi,\sigma]_{-\infty} = -1,\ \ [\varphi,\omega]_{-\infty} = 0,\ \ [\psi,\sigma]_{-\infty} = 0,\ \ [\psi,\omega]_{-\infty} = 1. \tag{4.1}$$

According to the Weyl's analysis, the function $M_{\infty\beta_2}(\lambda)$ of the self-adjoint operator $T^-_{\infty\beta_2}$ called the Weyl function is parametrized from the condition $[\psi + M_{\infty\beta_2}\varphi,\sigma]_{\infty} - \beta_2 [\psi + M_{\infty\beta_2}\varphi,\omega]_{\infty} = 0$ as

$$M_{\infty\beta_2}(\lambda) = -\frac{[\psi,\sigma]_{\infty} - \beta_2 [\psi,\omega]_{\infty}}{[\varphi,\sigma]_{\infty} - \beta_2 [\varphi,\omega]_{\infty}}. \tag{4.2}$$

(4.2) shows that $M_{\infty\beta_2}(\lambda)$ is a meromorphic function on the complex plane $\mathbb{C}$ with a countable number of poles on the real axis, and these poles coincide with the eigenvalues of the self-adjoint operator $T_{\infty\beta_2}$.

Further $M_{\infty\beta_2}(\lambda)$ has the following properties: $Im\lambda Im M_{\infty\beta_2}(\lambda) > 0$ for $Im\lambda \neq 0$ and $M_{\infty\beta_2}(\bar{\lambda}) = \overline{M_{\infty\beta_2}(\lambda)}$ for complex λ with the exception of the real poles of $M_{\infty\beta_2}(\lambda)$.

Let us adopt the following notation $\vartheta(\lambda) = \psi(\lambda) + M_{\infty\beta_2}(\lambda)\varphi(\lambda)$ and consider the function

$$\Theta^-_{\beta_1\beta_2}(\lambda) = \frac{M_{\infty\beta_2}(\lambda) - \beta_1}{M_{\infty\beta_2}(\lambda) - \overline{\beta}_1}, \tag{4.3}$$

and the vector

$$\mathcal{F}^-_{\lambda}(\xi,\varsigma) = \langle e^{-i\lambda\xi}, (M_{\infty\beta_2}(\lambda) - \beta_1)^{-1}\gamma\vartheta(\lambda), \overline{\Theta^-_{\beta_1\beta_2}(\lambda)}e^{-i\lambda\varsigma}\rangle.$$

The vector $\mathcal{F}^-_{\lambda}(\xi,\varsigma)$ satisfies the equation $\mathbf{T}\mathcal{F} = \lambda\mathcal{F}$ and the boundary conditions (3.2) for real λ, but it does not belong to the space $\mathbf{H}$ for real λ.

Let us define the transformation $\Psi_- : \mathbf{f} \to \tilde{\mathbf{f}}_-(\lambda)$ as $(\Psi_-\mathbf{f})(\lambda) := \tilde{\mathbf{f}}_-(\lambda) := \frac{1}{\sqrt{2\pi}}(\mathbf{f}, \mathcal{F}^-_{\lambda})_{\mathbf{H}}$ on the vector $\mathbf{f} = \langle \phi_-, u, \phi_+\rangle$ in which ϕ_-, ϕ_+ are smooth compactly supported functions, and $u = \{u_k\}(k \in \mathbb{Z})$ is a finite sequence.

Lemma 4.3. *The transformation Ψ_- maps $\mathbf{H}_-$ isometrically onto $\mathcal{L}^2(\mathbb{R})$. For all vectors $\mathbf{f}, \mathbf{g} \in \mathbf{H}^-_-$ the Parseval equality and the inversion formula hold:*

$$(\mathbf{f},\mathbf{g})_{\mathbf{H}} = (\tilde{\mathbf{f}}_-, \tilde{\mathbf{g}}_-)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \tilde{\mathbf{f}}_-(\lambda)\overline{\tilde{\mathbf{g}}_-(\lambda)}d\lambda, \mathbf{f} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\mathbf{f}}_-(\lambda)\mathcal{F}^-_{\lambda}d\lambda,$$

where $\tilde{\mathbf{f}}_-(\lambda) = (\Psi_- f)(\lambda)$ *and* $\tilde{\mathbf{g}}_-(\lambda) = (\Psi_-\mathbf{g})(\lambda)$.

Proof. For the vectors $\mathbf{f} = \langle\phi_-, 0, 0\rangle$, $\mathbf{g} = \langle\psi_-, 0, 0\rangle \in \mathbf{D}_{in}$, we have

$$\tilde{\mathbf{f}}_-(\lambda) := \frac{1}{\sqrt{2\pi}}(\mathbf{f}, \mathcal{F}^-_{\lambda})_{\mathbf{H}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\phi_-(\xi)e^{i\lambda\xi}d\xi \in \mathcal{H}^2_-.$$

Using the Parseval equality for Fourier integrals, we get that

$$(\mathbf{f},\mathbf{g})_{\mathbf{H}} = \int_{-\infty}^{0}\phi_-(\xi)\overline{\psi_-(\xi)}d\xi = \int_{-\infty}^{\infty}\tilde{\mathbf{f}}_-(\lambda)\overline{\tilde{\mathbf{g}}_-(\lambda)}d\lambda = (\Psi_-\mathbf{f}, \Psi_-\mathbf{g})_{\mathcal{L}^2}.$$

We should note that $\mathcal{H}^2_{\pm}$ denotes the Hardy classes in $\mathcal{L}^2(\mathbb{R})$ that consists of the functions analytically extendable to the upper and lower half-planes, respectively.

Now consider the dense set $\mathbf{H}'_-$ in $\mathbf{H}^-_-$ consisting of vectors $\mathbf{f} \in \mathbf{H}'_-$ such that $\mathbf{f} = \mathcal{V}^-(r)\mathbf{f}_0$ where $\mathbf{f}_0 = \langle\phi_-, 0, 0\rangle$, $\phi_- \in C_0^{\infty}(-\infty, 0)$. In this case, if $\mathbf{f}, \mathbf{g} \in \mathbf{H}'_-$, then for $r > r_{\mathbf{f}}$ and $r > r_{\mathbf{g}}$, where $r = r_{\mathbf{f}}$ is a non-negative number (depending on $\mathbf{f}$), we have $\mathcal{V}^-(-r)\mathbf{f}, \mathcal{V}^-(-r)\mathbf{g} \in \mathbf{D}_{in}$ and, moreover, the first components of these vectors belong to $C_0^\infty(-\infty, 0)$. Since the operators $\mathcal{V}^-(r)$ ($r \in \mathbb{R}$) are unitary and the equality $\Psi_- \mathcal{V}^-(r)\mathbf{f} = (\mathcal{V}^-(r)\mathbf{f}, \digamma_\lambda^-)_\mathbf{H} = e^{i\lambda r}(\mathbf{f}, \digamma_\lambda^-)_\mathbf{H} = e^{i\lambda r}\Psi_-\mathbf{f}$ holds, we have

$$(\mathbf{f},\mathbf{g})_\mathbf{H} = (\mathcal{V}^-(-r)\mathbf{f}, \mathcal{V}^-(-r)\mathbf{g})_\mathbf{H} = (\Psi_-\mathcal{V}^-(-r)\mathbf{f}, \Psi_-\mathcal{V}^-(-r)\mathbf{g})_{\mathcal{L}^2}$$

$$= (e^{-i\lambda r}\Psi_-\mathbf{f}, e^{-i\lambda r}\Psi_-\mathbf{g})_{\mathcal{L}^2} = (\Psi_-\mathbf{f}, \Psi_-\mathbf{g})_{\mathcal{L}^2}. \tag{4.4}$$

Taking closure in (4.4), we obtain the Parseval equality for the whole space $\mathbf{H}_-^-$. If all integrals in it are understood as limits in the mean of integrals over finite intervals, then the inversion formula follows from the Parseval equality. Finally we have,

$$\Psi_-\mathbf{H}_-^- = \overline{\bigcup_{r\geq 0} \Psi_-\mathcal{V}^-(r)\mathbf{D}_{in}} = \overline{\bigcup_{r\geq 0} e^{-i\lambda r}\mathcal{H}_-^2} = \mathcal{L}^2(\mathbb{R}),$$

i.e. Ψ_- maps $\mathbf{H}_-^-$ onto the whole of $\mathcal{L}^2(\mathbb{R})$. The lemma is proved. □

Consider the vector

$$\digamma_\lambda^+(\xi,\varsigma) = \langle \Theta_{\beta_1\beta_2}^-(\lambda)e^{-i\lambda\xi}, (M_{\infty\beta_2}(\lambda) - \overline{\beta}_1)^{-1}\gamma\theta(\lambda), e^{-i\lambda\varsigma}\rangle,$$

which do not belong to the space $\mathbf{H}$ for real λ, but satisfies the equation $\mathbf{T}\digamma = \lambda\digamma$ ($\lambda \in \mathbb{R}$) and the boundary conditions (3.2).

We define the transformation $\Psi_+ : \mathbf{f} \to \tilde{\mathbf{f}}_+(\lambda)$ as $(\Psi_+\mathbf{f})(\lambda) := \tilde{\mathbf{f}}_+(\lambda) := \frac{1}{\sqrt{2\pi}}(\mathbf{f}, \digamma_\lambda^+)_\mathbf{H}$, where $\mathbf{f} = \langle \phi_-, u, \phi_+\rangle$, and ϕ_-, ϕ_+ are compactly supported smooth functions, and $u = \{u_k\}$ ($k \in \mathbb{Z}$) is a finite sequence. The proof of the next result is analogous to that of Lemma 4.3.

Lemma 4.4. *The transformation Ψ_+ maps $\mathbf{H}_+^-$ isometrically onto $\mathcal{L}^2(\mathbb{R})$ and for all vectors $\mathbf{f},\mathbf{g} \in \mathbf{H}_+^-$, the Parseval equality and the inversion formula hold as follows:*

$$(\mathbf{f},\mathbf{g})_\mathbf{H} = (\tilde{\mathbf{f}}_+, \tilde{\mathbf{g}}_+)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \tilde{\mathbf{f}}_+(\lambda)\overline{\tilde{\mathbf{g}}_+(\lambda)}d\lambda,\ \mathbf{f} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\mathbf{f}}_+(\lambda)\digamma_\lambda^+ d\lambda,$$

where $\tilde{\mathbf{f}}_+(\lambda) = (\Psi_+\mathbf{f})(\lambda)$ and $\tilde{\mathbf{g}}_+(\lambda) = (\Psi_+\mathbf{g})(\lambda)$.

From (4.3) one can see that the function $\Theta_{\beta_1\beta_2}^-$ satisfies $\left|\Theta_{\beta_1\beta_2}^-(\lambda)\right| = 1$ for all $\lambda \in \mathbb{R}$. Hence we have

$$\digamma_\lambda^- = \overline{\Theta_{\beta_1\beta_2}^-(\lambda)}\digamma_\lambda^+ \ (\lambda \in \mathbb{R}). \tag{4.5}$$

Further, from Lemmas 4.3 and 4.4 we obtain that $\mathbf{H}_-^- = \mathbf{H}_+^-$ and together with Lemma 4.2 we have $\mathbf{H} = \mathbf{H}_-^- = \mathbf{H}_+^-$. This proves the property (**3**) for $\mathcal{V}^-(r)$.

Thus, Ψ_- is the incoming spectral representation for the group $\{\mathcal{V}^-(r)\}$. Because the transformation Ψ_- maps isometrically onto $\mathcal{L}^2(\mathbb{R})$ with the subspace $\mathbf{D}_{in}$ mapped onto $\mathcal{H}_-^2$ and the operators $\mathcal{V}^-(r)$ passing into the operators of multiplication by $e^{i\lambda r}$. Similarly Ψ_+ is the outgoing spectral representation for $\{\mathcal{V}^-(r)\}$. It follows from (4.5) that the passage from the Ψ_+-representation of an vector $\mathbf{f} \in \mathbf{H}$ to its Ψ_--representation is realized by multiplication by the function $\Theta^-_{\beta_1\beta_2} : \tilde{\mathbf{f}}_-(\lambda) = \Theta^-_{\beta_1\beta_2}(\lambda)\tilde{\mathbf{f}}_+(\lambda)$. Lax and Phillips showed that [8] the scattering function (matrix) of the group $\{\mathcal{V}^-(r)\}$ with respect to the subspaces $\mathbf{D}_{in}$ and $\mathbf{D}_{out}$ is the coefficient by which the Ψ_--representation of a vector $\mathbf{f} \in \mathbf{H}$ must be multiplied in order to get the corresponding Ψ_+-representation: $\tilde{\mathbf{f}}_+(\lambda) = \overline{\Theta^-_{\beta_1\beta_2}(\lambda)}\tilde{\mathbf{f}}_-(\lambda)$. According to [8], we have now proved the following result.

Theorem 4.5. *The function $\overline{\Theta}^-_{\beta_1\beta_2}$ is the scattering function (matrix) of the unitary group $\{\mathcal{V}^-(r)\}$ (of the self-adjoint operator $\mathbf{T}^-_{\beta_1\beta_2}$).*

It is known that the subspace $\mathcal{K} = \mathcal{H}_+^2 \ominus \Theta\mathcal{H}_+^2$ is a nontrivial subspace of the Hilbert space $\mathcal{H}_+^2$, where Θ is an arbitrary inner function ([9, 10]) on the upper half-plane. In the subspace $\mathcal{K}$, let us consider the semigroup of the operators $\mathcal{Z}_r$ $(r \geq 0)$ acting according to the formula $\mathcal{Z}_r u = \mathcal{P}[e^{i\lambda r}u]$, $u := u(\lambda) \in \mathcal{K}$, where $\mathcal{P}$ is the orthogonal projection from $\mathcal{H}_+^2$ onto $\mathcal{K}$. Let $\mathcal{A}$ denote the generator of the semigroup $\{\mathcal{Z}_r\}$: $\mathcal{A}u = \lim_{r\to+0}[(ir)^{-1}(\mathcal{Z}_r u - u)]$. The operator $\mathcal{A}$ is a dissipative operator acting in $\mathcal{K}$ and having domain $\mathfrak{D}(\mathcal{A})$ which consists of all functions $u \in \mathcal{K}$ for which that the limit exists. The operator $\mathcal{A}$ is called a *model dissipative operator.* It is better to note that this model dissipative operator, which is associated with the names of Lax and Phillips [8], is a special case of a more general model dissipative operator constructed by Sz.-Nagy and Foiaş [9]. It will be shown that Θ is the characteristic function of the operator $\mathcal{A}$.

Let us consider the space $\mathbf{H} = \mathbf{D}_{in} \oplus \mathcal{M} \oplus \mathbf{D}_{out}$, where $\mathcal{M} = \langle 0, H, 0\rangle$. It was shown that with the help of the unitary transformation Ψ_- the following mappings

$$\mathbf{H} \to \mathcal{L}^2(\mathbb{R}),\ \mathbf{f} \to \tilde{\mathbf{f}}_-(\lambda) = (\Psi_-\mathbf{f})(\lambda),\ \mathbf{D}_{in} \to \mathcal{H}_-^2, \mathbf{D}_{out} \to \Theta^-_{\beta_1\beta_2}\mathcal{H}_+^2,$$

(4.6)

$$\mathcal{M} \to \mathcal{H}_+^2 \ominus \Theta_{\beta_1\beta_2}^- \mathcal{H}_+^2,\ \mathcal{V}^-(r)\mathbf{f} \to (\Psi_- \mathcal{V}^-(r)\Psi_-^{-1}\tilde{\mathbf{f}}_-)(\lambda) = e^{i\lambda r}\tilde{\mathbf{f}}_-(\lambda),$$

hold. These mappings (4.6) allow us to know that our operator $T_{\beta_1\beta_2}^-$ is unitary equivalent to the model dissipative operator with characteristic function $\Theta_{\beta_1\beta_2}^-$. On the other hand it is well known that the characteristic functions of unitary equivalent dissipative operators coincide [9, 11, 12]. Hence we have proved the following theorem.

Theorem 4.6. *The characteristic function of the dissipative operator $T_{\beta_1\beta_2}^-$ coincides with the function $\Theta_{\beta_1\beta_2}^-$ defined by (4.3).*

Consider the solutions $\phi(\lambda)$ and $\chi(\lambda)$ of (2.1) satisfying the conditions

$$[\phi, \sigma]_{-\infty} = \frac{1}{\sqrt{1+\beta_1^2}},\ [\phi, \omega]_{-\infty} = -\frac{\beta_1}{\sqrt{1+\beta_1^2}},$$

$$[\chi, \sigma]_{-\infty} = \frac{\beta_1}{\sqrt{1+\beta_1^2}},\ [\chi, \omega]_{-\infty} = \frac{1}{\sqrt{1+\beta_1^2}}.$$

Let $M_{\beta_1\infty}$ be the Weyl function of the self-adjoint operator $T_{\beta_1\infty}$, which is obtained in terms of the Wronskians of the solutions as follows

$$M_{\beta_1\infty}(\lambda) = -\frac{[\chi, \omega]_\infty}{[\phi, \omega]_\infty}.$$

Let us adopt the following notation:

$$K(\lambda) := \frac{[\varphi, \sigma]_\infty}{[\psi, \omega]_\infty},\ M(\lambda) := M_{\beta_1\infty}(\lambda),$$

$$\Theta^+(\lambda) := \Theta_{\beta_1\beta_2}^+(\lambda) := \frac{M(\lambda)K(\lambda) - \beta_2}{M(\lambda)K(\lambda) - \overline{\beta}_2}. \tag{4.7}$$

Consider the vector

$$\Upsilon_\lambda^-(\xi, \varsigma) = \langle e^{-i\lambda\xi}, \gamma M(\lambda)[(M(\lambda)K(\lambda) - \beta_2)[\chi, \omega]_\infty]^{-1}\phi(\lambda), \overline{\Theta^+(\lambda)}e^{-i\lambda\varsigma}\rangle.$$

Note that the vector $\Upsilon_\lambda^-(\xi, \varsigma)$ does not belong to the $\mathbf{H}$ for $\lambda \in \mathbb{R}$. However, Υ_λ^- satisfies the equation $\mathbf{T}\Upsilon_\lambda^- = \lambda\Upsilon_\lambda^-$ ($\lambda \in \mathbb{R}$) and the boundary conditions (3.8).

Using the vector Υ_λ^-, we define the transformation $\Phi_- : \mathbf{f} \to \tilde{\mathbf{f}}_-(\lambda)$ as $(\Phi_-\mathbf{f})(\lambda) := \tilde{\mathbf{f}}_-(\lambda) := \frac{1}{\sqrt{2\pi}}(\mathbf{f}, \Upsilon_\lambda^-)_{\mathbf{H}}$, where $\mathbf{f} = \langle \phi_-, u, \phi_+ \rangle$, ϕ_-, ϕ_+ are smooth compactly supported functions, and $u = \{u_k\}$ ($k \in \mathbb{Z}$) is a finite sequence. The proof of the next result is similar to that of Lemma 4.2.

Lemma 4.7. *The transformation Φ_- maps $\mathbf{H}_-^+$ isometrically onto $\mathcal{L}^2(\mathbb{R})$. For all vectors $\mathbf{f}, \mathbf{g} \in \mathbf{H}_-^+$ the Parseval equality and the inversion formula hold:*

$$(\mathbf{f}, \mathbf{g})_{\mathbf{H}} = (\tilde{\mathbf{f}}_-, \tilde{\mathbf{g}}_-)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \tilde{\mathbf{f}}_-(\lambda)\overline{\tilde{\mathbf{g}}_-(\lambda)}d\lambda,\ \mathbf{f} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{\mathbf{f}}_-(\lambda)\Upsilon_\lambda^- d\lambda,$$

where $\tilde{\mathbf{f}}_-(\lambda) = (\Phi_-\mathbf{f})(\lambda)$ and $\tilde{\mathbf{g}}_-(\lambda) = (\Phi_-\mathbf{g})(\lambda)$.

Now let

$$\Upsilon_\lambda^+(\xi, \varsigma) = \langle \Theta^+(\lambda)e^{-i\lambda\xi}, \gamma M(\lambda)[(M(\lambda)K(\lambda) - \overline{\beta}_2)[\chi, \omega]_\infty]^{-1}\phi(\lambda), e^{-i\lambda\varsigma}\rangle.$$

It is clear that the vector $\Upsilon_\lambda^+(\xi, \varsigma)$ does not belong to $\mathbf{H}$ for $\lambda \in \mathbb{R}$. However, Υ_λ^+ satisfies the equation $\mathbf{T}\Upsilon_\lambda^+ = \lambda\Upsilon_\lambda^+$ ($\lambda \in \mathbb{R}$) and the boundary conditions (3.8).

With the help of the vector $\Upsilon_\lambda^+(\xi, \varsigma)$, we define the transformation $\Phi_+ : \mathbf{f} \to \tilde{\mathbf{f}}_+(\lambda)$ on vectors $\mathbf{f} = \langle \phi_-, u, \phi_+\rangle$, in which ϕ_-, ϕ_+ are compactly supported smooth functions, and $u = \{u_k\}$ ($k \in \mathbb{Z}$) is a finite sequence by setting $(\Phi_+\mathbf{f})(\lambda) := \tilde{\mathbf{f}}_+(\lambda) := \frac{1}{\sqrt{2\pi}}(\mathbf{f}, \Upsilon_\lambda^+)_{\mathbf{H}}$.

Lemma 4.8. *The transformation Φ_+ maps $\mathbf{H}_+^+$ isometrically onto $\mathcal{L}^2(\mathbb{R})$ and for all vectors $\mathbf{f}, \mathbf{g} \in \mathbf{H}_+^+$, the Parseval equality and the inversion formula hold:*

$$(\mathbf{f}, \mathbf{g})_{\mathbf{H}} = (\tilde{\mathbf{f}}_+, \tilde{\mathbf{g}}_+)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \tilde{\mathbf{f}}_+(\lambda)\overline{\tilde{\mathbf{g}}_+(\lambda)}d\lambda,\ \mathbf{f} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{\mathbf{f}}_+(\lambda)\Upsilon_\lambda^+ d\lambda,$$

where $\tilde{\mathbf{f}}_+(\lambda) = (\Phi_+\mathbf{f})(\lambda)$ and $\tilde{\mathbf{g}}_+(\lambda) = (\Phi_+\mathbf{g})(\lambda)$.

Using (4.7), one can see that the function $\Theta_{\beta_1\beta_2}^+(\lambda)$ satisfies $\left|\Theta_{\beta_1\beta_2}^+(\lambda)\right| = 1$ for $\lambda \in \mathbb{R}$. Therefore we get that

$$\Upsilon_\lambda^- = \overline{\Theta}_{\beta_1\beta_2}^+(\lambda)\Upsilon_\lambda^+,\ \lambda \in \mathbb{R}. \tag{4.8}$$

From Lemmas 4.7 and 4.8, we have $\mathbf{H}_-^+ = \mathbf{H}_+^+$ and therefore together with Lemma 4.2. we obtain that $\mathbf{H} = \mathbf{H}_-^+ = \mathbf{H}_+^+$. From the formula (4.8), it follows that passage from the $\mathcal{F}_-$-representation of a vector $\mathbf{f} \in \mathbf{H}$ to its Φ_+-representation is accomplished as follows: $\tilde{\mathbf{f}}_+(\lambda) = \overline{\Theta}_{\beta_1\beta_2}^+(\lambda)\tilde{\mathbf{f}}_-(\lambda)$. According to [8] we have now proved the next assertion.

Theorem 4.9. *The function $\overline{\Theta}_{\beta_1\beta_2}^+$ is the scattering function (matrix) of the unitary group $\{\mathcal{V}^+(r)\}$ (of the self-adjoint operator $\mathbf{T}_{\beta_1\beta_2}^+$).*

Unitary transformation $\mathcal{F}_-$ gives the following mappings

$$\mathbf{H} \to \mathcal{L}^2(\mathbb{R}),\ \mathbf{f} \to \tilde{\mathbf{f}}_-(\lambda) = (\Phi_-\mathbf{f})(\lambda),\ \mathbf{D}_{in} \to \mathcal{H}^2_-,\ \mathbf{D}_{out} \to \Theta^+_{\beta_1\beta_2}\mathcal{H}^2_+,$$

(4.9)

$$\mathcal{M} \to \mathcal{H}^2_+ \ominus \Theta^+_{\beta_1\beta_2}\mathcal{H}^2_+,\ \mathcal{V}^+(r)\mathbf{f} \to (\Phi_-\mathcal{V}^+(r)\Phi_-^{-1}\tilde{\mathbf{f}}_-)(\lambda) = e^{i\lambda r}\tilde{\mathbf{f}}_-(\lambda).$$

Hence from (4.9), we arrive at the result that the operator $T^+_{\beta_1\beta_2}$ is unitary equivalent to the model dissipative operator with characteristic function $\Theta^+_{\beta_1\beta_2}$. We have thus proved the following theorem.

Theorem 4.10. *The characteristic function of the maximal dissipative operator $T^+_{\beta_1\beta_2}$ coincides with the function $\Theta^+_{\beta_1\beta_2}$ defined by (4.7).*

Characteristic function of a dissipative operator $T^\pm_{\beta_1\beta_2}$ carries complete information about the spectral properties of the dissipative operator [9-12]. This can be done by showing the absence of a singular factor $s^\pm(\lambda)$ of the characteristic function $\Theta^\pm_{\beta_1\beta_2}$ in the factorization $\Theta^\pm_{\beta_1\beta_2}(\lambda) = s^\pm(\lambda)\mathcal{B}^\pm(\lambda)$, where $\mathcal{B}^\pm(\lambda)$ is a Blaschke product. This proves the completeness of the system of eigenvectors and associated vectors (or root vectors) of the dissipative operators $T^\pm_{\beta_1\beta_2}$.

Let $\mathcal{S}$ denote the linear operator in the Hilbert space $\mathcal{H}$ with the domain $\mathfrak{D}(\mathcal{S})$. The complex number λ_0 is called an *eigenvalue* of the operator $\mathcal{S}$ if there exists a nonzero element $u_0 \in \mathfrak{D}(\mathcal{S})$ such that $\mathcal{S}u_0 = \lambda_0 u_0$. Such element u_0 is called the *eigenvector* of the operator $\mathcal{S}$ corresponding to the eigenvalue λ_0. The elements $u_1, u_2, ..., u_k$ are called the *associated vectors* of the eigenvector u_0 if they belong to $\mathfrak{D}(\mathcal{S})$ and $\mathcal{S}u_j = \lambda_0 u_j + u_{j-1}$, $j = 1, 2, ..., k$. The element $u \in \mathfrak{D}(\mathcal{S})$, $u \neq 0$ is called a *root vector* of the operator $\mathcal{S}$ corresponding to the eigenvalue λ_0, if all powers of $\mathcal{S}$ are defined on this element and $(\mathcal{S} - \lambda_0 I)^n u = 0$ for some integer n. The set of all root vectors of $\mathcal{S}$ corresponding to the same eigenvalue λ_0 with the vector $u = 0$ forms a linear set $\mathcal{N}_{\lambda_0}$ and is called the root lineal. The dimension of the lineal $\mathcal{N}_{\lambda_0}$ is called the *algebraic multiplicity* of the eigenvalue λ_0. The root lineal $\mathcal{N}_{\lambda_0}$ coincides with the linear span of all eigenvectors and associated vectors of $\mathcal{S}$ corresponding to the eigenvalue λ_0. Consequently, the completeness of the system of all eigenvectors and associated vectors of $\mathcal{S}$ is equivalent to the completeness of the system of all root vectors of this operator.

Theorem 4.11. *For all values of β_1 with $Im\beta_1 > 0$, except possibly for a single value $\beta_1 = \beta_1^0$, and for fixed β_2 ($Im\beta_2 = 0$ or $\beta_2 = 0$), the characteristic function $\Theta^-_{\beta_1\beta_2}$ of the maximal dissipative operator $T^-_{\beta_1\beta_2}$*

is a Blaschke product and the spectrum of $T^{-}_{\beta_1\beta_2}$ is purely discrete, and belongs to the open upper half plane. The operator $T^{-}_{\beta_1\beta_2}$ $(\beta_1 \neq \beta_1^0)$ has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity, and the system of eigenvectors and associated (or root) vectors of this operator is complete in the space $\ell^2_\Omega(\mathbb{Z};E)$.

Proof. From (4.3), it is clear that $\Theta^{-}_{\beta_1\beta_2}$ is an inner function in the upper half-plane and it is meromorphic in the whole λ-plane. Thus, it can be factored in the form

$$\Theta^{-}_{\beta_1\beta_2}(\lambda) = e^{i\lambda c}\mathcal{B}_{\beta_1\beta_2}(\lambda),\ c = c\,(\beta_1) > 0, \tag{4.10}$$

where $\mathcal{B}_{\beta_1\beta_2}(\lambda)$ is a Blaschke product. Using (4.10) one obtains that

$$|\Theta_{\beta_1\beta_2}(\lambda)| \leq e^{-c(\beta_1)Im\lambda},\ Im\lambda \geq 0. \tag{4.11}$$

On the other hand expressing $M_{\infty\beta_2}(\lambda)$ in terms of $\Theta^{-}_{\beta_1\beta_2}(\lambda)$, we find from (4.3) that

$$M_{\infty\beta_2}(\lambda) = \frac{\overline{\beta}_1\Theta^{-}_{\beta_1\beta_2}(\lambda) - \beta_1}{\Theta^{-}_{\beta_1\beta_2}(\lambda) - 1}. \tag{4.12}$$

Now if $c\,(\beta_1) > 0$ for a given value β_1 $(Im\beta_1 > 0)$, then (4.11) implies that $\lim_{r\to+\infty}\Theta^{-}_{\beta_1\beta_2}(ir) = 0$, and then (4.12) gives us that $\lim_{r\to+\infty} M_{\infty\beta_2}(ir) = \beta_1$. $c(\beta_1)$ can be nonzero at not more then a single point $\beta_1 = \beta_1^0$ (and, further, $\beta_1^0 = \lim_{r\to+\infty} M_{\infty\beta_2}(ir)$) because $M_{\infty\beta_2}(\lambda)$ does not depend on β_1. Therefore the proof is completed. □

The proof of the next result is analogous to that of Theorem 4.11.

Theorem 4.12. *For all values of β_2 with $Im\beta_2 > 0$, except possibly for a single value $\beta_2 = \beta_2^0$, and for fixed β_1 ($Im\beta_1 = 0$ or $\beta_1 = \infty$), the characteristic function $\Theta^{+}_{\beta_1\beta_2}$ of the maximal dissipative operator $T^{+}_{\beta_1\beta_2}$ is a Blaschke product and the spectrum of $T^{+}_{\beta_1\beta_2}$ is purely discrete, and belongs to the open upper half-plane. The operator $T^{+}_{\beta_1\beta_2}$ $(\beta_2 \neq \beta_2^0)$ has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity, and the system of eigenvectors and associated (or root) vectors of this operators is complete in the space $\ell^2_\Omega(\mathbb{Z};E)$.*

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