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ITERATIVE ALGORITHM FOR THE GENERALIZED (P, Q)-REFLEXIVE SOLUTION OF A QUATERNION MATRIX EQUATION WITH j -CONJUGATE OF THE UNKNOWN

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ABSTRACT. In the present paper, we propose an iterative algorithm for solving the generalized (P, Q) -reflexive solution of the quaternion matrix equation $\sum_{l=1}^u A_l X B_l + \sum_{s=1}^v C_s \tilde{X} D_s = F$. By this iterative algorithm, the solvability of the problem can be determined automatically. When the matrix equation is consistent over a generalized (P, Q) -reflexive matrix X , a generalized (P, Q) -reflexive solution can be obtained within finite iteration steps in the absence of roundoff errors, and the least Frobenius norm generalized (P, Q) -reflexive solution can be obtained by choosing an appropriate initial iterative matrix. Furthermore, the optimal approximate generalized (P, Q) -reflexive solution to a given matrix X_0 can be derived by finding the least Frobenius norm generalized (P, Q) -reflexive solution of a new corresponding quaternion matrix equation. Finally, two numerical examples are given to illustrate the efficiency of the proposed methods.

Keywords: Quaternion matrix equation, generalized (P, Q) -reflexive solution, iterative method, optimal approximate solution.

MSC(2010): Primary: 65F10; Secondary: 15B33, 15A24.

1. Introduction

Throughout the paper, the notations $\mathbb{R}^{m \times n}$ and $\mathbb{H}^{m \times n}$ represent the set of all $m \times n$ real matrices and the set of all $m \times n$ matrices over the

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quaternion algebra $\mathbb{H} = \{a_1 + a_2i + a_3j + a_4k \mid i^2 = j^2 = k^2 = ijk = -1, a_1, a_2, a_3, a_4 \in \mathbb{R}\}$. We denote the identity matrix with the appropriate size by I . We use E to denote a matrix with the appropriate size whose elements are all equal to one. We denote the conjugate transpose, the transpose, the conjugate, the trace, the column space, the real part, the $mn \times 1$ vector formed by the vertical concatenation of the respective columns of a matrix A by A^H , A^T , \bar{A} , $\text{tr}(A)$, $R(A)$, $\text{Re}(A)$, $\text{vec}(A)$, respectively. If $A \in \mathbb{H}^{m \times n}$, let $A = A_1 + A_2i + A_3j + A_4k$, where $A_t \in \mathbb{R}^{m \times n}$, and define $\tilde{A} = A_1 - A_2i + A_3j - A_4k$ to be the j-conjugate of A . The Frobenius norm of A is denoted by $\|A\|$, that is, $\|A\| = \sqrt{\text{tr}(A^H A)}$. Moreover, $A \otimes B$ and $A \odot B$ stand for the Kronecker matrix product and Hadamard matrix product of the matrices A and B .

The definition of generalized reflexive matrix can be found in [2]. A complex matrix A is called generalized reflexive (generalized anti-reflexive) if $A = PAQ$ ($A = -PAQ$), where P and Q are two generalized reflection matrices, i.e., $P = P^H = P^{-1}$ and $Q = Q^H = Q^{-1}$. The generalized reflexive matrices have been widely used in engineering and scientific computations [2–4]. Throughout the article, we use $\mathbb{H}_r^{m \times n}(P, Q)$ ($\mathbb{H}_a^{m \times n}(P, Q)$) to denote the set of $m \times n$ generalized reflexive (generalized anti-reflexive) quaternion matrices with respect to the generalized reflection matrix pair (P, Q) .

In the field of matrix algebra, quaternion matrix equations have received much attention. Wang and Zhang [35] gave the expression of the reflexive re-nonnegative definite solution of the quaternion matrix equation $AXA^H = B$. Yuan and Wang [48] derived the expressions of the least squares η -Hermitian solution with the least norm and the expressions of the least squares anti- η -Hermitian solution with the least norm for the quaternion matrix equation $AXB + CXD = E$. Jiang and Wei [22] derived the explicit solution of the quaternion matrix equation $X - A\tilde{X}B = C$. Song, Chen and Wang [33] obtained the expressions of the explicit solutions of quaternion matrix equations $XF - AX = BY$ and $XF - A\tilde{X} = BY$. Song, Chen and Liu [31] gave the solution expressions of the quaternions $X - AXF = C$ and $X - A\tilde{X}F = C$. Yuan and Liao [49] derived the expressions of the least squares solution of the quaternion matrix equation $X - A\tilde{X}B = C$ with the least norm. Li and Wu [23] studied the expressions of symmetric and skew-antisymmetric solutions to the quaternion matrix equations $A_1X = C_1$ and $XB_3 = C_3$. Feng and Cheng [21] gave a clear description of the solution set to the quaternion matrix equation $AX - \bar{X}B = 0$.

The iterative method is a very important method to solve matrix equations. Peng, Hu and Zhang [25, 26] constructed iteration methods to solve the symmetric and reflexive solutions of matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$. Ding and Chen proposed the hierarchical gradient-based iterative (HGI) algorithms [12] and hierarchical least-squares iterative (HLSI) algorithms [13] for solving general (coupled) matrix equations, based on the hierarchical identification principle [14]. Peng [28–30] presented several efficient iterative methods to solve the constrained least squares solutions of linear matrix equations $AXB = C$ and $AXB + CYD = E$, by using Paige’s algorithm [24] as the frame method. Wang, Cheng and Wei [44] proposed iterative algorithms for solving the matrix equation $AXB + CX^TD = E$. Duan et al. [15–19] proposed iterative algorithms for the (Hermitian) positive definite solutions of some nonlinear matrix equations. Dehghan and Hajarian constructed iterative algorithms to solve several linear matrix equations over (anti-)reflexive [6, 10], generalized centro-symmetric [5, 9] and generalized bisymmetric [7, 8] matrices. Zhou et al. [51] reported gradient-based iterative algorithms for solving the general coupled Sylvester matrix equations with weighted least squares solutions. Wu et al. [45–47] proposed iterative algorithms for solving various complex matrix equations. Wang, Wei and Feng [43] derived an iterative method for finding the minimum-norm solution of the QLS problem in quaternionic quantum theory.

However, to our best knowledge, the generalized (P, Q) -reflexive solution of

$$(1.1) \quad \sum_{l=1}^u A_l X B_l + \sum_{s=1}^v C_s \tilde{X} D_s = F$$

over the quaternion algebra \mathbb{H} have not been considered so far. Moreover, the matrix equation (1.1) obviously includes the quaternion matrix equations $X - A\tilde{X}B = C$, $XF - A\tilde{X} = BY$, $AXB + CXD = E$ and $AXB = C$ as special cases, which have been investigated in [22, 31, 33, 39, 48, 49]. Motivated by the work mentioned above and keeping the interests and wide applications of quaternion matrices in view (e.g. [1, 11, 20, 32, 34, 36–42, 48, 50]), we, in this paper, consider an iterative algorithm for the following two problems:

Problem 1.1. For given matrices $A_l, C_s \in \mathbb{H}^{p \times m}$, $B_l, D_s \in \mathbb{H}^{n \times q}$, $l = 1, 2, \dots, u$, $s = 1, 2, \dots, v$, $F \in \mathbb{H}^{p \times q}$ and the generalized reflection

matrices $P \in \mathbb{H}^{m \times m}$, $Q \in \mathbb{H}^{n \times n}$, find $X \in \mathbb{H}_r^{m \times n}(P, Q)$, such that

$$\sum_{l=1}^u A_l X B_l + \sum_{s=1}^v C_s \tilde{X} D_s = F.$$

Problem 1.2. When Problem 1.1 is consistent, let its solution set be denoted by S_H . For a given matrix $X_0 \in \mathbb{H}^{m \times n}$, find $\check{X} \in S_H$, such that

$$\|\check{X} - X_0\| = \min_{X \in S_H} \|X - X_0\|.$$

The paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we introduce an iterative algorithm for solving Problem 1.1. Then we prove that the given algorithm can be used to obtain a generalized (P, Q) -reflexive solution for any generalized (P, Q) -reflexive initial matrix within finite steps in the absence of roundoff errors. Also, we prove that the least Frobenius norm generalized (P, Q) -reflexive solution can be obtained by choosing a special kind of initial matrix. In addition, the solution of Problem 1.2 by finding the least Frobenius norm generalized (P, Q) -reflexive solution of a new matrix equation is given. In Section 4, we give two numerical examples to verify our results. In Section 5, we give some conclusions to end this paper.

2. Preliminaries

In this section, we provide some results which will play important roles in this paper.

First, we recall some results about quaternion matrix arithmetic. Due to the non-commutativity of \mathbb{H} , some well-known equalities for complex and real matrices no longer hold for quaternion matrices. The following theorem gives a list of facts for quaternion matrix arithmetic.

Theorem 2.1. [50] Let $A \in \mathbb{H}^{m \times n}$, $B \in \mathbb{H}^{n \times p}$. Then

- (1) $(\overline{A})^T = \overline{(A^T)}$;
- (2) $(AB)^H = B^H A^H$;
- (3) $\overline{AB} \neq \overline{A} \overline{B}$ in general;
- (4) $(AB)^T \neq B^T A^T$ in general;
- (5) $(AB)^{-1} = B^{-1} A^{-1}$ if A and B are invertible;
- (6) $(A^H)^{-1} = (A^{-1})^H$ if A is invertible;
- (7) $(\overline{A})^{-1} \neq \overline{A^{-1}}$ in general;
- (8) $(A^T)^{-1} \neq (A^{-1})^T$ in general.

Moreover, it is easy to verify that for $A \in \mathbb{H}^{m \times n}$, $B \in \mathbb{H}^{n \times p}$ we have $\widetilde{A^H} = (\widetilde{A})^H$ and $\widetilde{AB} = \widetilde{A}\widetilde{B}$.

Then, we introduce a real inner product for the space $\mathbb{H}^{m \times n}$ over the real field \mathbb{R} . In [45–47], the following real inner product was presented to solve some complex matrix equations:

$$(2.1) \quad \langle A, B \rangle = \operatorname{Re}[\operatorname{tr}(B^H A)] ,$$

for $A, B \in \mathbb{C}^{m \times n}$. It is easy to verify that if we let $A, B \in \mathbb{H}^{m \times n}$, (2.1) also defines a real inner product in $\mathbb{H}^{m \times n}$ over \mathbb{R} . We denote this real inner product space as $(\mathbb{H}^{m \times n}, \mathbb{R}, \langle \cdot, \cdot \rangle)$. Let $\|\cdot\|_\Delta$ represent the matrix norm induced by the inner product $\langle \cdot, \cdot \rangle$. For an arbitrary quaternion matrix $A \in \mathbb{H}^{m \times n}$, it is obvious that the following equalities hold

$$\|A\|_\Delta = \sqrt{\langle A, A \rangle} = \sqrt{\operatorname{Re}[\operatorname{tr}(A^H A)]} = \sqrt{\operatorname{tr}(A^H A)} = \|A\| ,$$

which reveals that the induced matrix norm is exactly the Frobenius norm. For convenience, we still use $\|\cdot\|$ to denote the induced matrix norm.

Let E_{ij} denote the $m \times n$ matrix whose (i, j) entry is 1, and the other elements are zeros. In inner product space $(\mathbb{H}^{m \times n}, \mathbb{R}, \langle \cdot, \cdot \rangle)$, it is easy to verify that $E_{ij}, E_{ij}i, E_{ij}j, E_{ij}k, i = 1, 2, \dots, m, j = 1, 2, \dots, n$, is an orthonormal basis, which reveals that the dimension of the inner product space $(\mathbb{H}^{m \times n}, \mathbb{R}, \langle \cdot, \cdot \rangle)$ is $4mn$.

Lemma 2.2. *Let $P \in \mathbb{H}^{m \times m}$ and $Q \in \mathbb{H}^{n \times n}$ be two generalized reflection matrices. If $A \in \mathbb{H}_r^{m \times n}(P, Q)$, $B \in \mathbb{H}_a^{m \times n}(P, Q)$, then we have $\langle A, B \rangle = 0$.*

Proof. By the equalities

$$\begin{aligned} \langle A, B \rangle &= \operatorname{Re}[\operatorname{tr}(B^H A)] = -\operatorname{Re}[\operatorname{tr}((PBQ)^H PAQ)] = -\operatorname{Re}[\operatorname{tr}(QB^H AQ)] \\ &= -\operatorname{Re}[\operatorname{tr}(B^H A)] = -\langle A, B \rangle, \end{aligned}$$

the proof is trivial. \square

Finally, we introduce a real representation of a quaternion matrix. For an arbitrary quaternion matrix $M = M_1 + M_2i + M_3j + M_4k$, a map $\phi(\cdot)$, from $\mathbb{H}^{m \times n}$ to $\mathbb{R}^{4m \times 4n}$, can be defined as

$$\phi(M) = \begin{bmatrix} M_1 & -M_2 & -M_3 & -M_4 \\ M_2 & M_1 & -M_4 & M_3 \\ M_3 & M_4 & M_1 & -M_2 \\ M_4 & -M_3 & M_2 & M_1 \end{bmatrix} .$$

Let M and N be two arbitrary quaternion matrices with appropriate size. From [42], we know that $\phi(\cdot)$ satisfies the following properties:

- (1) $M = N \iff \phi(M) = \phi(N)$;
- (2) $\phi(M + N) = \phi(M) + \phi(N)$, $\phi(MN) = \phi(M)\phi(N)$, $\phi(kM) = k\phi(M)$, $k \in \mathbb{R}$;
- (3) $\phi(M^H) = \phi^T(M)$;
- (4) $\phi(M) = T_m^{-1}\phi(M)T_n = R_m^{-1}\phi(M)R_n = S_m^{-1}\phi(M)S_n$, where

$$R_t = \begin{bmatrix} 0 & -I_{2t} \\ I_{2t} & 0 \end{bmatrix}, S_t = \begin{bmatrix} 0 & 0 & 0 & -I_t \\ 0 & 0 & I_t & 0 \\ 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \end{bmatrix}, T_t = \begin{bmatrix} 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \\ 0 & 0 & 0 & I_t \\ 0 & 0 & -I_t & 0 \end{bmatrix},$$

$t = m, n$.

Through a simple verification, we derive that $\phi(\cdot)$ also satisfies the following two properties which is useful for some deduction in this paper:

- (5) $\|\phi(M)\| = 2\|M\|$;
- (6) $\phi(\widetilde{M}) = -\phi(M) + 2\phi(M) \odot W$, where

$$W = \begin{bmatrix} E & 0 & E & 0 \\ 0 & E & 0 & E \\ E & 0 & E & 0 \\ 0 & E & 0 & E \end{bmatrix}.$$

3. Main results

3.1. The solution of Problem 1.1. In this subsection, we will construct an algorithm for solving Problem 1.1. Then some lemmas will be given to analyse the properties of the proposed algorithm. Using these lemmas, we prove that the proposed algorithm is convergent.

Algorithm 1. 1. Choose an initial matrix $X(1) \in \mathbb{H}_r^{m \times n}(P, Q)$;

2. Calculate

$$R(1) = F - \sum_{l=1}^u A_l X(1) B_l - \sum_{s=1}^v C_s \widetilde{X(1)} D_s;$$

$$P(1) = \frac{1}{2} \left(\sum_{l=1}^u A_l^H R(1) B_l^H + \sum_{s=1}^v \widetilde{C_s^H} \widetilde{R(1)} \widetilde{D_s^H} + \sum_{l=1}^u P A_l^H R(1) B_l^H Q \right. \\ \left. + \sum_{s=1}^v P \widetilde{C_s^H} \widetilde{R(1)} \widetilde{D_s^H} Q \right);$$

$k := 1$;

3. If $R(k) = 0$, or $R(k) \neq 0$ and $P(k) = 0$, stop; else $k := k + 1$;
 4. Calculate

$$X(k) = X(k-1) + \frac{\|R(k-1)\|^2}{\|P(k-1)\|^2} P(k-1);$$

$$R(k) = R(k-1) - \frac{\|R(k-1)\|^2}{\|P(k-1)\|^2} \left(\sum_{l=1}^u A_l P(k-1) B_l + \sum_{s=1}^v C_s \widetilde{P(k-1)} D_s \right);$$

$$\begin{aligned} P(k) &= \frac{1}{2} \left(\sum_{l=1}^u A_l^H R(k) B_l^H + \sum_{s=1}^v \widetilde{C_s^H} \widetilde{R(k)} \widetilde{D_s^H} + \sum_{l=1}^u P A_l^H R(k) B_l^H Q \right. \\ &\quad \left. + \sum_{s=1}^v P \widetilde{C_s^H} \widetilde{R(k)} \widetilde{D_s^H} Q \right) + \frac{\|R(k)\|^2}{\|R(k-1)\|^2} P(k-1); \end{aligned}$$

5. Go to Step 3.

Lemma 3.1. Assume that the sequences $\{R(i)\}$ and $\{P(i)\}$ are generated by Algorithm 1, then

$$\langle R(i), R(j) \rangle = 0 \text{ and } \langle P(i), P(j) \rangle = 0 \text{ for } i, j = 1, 2, \dots, i \neq j.$$

Proof. We only need to prove that $\langle R(i), R(j) \rangle = 0$ and $\langle P(i), P(j) \rangle = 0$ for $1 \leq i < j$.

Now we prove this conclusion by induction. First, we show that

$$(3.1) \quad \langle R(i), R(i+1) \rangle = 0 \text{ and } \langle P(i), P(i+1) \rangle = 0 \text{ for } i = 1, 2, \dots$$

When $i = 1$, from Algorithm 1, we have

$$\begin{aligned} &\langle R(1), R(2) \rangle \\ &= \|R(1)\|^2 - \frac{\|R(1)\|^2}{\|P(1)\|^2} \operatorname{Re}\{\operatorname{tr}[P^H(1) \left(\sum_{l=1}^u A_l^H R(1) B_l^H + \sum_{s=1}^v \widetilde{C_s^H} \widetilde{R(1)} \widetilde{D_s^H} \right)]\} \\ &= \|R(1)\|^2 - \frac{\|R(1)\|^2}{\|P(1)\|^2} \operatorname{Re}\{\operatorname{tr}[P^H(1) \\ &\quad \left. \frac{\sum_{l=1}^u A_l^H R(1) B_l^H + \sum_{s=1}^v \widetilde{C_s^H} \widetilde{R(1)} \widetilde{D_s^H} + \sum_{l=1}^u P A_l^H R(1) B_l^H Q + \sum_{s=1}^v P \widetilde{C_s^H} \widetilde{R(1)} \widetilde{D_s^H} Q}{2} \right)]\} \\ &= \|R(1)\|^2 - \frac{\|R(1)\|^2}{\|P(1)\|^2} \|P(1)\|^2 \\ &= 0. \end{aligned}$$

Also we can write

$$\begin{aligned}
& \langle P(1), P(2) \rangle \\
&= \frac{\|R(2)\|^2}{\|R(1)\|^2} \|P(1)\|^2 + \operatorname{Re}\{\operatorname{tr}[P^H(1)(\sum_{l=1}^u A_l^H R(2) B_l^H + \sum_{s=1}^v \widetilde{C}_s^H \widetilde{R(2)} \widetilde{D}_s^H)]\} \\
&= \frac{\|R(2)\|^2}{\|R(1)\|^2} \|P(1)\|^2 - \frac{\|P(1)\|^2}{\|R(1)\|^2} \|R(2)\|^2 \\
&= 0.
\end{aligned}$$

Now, assume that conclusion (3.1) holds for $1 \leq i \leq t-1$, then

$$\begin{aligned}
& \langle R(t), R(t+1) \rangle \\
&= \|R(t)\|^2 - \frac{\|R(t)\|^2}{\|P(t)\|^2} \operatorname{Re}\{\operatorname{tr}[P^H(t)(\sum_{l=1}^u A_l^H R(t) B_l^H + \sum_{s=1}^v \widetilde{C}_s^H \widetilde{R(t)} \widetilde{D}_s^H)]\} \\
&= \|R(t)\|^2 - \frac{\|R(t)\|^2}{\|P(t)\|^2} \operatorname{Re}\{\operatorname{tr}[P^H(t)(P(t) - \frac{\|R(t)\|^2}{\|R(t-1)\|^2} P(t-1))]\} \\
&= 0.
\end{aligned}$$

And we also have

$$\begin{aligned}
& \langle P(t), P(t+1) \rangle \\
&= \frac{\|R(t+1)\|^2}{\|R(t)\|^2} \|P(t)\|^2 + \operatorname{Re}\{\operatorname{tr}[P^H(t)(\sum_{l=1}^u A_l^H R(t+1) B_l^H + \sum_{s=1}^v \widetilde{C}_s^H \widetilde{R(t+1)} \widetilde{D}_s^H)]\} \\
&= \frac{\|R(t+1)\|^2}{\|R(t)\|^2} \|P(t)\|^2 + \frac{\|P(t)\|^2}{\|R(t)\|^2} \operatorname{Re}\{\operatorname{tr}[R^H(t+1)((R(t) - R(t+1)))]\} \\
&= 0.
\end{aligned}$$

Therefore, the conclusion (3.1) holds for $i \geq 1$.

Assume that $\langle R(i), R(i+r) \rangle = 0$ and $\langle P(i), P(i+r) \rangle = 0$ for $i \geq 1$ and $r \geq 1$. We will show that

$$(3.2) \quad \langle R(i), R(i+r+1) \rangle = 0 \text{ and } \langle P(i), P(i+r+1) \rangle = 0.$$

It follows from Algorithm 1 that

$$\begin{aligned}
& \langle R(1), R(r+2) \rangle \\
&= \operatorname{Re}\{\operatorname{tr}[R^H(r+1)R(1)]\} \\
&- \frac{\|R(r+1)\|^2}{\|P(r+1)\|^2} \operatorname{Re}\{\operatorname{tr}[P^H(r+1)(\sum_{l=1}^u A_l^H R(1)B_l^H + \sum_{s=1}^v \widetilde{C}_s^H \widetilde{R}(1)\widetilde{D}_s^H)]\} \\
&= \operatorname{Re}\{\operatorname{tr}[R^H(r+1)R(1)]\} - \frac{\|R(r+1)\|^2}{\|P(r+1)\|^2} \operatorname{Re}\{\operatorname{tr}[P^H(r+1)P(1)]\} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
& \langle P(1), P(r+2) \rangle \\
&= \operatorname{Re}\{\operatorname{tr}[R^H(r+2)(\sum_{l=1}^u A_l P(1)B_l \\
&+ \sum_{s=1}^v C_s \widetilde{P}(1)\widetilde{D}_s)]\} + \frac{\|R(r+2)\|^2}{\|R(r+1)\|^2} \operatorname{Re}\{\operatorname{tr}[P^H(r+1)P(1)]\} \\
&= \frac{\|P(1)\|^2}{\|R(1)\|^2} \operatorname{Re}\{\operatorname{tr}[R^H(r+2)(R(1) - R(2))]\} \\
&+ \frac{\|R(r+2)\|^2}{\|R(r+1)\|^2} \operatorname{Re}\{\operatorname{tr}[P^H(r+1)P(1)]\} \\
(3.3) \quad &= 0.
\end{aligned}$$

Also, we have

$$\begin{aligned}
& \langle R(i), R(i+r+1) \rangle \\
&= \operatorname{Re}\{\operatorname{tr}[R^H(i+r)R(i)]\} \\
&- \frac{\|R(i+r)\|^2}{\|P(i+r)\|^2} \operatorname{Re}\{\operatorname{tr}[P^H(i+r)(\sum_{l=1}^u A_l^H R(i)B_l^H + \sum_{s=1}^v \widetilde{C}_s^H \widetilde{R}(i)\widetilde{D}_s^H)]\} \\
&= -\frac{\|R(i+r)\|^2}{\|P(i+r)\|^2} \operatorname{Re}\{\operatorname{tr}[P^H(i+r)(P(i) - \frac{\|R(i)\|^2}{\|R(i-1)\|^2} P(i-1))]\} \\
(3.4) \quad &= \frac{\|R(i+r)\|^2 \|R(i)\|^2}{\|P(i+r)\|^2 \|R(i-1)\|^2} \operatorname{Re}\{\operatorname{tr}[P^H(i+r)P(i-1)]\}
\end{aligned}$$

and

$$\begin{aligned}
& \langle P(i), P(i+r+1) \rangle \\
&= \operatorname{Re}\{\operatorname{tr}[R^H(i+r+1)(\sum_{l=1}^u A_l P(i) B_l + \sum_{s=1}^v C_s \widetilde{P(i)} D_s)]\} \\
&= \frac{\|P(i)\|^2}{\|R(i)\|^2} \operatorname{Re}\{\operatorname{tr}[R^H(i+r+1)(R(i) - R(i+1))]\} \\
(3.5) \quad &= \frac{\|P(i)\|^2 \|R(i+r)\|^2 \|R(i)\|^2}{\|R(i)\|^2 \|P(i+r)\|^2 \|R(i-1)\|^2} \operatorname{Re}\{\operatorname{tr}[P^H(i+r)P(i-1)]\}.
\end{aligned}$$

Repeating the above process (3.4) and (3.5), we obtain

$$\begin{aligned}
\langle R(i), R(i+r+1) \rangle &= \dots = \alpha \operatorname{Re}\{\operatorname{tr}[P^H(r+2)P(1)]\}; \\
\langle P(i), P(i+r+1) \rangle &= \dots = \beta \operatorname{Re}\{\operatorname{tr}[P^H(r+2)P(1)]\}.
\end{aligned}$$

Combining these two relations with (3.3), implies that (3.2) holds.

So, by the principle of induction, we know that Lemma 3.1 holds. \square

Lemma 3.2. *Assume that Problem 1.1 is consistent, and let $X^* \in \mathbb{H}_r^{m \times n}(P, Q)$ be a solution. Then, for any initial matrix $X(1) \in \mathbb{H}_r^{m \times n}(P, Q)$, the sequences $\{R(i)\}$, $\{P(i)\}$ and $\{X(i)\}$ generated by Algorithm 1 satisfy*

$$(3.6) \quad \langle P(i), X^* - X(i) \rangle = \|R(i)\|^2, i = 1, 2, \dots$$

Proof. We prove this conclusion by induction.

When $i = 1$, it follows from Algorithm 1 that

$$\begin{aligned}
& \langle P(1), X^* - X(1) \rangle \\
&= \operatorname{Re}\{\operatorname{tr}[(X^* - X(1))^H (\sum_{l=1}^u A_l^H R(1) B_l^H + \sum_{s=1}^v \widetilde{C_s^H} \widetilde{R(1)} \widetilde{D_s^H})]\} \\
&= \operatorname{Re}\{\operatorname{tr}[R^H(1) (\sum_{l=1}^u A_l (X^* - X(1)) B_l + \sum_{s=1}^v C_s (\widetilde{X^*} - \widetilde{X(1)}) D_s)]\} \\
&= \|R(1)\|^2.
\end{aligned}$$

This implies that (3.6) holds for $i = 1$.

Now it is assumed that (3.6) holds for $i = t$, that is

$$\langle P(t), X^* - X(t) \rangle = \|R(t)\|^2.$$

Then, when $i = t + 1$

$$\begin{aligned}
& \langle P(t+1), X^* - X(t+1) \rangle \\
&= \operatorname{Re}\{\operatorname{tr}[R^H(t+1)(\sum_{l=1}^u A_l(X^* - X(t+1))B_l + \sum_{s=1}^v C_s(\widetilde{X^*} - \widetilde{X(t+1)})D_s)]\} + \\
& \frac{\|R(t+1)\|^2}{\|R(t)\|^2} \{\operatorname{Re}\{\operatorname{tr}[(X^* - X(t))^H P(t)]\} - \operatorname{Re}\{\operatorname{tr}[(X(t+1) - X(t))^H P(t)]\}\} \\
&= \|R(t+1)\|^2 + \frac{\|R(t+1)\|^2}{\|R(t)\|^2} \{\|R(t)\|^2 - \frac{\|R(t)\|^2}{\|P(t)\|^2} \operatorname{Re}\{\operatorname{tr}[P^H(t)P(t)]\}\} \\
&= \|R(t+1)\|^2.
\end{aligned}$$

Therefore, Lemma 3.2 holds by the principle of induction. \square

From the above two lemmas, we have the following conclusions.

Remark 3.3. *If there exists a positive number i such that $R(i) \neq 0$ and $P(i) = 0$, then by Lemma 3.2 Problem 1.1 is not consistent. Hence, the solvability of Problem 1.1 can be determined by Algorithm 1 automatically in the absence of roundoff errors.*

Theorem 3.4. *Suppose that Problem 1.1 is consistent. Then for any initial matrix $X(1) \in \mathbb{H}_r^{m \times n}(P, Q)$, a solution of Problem 1.1 can be obtained within finite iteration steps in the absence of roundoff errors.*

Proof. It is known that the inner product space $(\mathbb{H}^{p \times q}, R, \langle \cdot, \cdot \rangle)$ is $4pq$ -dimensional. According to Lemma 3.2, if $R(i) \neq 0, i = 1, 2, \dots, 4pq$, then we have $P(i) \neq 0, i = 1, 2, \dots, 4pq$. Hence $R(4pq+1)$ and $P(4pq+1)$ can be computed. From Lemma 3.1, it is not difficult to get

$$\langle R(i), R(j) \rangle = 0 \text{ for } i, j = 1, 2, \dots, 4pq, i \neq j.$$

Then $R(1), R(2), \dots, R(4pq)$ is an orthogonal basis of the inner product space $(\mathbb{H}^{p \times q}, R, \langle \cdot, \cdot \rangle)$. In addition, we can get from Lemma 3.1 that

$$\langle R(i), R(4pq+1) \rangle = 0 \text{ for } i = 1, 2, \dots, 4pq.$$

It follows that $R(4pq+1) = 0$, which implies that $X(4pq+1)$ is a solution of Problem 1.1. \square

3.2. The solution of Problem 1.2. In this subsection, firstly we introduce some lemmas. Then, we will prove that the least Frobenius norm generalized (P, Q) -reflexive solution of (1.1) can be derived by choosing a suitable initial iterative matrix. Finally, we solve Problem 1.2 by computing the least Frobenius norm generalized (P, Q) -reflexive solution of a new-constructed quaternion matrix equation.

Lemma 3.5. [27] *Assume that the consistent system of linear equations $My = b$ has a solution $y_0 \in R(M^T)$. Then y_0 is the unique least Frobenius norm solution of the system of linear equations.*

Lemma 3.6. *If Problem 1.1 is consistent, then the following system of real linear equations is consistent*

$$(3.7) \quad \begin{bmatrix} \left(\begin{array}{c} \sum_{l=1}^u \phi^T(B_l) \otimes \phi(A_l) - \sum_{s=1}^v \phi^T(D_s) \otimes \phi(C_s) \\ + 2 \sum_{s=1}^v (\phi^T(D_s) \otimes \phi(C_s)) \text{diag}(\text{vec}(W)) \end{array} \right) \\ \left(\begin{array}{c} \sum_{l=1}^u \phi^T(B_l) \phi(Q) \otimes \phi(A_l) \phi(P) - \sum_{s=1}^v \phi^T(D_s) \phi(Q) \otimes \phi(C_s) \phi(P) \\ + 2 \sum_{s=1}^v (\phi^T(D_s) \otimes \phi(C_s)) \text{diag}(\text{vec}(W)) (\phi(Q) \otimes \phi(P)) \end{array} \right) \end{bmatrix}^Y = \begin{bmatrix} \text{vec}(\phi(F)) \\ \text{vec}(\phi(F)) \end{bmatrix}.$$

Furthermore, if the solution sets of Problem 1.1 and (3.7) are denoted by S_X and S_Y respectively, then,

$$(3.8) \quad \text{vec}(\phi(S_X)) \subseteq S_Y.$$

Proof. If Problem 1.1 is consistent, let X be a solution of Problem 1.1, then we have $\sum_{l=1}^u A_l X B_l + \sum_{s=1}^v C_s \tilde{X} D_s = F$ and $PXQ = X$, which implies that X is a solution of quaternion matrix equations

$$(3.9) \quad \begin{cases} \sum_{l=1}^u A_l X B_l + \sum_{s=1}^v C_s \tilde{X} D_s = F, \\ \sum_{l=1}^u A_l P X Q B_l + \sum_{s=1}^v C_s \tilde{P} \tilde{X} \tilde{Q} D_s = F. \end{cases}$$

From properties (1), (2) and (6) of $\phi(\cdot)$, we derive that the quaternion matrix equations (3.9) is equivalent to the following real matrix equations

$$\begin{bmatrix} \left(\begin{array}{c} \sum_{l=1}^u \phi^T(B_l) \otimes \phi(A_l) - \sum_{s=1}^v \phi^T(D_s) \otimes \phi(C_s) \\ + 2 \sum_{s=1}^v (\phi^T(D_s) \otimes \phi(C_s)) \text{diag}(\text{vec}(W)) \end{array} \right) \\ \left(\begin{array}{c} \sum_{l=1}^u \phi^T(B_l) \phi(Q) \otimes \phi(A_l) \phi(P) - \sum_{s=1}^v \phi^T(D_s) \phi(Q) \otimes \phi(C_s) \phi(P) \\ + 2 \sum_{s=1}^v (\phi^T(D_s) \otimes \phi(C_s)) \text{diag}(\text{vec}(W)) (\phi(Q) \otimes \phi(P)) \end{array} \right) \end{bmatrix} \text{vec}(\phi(X)) = \begin{bmatrix} \text{vec}(\phi(F)) \\ \text{vec}(\phi(F)) \end{bmatrix}.$$

which implies that $\text{vec}(\phi(X))$ is a solution of (3.7). So system of real linear equations (3.7) is consistent.

From the above procedure, the proof of (3.8) is trivial. \square

Theorem 3.7. *If $\overset{\circ}{X}$ is a solution of Problem 1.1, and $\overset{\circ}{X}$ can be expressed as*

$$\overset{\circ}{X} = \sum_{l=1}^u A_l^H G B_l^H + \sum_{s=1}^v \widetilde{C}_s^H \widetilde{G} \widetilde{D}_s^H + \sum_{l=1}^u P A_l^H G B_l^H Q + \sum_{s=1}^v P \widetilde{C}_s^H \widetilde{G} \widetilde{D}_s^H Q, \\ G \in \mathbb{H}^{p \times q},$$

then, $\overset{\circ}{X}$ is the least Frobenius norm solution of Problem 1.1.

Proof. By properties (1), (2), (3) and (6) of $\phi(\cdot)$, we have

$$\begin{aligned} & \text{vec}(\phi(\overset{\circ}{X})) \\ &= \text{vec} \left(\sum_{l=1}^u \phi^T(A_l) \phi(G) \phi^T(B_l) - \sum_{s=1}^v \phi^T(C_s) \phi(G) \phi^T(D_s) \right. \\ & \quad \left. + 2 \sum_{s=1}^v (\phi^T(C_s) \phi(G) \phi^T(D_s)) \odot W + \sum_{l=1}^u \phi(P) \phi^T(A_l) \phi(G) \phi^T(B_l) \phi(Q) \right. \\ & \quad \left. - \sum_{s=1}^v \phi(P) \phi^T(C_s) \phi(G) \phi^T(D_s) \phi(Q) + 2 \sum_{s=1}^v \phi(P) ((\phi^T(C_s) \phi(G) \phi^T(D_s)) \odot W) \phi(Q) \right) \\ &= \left[\sum_{l=1}^u \phi(B_l) \otimes \phi^T(A_l) - \sum_{s=1}^v \phi(D_s) \otimes \phi^T(C_s) + 2 \sum_{s=1}^v \text{diag}(\text{vec}(W)) (\phi(D_s) \otimes \phi^T(C_s)), \right. \\ & \quad \left. \sum_{l=1}^u \phi(Q) \phi(B_l) \otimes \phi(P) \phi^T(A_l) - \sum_{s=1}^v \phi(Q) \phi(D_s) \otimes \phi(P) \phi^T(C_s) \right. \\ & \quad \left. + 2 \sum_{s=1}^v (\phi(Q) \otimes \phi(P)) \text{diag}(\text{vec}(W)) (\phi(D_s) \otimes \phi^T(C_s)) \right] \begin{bmatrix} \text{vec}(\phi(G)) \\ \text{vec}(\phi(G)) \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \left(\begin{array}{c} \sum_{l=1}^u \phi^T(B_l) \otimes \phi(A_l) - \sum_{s=1}^v \phi^T(D_s) \otimes \phi(C_s) \\ + 2 \sum_{s=1}^v (\phi^T(D_s) \otimes \phi(C_s)) \text{diag}(\text{vec}(W)) \end{array} \right) \\ \left(\begin{array}{c} \sum_{l=1}^u \phi^T(B_l)\phi(Q) \otimes \phi(A_l)\phi(P) - \sum_{s=1}^v \phi^T(D_s)\phi(Q) \otimes \phi(C_s)\phi(P) \\ + 2 \sum_{s=1}^v (\phi^T(D_s) \otimes \phi(C_s)) \text{diag}(\text{vec}(W)(\phi(Q) \otimes \phi(P))) \end{array} \right) \end{bmatrix}^T \begin{bmatrix} \text{vec}(\phi(G)) \\ \text{vec}(\phi(G)) \end{bmatrix}$$

By Lemma 3.5, $\phi(\hat{X})$ is the least Frobenius norm solution of matrix equations (3.7). By property (5) of $\phi(\cdot)$, we derive from Lemma 3.6 that \hat{X} is the least Frobenius norm solution of Problem 1.1. \square

From Algorithm 1, it is obvious that, if we consider

$$X(1) = \sum_{l=1}^u A_l^H G B_l^H + \sum_{s=1}^v \widetilde{C}_s^H \widetilde{G} \widetilde{D}_s^H + \sum_{l=1}^u P A_l^H G B_l^H Q + \sum_{s=1}^v P \widetilde{C}_s^H \widetilde{G} \widetilde{D}_s^H Q,$$

$G \in \mathbb{H}^{p \times q},$

then all $X(k)$ generated by Algorithm 1 can be expressed as

$$X(k) = \sum_{l=1}^u A_l^H G_k B_l^H + \sum_{s=1}^v \widetilde{C}_s^H \widetilde{G}_k \widetilde{D}_s^H + \sum_{l=1}^u P A_l^H G_k B_l^H Q + \sum_{s=1}^v P \widetilde{C}_s^H \widetilde{G}_k \widetilde{D}_s^H Q,$$

$G_k \in \mathbb{H}^{p \times q}.$

Using the above conclusion and considering Theorem 3.7, we propose the following theorem.

Theorem 3.8. *Suppose that Problem 1.1 is consistent. Let the initial iteration matrix be*

$$X(1) = \sum_{l=1}^u A_l^H G B_l^H + \sum_{s=1}^v \widetilde{C}_s^H \widetilde{G} \widetilde{D}_s^H + \sum_{l=1}^u P A_l^H G B_l^H Q + \sum_{s=1}^v P \widetilde{C}_s^H \widetilde{G} \widetilde{D}_s^H Q,$$

where G is an arbitrary quaternion matrix, or especially, $X(1) = 0$, then the solution X^* , generated by Algorithm 1, is the least Frobenius norm solution of Problem 1.1.

Now, we study Problem 1.2. When Problem 1.1 is consistent, the solution set of Problem 1.1 denoted by S_H is not empty. For a given matrix $X_0 \in \mathbb{H}^{m \times n}$ and $X \in S_H$, it is easy to verify that $X - \frac{X_0 + P X_0 Q}{2} \in$

$\mathbb{H}_r^{m \times n}(P, Q)$ and $\frac{X_0 - PX_0Q}{2} \in \mathbb{H}_a^{m \times n}(P, Q)$. By Lemma 2.2, we have

$$\begin{aligned} \|X - X_0\|^2 &= \left\| X - \frac{X_0 + PX_0Q}{2} - \frac{X_0 - PX_0Q}{2} \right\|^2 \\ &= \left\| X - \frac{X_0 + PX_0Q}{2} \right\|^2 + \left\| \frac{X_0 - PX_0Q}{2} \right\|^2. \end{aligned}$$

Hence, Problem 1.2 is equivalent to finding $\check{X} \in S_H$, such that $\left\| \check{X} - \frac{X_0 + PX_0Q}{2} \right\| = \min_{X \in S_H} \left\| X - \frac{X_0 + PX_0Q}{2} \right\|$. Let $\dot{X} = X - \frac{X_0 + PX_0Q}{2}$, $\dot{F} = F - \sum_{l=1}^u A_l \frac{X_0 + PX_0Q}{2} B_l - \sum_{s=1}^v C_s \frac{\tilde{X}_0 + \tilde{P}\tilde{X}_0\tilde{Q}}{2} D_s$, we have

$$\sum_{l=1}^u A_l X B_l + \sum_{s=1}^v C_s \tilde{X} D_s = F \iff \sum_{l=1}^u A_l \dot{X} B_l + \sum_{s=1}^v C_s \tilde{\dot{X}} D_s = \dot{F}.$$

Therefore, Problem 1.2 is equivalent to finding the least Frobenius norm generalized (P, Q) -reflexive solution of the quaternion matrix equation

$$(3.10) \quad \sum_{l=1}^u A_l \dot{X} B_l + \sum_{s=1}^v C_s \tilde{\dot{X}} D_s = \dot{F}.$$

By using Algorithm 1, let the initial iteration matrix $\dot{X}(1)$ $= \sum_{l=1}^u A_l^H G B_l^H + \sum_{s=1}^v \widetilde{C_s^H} \widetilde{G} \widetilde{D_s^H} + \sum_{l=1}^u P A_l^H G B_l^H Q + \sum_{s=1}^v P \widetilde{C_s^H} \widetilde{G} \widetilde{D_s^H} Q$, where G is an arbitrary quaternion matrix in $\mathbb{H}^{p \times q}$, or especially, $\dot{X}(1) = 0$, we can obtain the least Frobenius norm generalized (P, Q) -reflexive solution \dot{X}^* of (3.10). Then we can obtain the solution of Problem 1.2, which is

$$\check{X} = \dot{X}^* + \frac{X_0 + PX_0Q}{2}.$$

4. Examples

In this section, we give two examples to illustrate the efficiency of the theoretical results.

Example 4.1. Consider the quaternion matrix equation

$$(4.1) \quad A_1 X B_1 + C_1 \tilde{X} D_1 + A_2 X B_2 + C_2 \tilde{X} D_2 = F,$$

with

$$A_1 = \begin{bmatrix} 19 - 5i + j + 6k & 10 & 4 + 4i + j + 3k \\ -9 + 3i + 2j + k & 2 - i - j & 3 + 3i + 3j - 7k \end{bmatrix},$$

$$\begin{aligned}
A_2 &= \begin{bmatrix} -2+i+j & 1+i+k & 3+2i+7j \\ 1-5i-4j+k & 2j+k & 5-i+6j+8k \end{bmatrix}, \\
B_1 &= \begin{bmatrix} 2j & -2+9i+12j+k \\ -5+5i+5j-k & 7+11k \\ 1+2i-j-8k & 5+i+4j+4k \\ 3+4i+j+k & 2i-2j+4k \end{bmatrix}, \\
B_2 &= \begin{bmatrix} 5+2i+j+4k & i+2j-9k \\ -i+3j+k & 2+7i+2k \\ 3+3i-6j+3k & -2+2i+2j+k \\ i+j & 13+2i+2j+7k \end{bmatrix}, \\
C_1 &= \begin{bmatrix} 2j+k & 1+2i+j-2k & 9+3j+5k \\ 3+i+2j+2k & -2+2i+k & 2+4i+3j+3k \end{bmatrix}, \\
C_2 &= \begin{bmatrix} 1+3i+10j & -3+i+3j+13k & i-5j+2k \\ 1+2i+2j+3k & -2-i-7j+2k & 3i+3j+2k \end{bmatrix}, \\
D_1 &= \begin{bmatrix} 1+3i+4j+k & i+3j+4k \\ i-3j+6k & 2-4j+8k \\ 4+i-5j+k & 5+2i+4j \\ 1-i+2j+2k & 1+i-j+8k \end{bmatrix}, \\
D_2 &= \begin{bmatrix} 1+i+j+k & 2+6i+4j+k \\ 12+i+13j & 1+6j+2k \\ 4-16i+8j+19k & 2-2i+7j+2k \\ -1-i+5j+k & 12+i-2j \end{bmatrix},
\end{aligned}$$

and

$$F = \begin{bmatrix} 3+4i-j+3k & 3j+8k \\ 6-8i+4j+k & 1+2i+j-5k \end{bmatrix}.$$

We apply Algorithm 1 to find the generalized (P, Q) -reflexive solution of (4.1), where

$$P = \begin{bmatrix} 0.28 & 0 & 0.96k \\ 0 & 1 & 0 \\ -0.96k & 0 & -0.28 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -0.28 & 0 & -0.96k & 0 \\ 0 & 0.28 & 0 & -0.96j \\ 0.96k & 0 & 0.28 & 0 \\ 0 & 0.96j & 0 & -0.28 \end{bmatrix}$$

are two generalized reflection matrices. For the initial matrix $X(1) =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.64 & 0 & -0.48j \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{H}_r^{3 \times 4}(P, Q), \text{ we obtain a solution, that is}$$

$$X^* = X(21) =$$

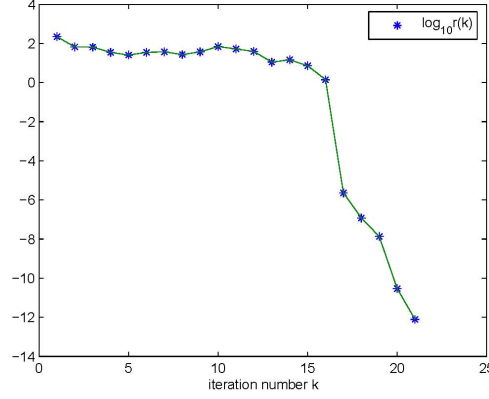


FIGURE 1. The convergence curve for the Frobenius norm of the residuals from Example 4.1.

$$\begin{bmatrix} -0.06834 + 0.1187i + 0.04175j - 0.05065k & -0.04482 + 0.01410i - 0.04205j + 0.01970k \\ 0.1035 - 0.004782i - 0.01668j - 0.04294k & 0.3454 - 0.2330i + 0.02003j - 0.1233k \\ 0.1207 + 0.1768i + 0.05934j - 0.1828k & 0.05732 - 0.04482i + 0.07723j - 0.02195k \\ 0.09114 - 0.2012i + 0.009891j - 0.1429k & -0.04482 + 0.05732i - 0.02195j + 0.07723k \\ -0.05725 + 0.02224i - 0.006376j - 0.1380k & 0.01502 - 0.09249i - 0.2591j + 0.1748k \\ 0.06834 + 0.1187i + 0.04175j + 0.05065k & -0.05915 + 0.05762i + 0.01373j + 0.01590k \end{bmatrix}$$

with corresponding residual $\|R(21)\| = 7.6047 \times 10^{-13}$. The convergence curve for the Frobenius norm of the residuals $R(k)$ is given in Figure 1, where $r(k) = \|R(k)\|$.

Example 4.2. In this example, we choose the matrices $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, F, P$ and Q the same as in Example 4.1. Let

$$X_0 = \begin{bmatrix} 1 & i & j & k \\ i & j & k & 1 \\ j & k & 1 & i \end{bmatrix}.$$

In order to find the optimal approximation generalized (P, Q) -reflexive solution to the given matrix X_0 , let $\dot{X} = X - \frac{X_0 + PX_0Q}{2}$ and $\dot{F} = F - A_1 \frac{X_0 + PX_0Q}{2} B_1 - A_2 \frac{X_0 + PX_0Q}{2} B_2 - C_1 \frac{\tilde{X}_0 + \tilde{P}\tilde{X}_0\tilde{Q}}{2} D_1 - C_2 \frac{\tilde{X}_0 + \tilde{P}\tilde{X}_0\tilde{Q}}{2} D_2$. We obtain the least Frobenius norm generalized (P, Q) -reflexive solution \dot{X}^* of the quaternion matrix equation $A_1 \dot{X} B_1 + C_1 \dot{X} D_1 + A_2 \dot{X} B_2 + C_2 \dot{X} D_2 = \dot{F}$, after 23 steps, by choosing the initial matrix $\dot{X}(1) = 0$, with corresponding residual $\|\dot{R}(23)\| = 7.718 \times 10^{-14}$. The convergence curve for

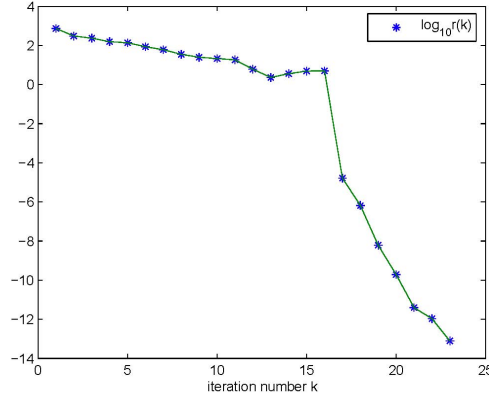


FIGURE 2. The convergence curve for the Frobenius norm of the residuals from Example 4.2.

the Frobenius norm of the residuals $\dot{R}(k)$ is given in Figure 2, where $r(k) = \left\| \dot{R}(k) \right\|$.

Therefore, the optimal approximation generalized (P, Q) -reflexive solution to the given matrix X_0 is

$$\check{X} = \dot{X}^* + \frac{X_0 + PX_0Q}{2} = \begin{bmatrix} -0.007483 + 0.1251 i - 0.004042 j + 0.04138 k & 0.06469 + 0.1681 i - 0.2289 j + 0.05607 k \\ -0.01278 - 0.01532 i + 0.09021 j - 0.1126 k & -0.06220 + 0.1145 i + 0.3711 j - 0.3495 k \\ 0.1265 + 0.1148 i + 0.1222 j + 0.03131 k & 0.02879 - 0.02258 i + 0.07145 j - 0.1985 k \\ 0.1507 - 0.1124 i - 0.04924 j + 0.03568 k & -0.02258 + 0.02879 i - 0.1985 j + 0.07145 k \\ -0.1501 - 0.1203 i - 0.02042 j + 0.01703 k & 0.2783 - 0.2621 i + 0.04665 j - 0.08584 k \\ 0.007483 + 0.1251 i - 0.004042 j - 0.04138 k & -0.2098 + 0.05112 i - 0.03928 j + 0.2157 k \end{bmatrix}.$$

The results show that Algorithm 1 is quite efficient.

5. Conclusions

In this paper, an algorithm has been presented for solving the generalized (P, Q) -reflexive solution of quaternion matrix equation $\sum_{l=1}^u A_l X B_l + \sum_{s=1}^v C_s \tilde{X} D_s = F$. By this algorithm, the solvability of the problem can be determined automatically. Also, when the problem is consistent, for any generalized (P, Q) -reflexive initial iterative matrix, a generalized

(P, Q) -reflexive solution can be obtained within finite iteration steps in the absence of roundoff errors. It has been proven that by choosing a suitable initial iterative matrix, we can derive the least Frobenius norm generalized (P, Q) -reflexive solution of the quaternion matrix equation $\sum_{l=1}^u A_l X B_l + \sum_{s=1}^v C_s \tilde{X} D_s = F$ through Algorithm 1. And then, by using Algorithm 1, we solved Problem 1.2. Finally, two numerical examples were given to show the efficiency of the presented algorithm.

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