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# ON IDEALS OF IDEALS IN $C(X)$

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Abstract. In this article, we have characterized ideals in $C(X)$ in which every ideal is also an ideal (a $z$-ideal) of $C(X)$. Motivated by this characterization, we observe that $C_\infty(X)$ is a regular ring if and only if every open locally compact $\sigma$-compact subset of $X$ is finite. Concerning prime ideals, it is shown that the sum of every two prime (semiprime) ideals of each ideal in $C(X)$ is prime (semiprime) if and only if $X$ is an $F$-space. Concerning maximal ideals of an ideal, we generalize the notion of separability to ideals and we have proved the coincidence of separability of an ideal with dense separability of a subspace of $\beta X$. Finally, we have shown that the Goldie dimension of an ideal $I$ in $C(X)$ coincide with the cellularity of $X \setminus \Delta(I)$.

**Keywords:** Dense separable, cellularity, $\sigma$-compact, $F$-space, Goldie dimension.
**MSC(2010):** Primary: 54C40; Secondary: 13A30.

## 1. Introduction

Throughout this paper, we denote we denote a completely regular Hausdorff space by $X$. We denote the ring of all real valued continuous functions on $X$ by $C(X)$ and $C^*(X)$ is the subring of $C(X)$ consisting of bounded functions.

Whenever $I$ is an ideal of a ring $R$ and $J$ is an ideal of $I$, then $J$ is not necessarily an ideal of $R$, see Section 5 in [12]. But, in this case, the subset $J$ of $R$ can not be any arbitrary set. In fact, whenever a subset $S$ of a ring $(R, +, .)$ is an ideal of an ideal in $R$ but not an ideal of $R$, then we have: (i) $S$ is a group under $+$, (ii) for all $s \in S$,

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$(s)S \subseteq S$, where $(s)$ is the principal ideal in $R$ generated by $s$ and (iii) $S \subsetneqq < S > \subsetneqq R$, where $< S >$ is the ideal generated by $S$. The converse is also true, i.e., whenever $S$ is a subset of a ring $R$ satisfying the above three conditions, then $S$ can be an ideal of an ideal in $R$ which is not necessarily an ideal of $R$. In fact $S$ is an ideal of $< S >$ and $< S >$ is the smallest ideal containing $S$ in which $S$ is an ideal. Moreover, if we take $\prod_S = \{a \in R : (a)S \subseteq S\}$, then for each $a \in \prod_S$, $S$ is an ideal of $< S, a >$ as well. We note that whenever $S$ is an ideal of an ideal $I_\alpha$ for each $\alpha$, then $S$ is also an ideal of $\sum_\alpha I_\alpha$. This implies that $S$ is an ideal of $\sum_{a\in\prod_S} < S, a >$ and the latter ideal is the largest ideal containing $S$ in which $S$ is an ideal. Whenever $I$ is an ideal of a ring $R$, it is well-known that every prime ideal of $I$ is also an ideal of $R$, see Theorem 1.1, below. Now it is natural to ask when every ideal of a given ideal in $C(X)$ is an ideal (a $z$-ideal) of $C(X)$. We will answer this question in Section 2 and it turns out in this section that such ideals should be pure ideals ($P$-ideals).

Ideals of ideals in rings and algebras are studied in [18] and [19] and prime ideals and maximal ideals of any ideal are characterized in the same references. For prime ideals we have the following result from [12].

**Theorem 1.1.** *Let $R$ be a ring and $J$ be an ideal of $R$. An ideal $P$ of $J$ is prime in $J$ if and only if $P = J \cap Q$ for some prime ideal $Q$ in $R$. Furthermore, if $P$ is proper, then $Q$ is unique.*

But a similar characterization is not true for maximal ideals of an ideal, i.e., maximal ideals of a given ideal $I$ in a ring $R$ are not necessarily of the form $I \cap M$, where $M$ is a maximal ideal of $R$. In fact, for a maximal ideal $M$ in $R$, $I \cap M$ may not be a maximal ideal of $I$, see Example 3.5 in [18]. We also cite the next result from [18] which characterizes the maximal ideals of an ideal in a commutative algebra over the rationals and so in a $C(X)$.

**Theorem 1.2.** *Let $\mathcal{A}$ be a commutative algebra over the rationals with unity and $K$ be an ideal of $\mathcal{A}$. Then an ideal $D$ of $K$ is a maximal ideal of $K$ if and only if $D = M \cap K$ for some maximal ideal $M$ in $\mathcal{A}$ with $K \nsubseteq M$.*

In $C(X)$ it is well known that the sum of every two prime (semiprime) ideals in $C(X)$ is a prime (semiprime) $z$-ideal, see [15]. Prime ideals in $C(X)$ containing a given prime ideal form a chain, see 14.3(c) in [9]. It is also easy to see that the product of two maximal ideals ($z$-ideals) coincide with their intersection, see 2D in [9]. Do these facts hold for prime and maximal ideals of ideals in $C(X)$? We answer this question in Section 3 and we observe in this section that the sum of every two prime (semiprime) ideals of each ideal in $C(X)$ is a prime (semiprime) ideal or all of the ideal if and only if $X$ is an $F$-space.

In Section 4, we have borrowed the concept of separability from [16] and we have generalized it to ideals of $C(X)$. Next, we have shown that the separability of an ideal $I$ of $C(X)$ is equivalent to dense separability (which is borrowed from [14]) of $X \setminus \Delta(I)$. The separability of $P$-ideals are also investigated in this section and it is shown that every $P$-ideal of $C(X)$ is separable if and only if the set $\mathbb{I}(X)$ has cardinality less than or equal to $\aleph_0$ and the set of non-$P$-points of $X$ is dense in $X \setminus \mathbb{I}(X)$, where $\mathbb{I}(X)$ is the set of all isolated points of $X$.

Finally, Section 5 is devoted to the Goldie dimension of ideals in $C(X)$. In this section, the coincidence of the Goldie dimension of an ideal $I$ in $C(X)$ and the cellularity of $X \setminus \Delta(I)$ is proved. We also observe in this section that the Goldie dimension of an ideal in $C(X)$ is finite if and only if it is a finite direct sum of minimal ideals of that ideal (note, every minimal ideal of an ideal in $C(X)$ is also a minimal ideal of $C(X)$, see Lemma 4.1).

In this paper, we denote by $\beta X$ the Stone-Čech compactification of a space $X$. For each $f \in C(X)$, $Z(f)$ is the set of zeros of $f$ and for every ideal $I$ in $C(X)$, $\Delta(I) = \bigcap_{f\in I} Z(f)$ and $\theta(I) = \bigcap_{f\in I} \mathrm{cl}_{\beta X} Z(f)$. For each prime ideal $P$ in $C(X)$, there exists a unique $x \in \beta X$ such that $O^x \subseteq P \subseteq M^x$, see Theorem 7.15 in [9], where $M^x = \{f \in C(X) : x \in \mathrm{cl}_{\beta X} Z(f)\}$ and $O^x = \{f \in C(X) : x \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)\}$. More generally, for each subset $A$ of $\beta X$, we note that $M^A = \{f \in C(X) : A \subseteq \mathrm{cl}_{\beta X} Z(f)\} = \bigcap_{x\in A} M^x$ and $O^A = \{f \in C(X) : A \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)\} = \bigcap_{x\in A} O^x$. Finally $\{M^x : x \in \beta X\}$ is the collection of all maximal ideals of $C(X)$ and whenever $x \in X$, then $M_x$ and $O_x$ denote $M^x$ and $O^x$ respectively. In fact, if $M_x = \{f \in C(X) : f(x) = 0\}$, then $\{M_x : x \in X\}$ is the set of all fixed maximal ideals of $C(X)$ (note, an ideal $I$ in $C(X)$ is called fixed if $\bigcap_{f\in I} Z(f) \neq \emptyset$, else free). A point $x \in X$ is said to be a $P$-point if $M_x = O_x$ and whenever every point of $X$ is a $P$-point, then $X$ is called a $P$- space. It is easy to see that $X$ is a $P$-space if and only if the zeroset $Z(f)$ for each $f \in C(X)$ is open, see 4J in [9].

If $E$ is an ideal of a ring $R$, then $E$ is called an essential ideal in $R$ if $E$ intersects every nonzero ideal of $R$ nontrivially. The socle of $R$ is the intersection of all essential ideals of $R$. The socle of $C(X)$ is denoted by

$C_F(X)$ and it is characterized in [11] by the set of all functions in $C(X)$ vanishing everywhere except on a finite number of points of $X$. An ideal $I$ in $C(X)$ is called a $z$-ideal if $Z(f) \subseteq Z(g)$, $f \in I$ and $g \in C(X)$ imply that $g \in I$. The socle of $C(X)$ is an example of a $z$-ideal in $C(X)$ and related to this ideal, we have the ideal $C_K(X)$ ($C_\psi(X)$) consisting of all continuous functions with compact (pseudocompact) support. It is well-known that $C_K(X) = O^{\beta X \setminus X}$ and $C_\psi(X) = M^{\beta X \setminus \upsilon X}$, where $\upsilon X$ is the real compactification of $X$, see [9] and [10] for more details. The support of $f \in C(X)$ is $\mathrm{cl}_X(X \setminus Z(f)$ and the reader is referred to [8, 9] and [22] for undefined terms and notations.

## 2. Ideals in $C(X)$ whose ideals are also ideals of $C(X)$

Whenever $I$ is an ideal in a ring $R$ and $J$ is an ideal of $I$, Then $J$ is not necessarily an ideal of the whole ring $R$, see an example in Section 5 of [12]. Using Theorems 1.1 and 1.2, we observe that all prime and all maximal ideals of a given ideal in $C(X)$ are also ideals of $C(X)$. In [13], it is also shown that for each ideal $I$ of $C(X)$, every ideal of $I$ is also an ideal of $C(X)$ if and only if $X$ is a $P-$space. Now it is natural to ask when every ideal of a given ideal $I$ in $C(X)$ is an ideal of $C(X)$. First we recall that an ideal $I$ of a ring $R$ is called pure if for each $f \in I$, there exists $g \in I$ such that $f = fg$. It is proved in [6] that such ideals are exactly of the form $O^A$, where $A$ is a closed subset of $\beta X$. In the following proposition, we answer our question which is also a new characterization for pure ideals.

**Proposition 2.1.** *Let $I$ be an ideal in $C(X)$. Then every ideal of $I$ is an ideal of $C(X)$ if and only if $I$ is a pure ideal.*

*Proof.* Let $I$ be a pure ideal and $J$ be an ideal of $I$. For $g \in C(X)$ and $j \in J$, it is enough to show that $gj \in J$. By our hypothesis, there exists $i \in I$ such that $j = ij$. Now we have $gj = gij = (gi)j \in J$. Conversely, suppose that every ideal of $I$ is an ideal in $C(X)$. Fix $p \in \theta(I)$, we are going to show $I \subseteq O^p$. Suppose on the contrary that $I \nsubseteq O^p$ and take $f \in (M^p \setminus O^p) \cap I$ (if $O^p = M^p$, then $I \subseteq M^p = O^p$). Without loss of generality, we consider $f$ to be bounded (in fact, whenever $f \in (M^p \setminus O^p) \cap I$, then $f\frac{1}{1+|f|} \in (M^p \setminus O^p) \cap I$ is bounded). Define

$$J = \{fg + nf : g \in M^p, n \in \mathbb{Z}\}.$$

Obviously $J$ is an ideal of $I$ and hence it is an ideal of $C(X)$ by our assumption. Since $f \in J$ and $J$ is an ideal of $C(X)$, $\frac{1}{2}f \in J$ and therefore there exist $n \in \mathbb{Z}$, $g \in M^p$ such that

$$\frac{1}{2}f = fg + nf$$

and hence $f(g + n - \frac{1}{2}) = 0$. Since $f \in M^p \setminus O^p$, we may consider a net $(x_\lambda)_{\lambda \in \Lambda}$ such that $f(x_\lambda) \neq 0$, $\forall \lambda \in \Lambda$ and $x_\lambda \to p$. Hence we have $g(x_\lambda) = \frac{1}{2} - n, \forall \lambda \in \Lambda$ and therefore $g^*(x_\lambda) = \frac{1}{2} - n, \forall \lambda \in \Lambda$, where $g^*$ is an Stone extension of $g$ to the one-point compactification of $\mathbb{R}$, defined in 7.5 of [9]. This means that $g^*(p) \neq 0$ which contradicts $g \in M^p$. Thus, we have shown that $I \subseteq O^p$, $\forall p \in \theta(I)$, and by Theorem 1.3 in [7], we observe that $I = O^{\theta(I)}$, i.e., $I$ is a pure ideal. Theorem 1.3 in [7] is due to Mc Knight and states that whenever $J$ is an ideal of $C(X)$ and $A = \bigcap_{f \in J} \mathrm{cl}_{\beta X} Z(f)$, then $O^A \subseteq J \subseteq M^A$. □

The above theorem is not true in arbitrary commutative rings. For example, if we consider the ring of integers $\mathbb{Z}$, then obviously every ideal of each ideal in $\mathbb{Z}$ is also an ideal of the whole ring $\mathbb{Z}$, but the only pure ideal of $\mathbb{Z}$ is (0).

A nonzero ideal $I$ is said to be a $P$-ideal, if every proper prime ideal of $I$ is maximal in $I$. These ideals are first introduced and studied in $C(X)$ by D. Rudd in [20]. In Theorem 1.5 of the same reference, it is shown that an ideal $I$ in $C(X)$ is a $P$-ideal if and only if $Z(f)$ is open for each $f \in I$. Using this, it is manifest, to see that an ideal $I$ in $C(X)$ is a $P$-ideal if and only if it is pure and every point of $X \setminus \Delta(I)$ is a $P$-point. The following proposition gives a new characterization for $P$-ideals in $C(X)$.

**Proposition 2.2.** *Let $I$ be an ideal in $C(X)$. Then every ideal of $I$ is a $z$-ideal of $C(X)$ if and only if $I$ is a $P$-ideal.*

*Proof.* If $I$ is a $P$-ideal, then it is pure and by Proposition 2.1, every ideal of $I$ is an ideal of $C(X)$. Now, suppose that $J$ is an ideal of $I$, $f \in J$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$. Since $Z(f)$ is open, $g$ is a multiple of $f$ by 1D in [9], hence $g \in J$, i.e., $J$ is a $z$-ideal of $C(X)$. Conversely, let every ideal of $I$ be a $z$-ideal of $C(X)$ and $f \in I$. We show that $Z(f)$ is open. By Proposition 2.1, $I$ should be a pure ideal, i.e., there exists $i \in I$ such that $f = if$. This shows that $f \in fI$. Since $fI$ is an ideal of $I$, it is a $z$-ideal in $C(X)$, by our hypothesis. Now $Z(f^{\frac{1}{3}}) = Z(f)$ implies that $f^{\frac{1}{3}} = fj$ for some $j \in I$. Thus $f^{\frac{1}{3}}(1 - f^{\frac{2}{3}}j) = 0$, hence $Z(f) \cup Z(1 - f^{\frac{2}{3}}j) = X$. But $Z(f) \cap Z(1 - f^{\frac{2}{3}}j) = \emptyset$ implies that $Z(f)$ is open. □

**Remark 2.3.** *Whenever $x \in \beta X$ and $P$ is a prime ideal in $C(X)$ containing $O^x$, then by Proposition 2.1, every ideal of $P$ is an ideal of $C(X)$ if and only if $P$ is a pure ideal, i.e., if and only if $O^x = P$ or equivalently $x$ is an $F$-point. This implies that every ideal of each prime ideal of $C(X)$ is an ideal of $C(X)$ if and only if $X$ is an $F$-space. Using Proposition 2.2, if $M$ is a maximal ideal of $C(X)$ whose every ideal is a $z$-ideal of $C(X)$, then $M$ is a $P$-ideal. It is easy to see that whenever $C(X)$ has a maximal $P$-ideal, then $X$ is a $P$-space. So, there exists a maximal ideal in $C(X)$ whose every ideal is a $z$-ideal of $C(X)$ if and only if $X$ is a $P$-space.*

Since $Z(f)$, for each $f \in C_F(X)$ is open, by its characterization in [11], the socle $C_F(X)$ of $C(X)$ is a $P$-ideal, see Theorem 1.5 in [20]. But what about the other familiar ideals such as $C_K(X)$ and $C_\psi(X)$ which are intersections of a set of pure ideals and a set of maximal ideals respectively? Next, we answer these questions, but first we give the following lemma which is needed in the sequel.

**Lemma 2.4.** *Let $\mathbb{I}(X)$ be the set of isolated points of $X$, then $C_F(X) = O^{\beta X \setminus \mathbb{I}(X)}$.*

*Proof.* Since $C_F(X)$ is a $P$-ideal, it is pure and hence $C_F(X) = O^A$ for some closed subset $A$ of $\beta X$. But $A$ is compact, then $\theta(O^A) = A$, and it is enough to show that $A = \beta X \setminus \mathbb{I}(X)$. If $f \in C_F(X)$, then $Z(f) = X \setminus \{x_1, ..., x_n\}$, where $x_1, ..., x_n \in \mathbb{I}(X)$. Hence $\mathrm{cl}_{\beta X} Z(f) = \beta X \setminus \{x_1, ..., x_n\}$ and consequently

$$A = \theta(C_F(X)) = \bigcap_{f \in C_F(X)} \mathrm{cl}_{\beta X} Z(f) = \beta X \setminus \mathbb{I}(X).$$

Therefore $C_F(X) = O^{\beta X \setminus \mathbb{I}(X)}$. $\square$

The ideals $C_K(X)$ and $C_\psi(X)$ may be $P$-ideals. The following proposition states that every ideal of $C_\psi(X)$ is a $z$-ideal of $C(X)$ if and only if $X$ is pseudodiscrete, $\psi-$compact. We recall that a space $X$ is said to be pseudodiscrete if every compact subset of $X$ has finite interior and $X$ is $\psi-$compact if $C_K(X) = C_\psi(X)$. These concepts are first introduced in [3] and [10] respectively.

**Proposition 2.5.** *(i) $C_K(X)$ is a $P$-ideal if and only if $X$ is pseudodiscrete.*
*(ii) $C_\psi(X)$ is a $P$-ideal if and only if $X$ is pseudodiscrete, $\psi-$compact.*

*Moreover, if $C_K(X)(C_\psi(X))$ is a $P$-ideal, then $C_K(X)(C_\psi(X))$ and $C_F(X)$ coincide.*

*Proof.* (i) In Theorem 4.5 in [3], it is shown that $X$ is pseudodiscrete if and only if $C_K(X) = C_F(X)$. Hence it is sufficient to show that $C_K(X)$ is a $P$-ideal if and only if $C_K(X) = C_F(X)$. Clearly the coincidence of $C_K(X)$ and $C_F(X)$ implies that $C_K(X)$ is a $P$-ideal. Now suppose that $C_K(X)$ is a $P$-ideal. Then for each $f \in C_K(X)$, we have $\mathrm{cl}_X(X \setminus Z(f)) \subseteq X \setminus \Delta(C_K(X))$, by purity of $C_K(X)$. Since all points of X $\setminus \Delta(C_K(X))$ are $P$-points, $\mathrm{cl}_X(X \setminus Z(f))$ is a compact $P$-space and hence $X \setminus Z(f)$ is finite by 4k in [9]. Thus $C_K(X) \subseteq C_F(X)$ and we are done.

(ii) By part (i) and the definition of a $\psi-$compact space, whenever $X$ is pseudodiscrete and $\psi-$compact, then $C_\psi(X) = C_F(X)$ is a $P$-ideal. Conversely, suppose that $C_\psi(X)$ is a $P$-ideal. Using a similar argument and by purity of $C_\psi(X)$, for each $f \in C_\psi(X)$, $\mathrm{cl}_X(X \setminus Z(f))$ is contained in $X \setminus \Delta(C_\psi(X))$ which should be a pseudocompact $P$-space and hence it is finite, again by 4k in [9]. We have thus shown that $C_\psi(X) \subseteq C_F(X)$, which implies that $C_K(X) = C_F(X)$, i.e., $X$ is pseudodiscrete $\psi-$compact. □

Since $C_\psi(X)$ is the intersection of all hyper-real maximal ideals and every isomorphism takes hyper-real maximal ideals to hyper-real maximal ideals, the following result is an immediate corollary of Proposition 2.5.

**Corollary 2.6.** *(i) If $C(X) \cong C(Y)$ and $X$ is a pseudodiscrete, $\psi$-compact space, then $Y$ is too.*
*(ii) $\upsilon X$ is pseudodiscrete if and only if $X$ is $\psi-$compact, pseudodiscrete.*

In the second part of the above corollary, $\psi-$compactness of $X$ is needed. For an example of a pseudodiscrete space $X$ such that $\upsilon X$ is not pseudodiscrete, see example 2 in [21].

The following proposition characterizes $\mathrm{cl}_u C_F(X)$, the closure of $C_F(X)$ with respect to the uniform norm topology on $C^*(X)$, see [8,22] and Exercise 2M in [9] for definition and properties of the uniform norm topology on $C^*(X)$. First for each $f \in C(X)$ and each $\epsilon > 0$, we consider $A_f^\epsilon = \{x \in X : |f(x)| > \epsilon\}$ and define

$$C_F^\infty(X) = \{f \in C(X) : A_f^\epsilon \text{ is finite, } \forall \epsilon > 0\}.$$

**Corollary 2.7.** *$cl_u C_F(X) = M^{*\beta X \setminus \mathbb{I}(X)} = C_F^\infty(X)$.*

*Proof.* The first equality is evident. For the second equality, let $f \in C_F^\infty(X) \subseteq C^*(X)$. Hence, for each $\epsilon > 0$, $A_f^\epsilon$ is finite set consisting entirely of isolated points, say $\{x_1, ..., x_n\}$. Hence for all $x \in \beta X \setminus \{x_1, ..., x_n\}$, we have $|f^\beta(x)| \leq \epsilon$. This implies that $f^\beta(\beta X \setminus \mathbb{I}(X)) = 0$, i.e., $f \in M^{*\beta X \setminus \mathbb{I}(X)}$. For the reverse inclusion, suppose that $f \in M^{*\beta X \setminus \mathbb{I}(X)}$, i.e., $\beta X \setminus \mathbb{I}(X) \subseteq Z(f^\beta)$. If $A_f^\epsilon$ is infinite for some $\epsilon > 0$, then $A_f^\epsilon$ has a cluster point $x$ in $\beta X \setminus \mathbb{I}(X)$ which implies that $|f^\beta(x)| \geq \epsilon$ and this contradicts $f^\beta(x) = 0$. □

In [1], the concept of a $P_\infty$-space is defined as a space $X$ in which $Z(f)$ is open for each $f \in C_\infty(X)$. $C_\infty(X)$ is the subring of $C(X)$ consisting of all continuous functions that vanish at infinity, i.e., consisting of all $f \in C(X)$ such that the set $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact in $X$, $\forall n \in \mathbb{N}$. In that article, the regularity of $C_\infty(X)$ is investigated as a ring and it is shown that $C_\infty(X)$ is regular if and only if every open locally compact $\sigma$-compact subset of $X$ is compact. Using our Corollary 2.7, we conclude this section by the fact that $C_\infty(X)$ is regular if and only if every open locally compact $\sigma$-compact subset of $X$ is (in fact) finite. First we need the following result which gives another characterization for $P_\infty$-spaces.

**Proposition 2.8.** *A space $X$ is a $P_\infty$-space if and only if $X$ is pseudodiscrete and every countable subset of $X$ consisting entirely of isolated points is closed.*

*Proof.* Let $X$ be a $P_\infty$-space, i.e., $Z(f)$ is open for each $f \in C_\infty(X)$. Since $C_K(X) \subseteq C_\infty(X)$, $C_K(X)$ should be a $P$-ideal and Proposition 2.5(i) yields that $X$ is pseudodiscrete, i.e., $C_K(X) = C_F(X)$. Now, suppose that $\{p_1, \ldots, p_n, \ldots\}$ is a sequence of isolated points and define

$$f(x) = \begin{cases} \frac{1}{n} & x = p_n,\ n \in \mathbb{N}, \\ 0 & x \neq p_n. \end{cases}$$

Clearly $f$ is continuous, in fact $f \in C_\infty(X)$. But $X$ is a $P_\infty$-space, then $Z(f) = X \setminus \{p_1, \ldots, p_n, \ldots\}$ is open and therefore $\{p_1, \ldots, p_n, \ldots\}$ is closed. Conversely, suppose that $X$ is pseudodiscrete and every countable set of isolated points in $X$ is closed. Hence we have $C_K(X) = C_F(X)$ and by Corollary 2.7 and Exercise 24A(2) in [22], $C_\infty(X) = C_F^\infty(X)$. So it suffices to show that $Z(f)$ is open for each $f \in C_F^\infty(X)$. By definition of $C_F^\infty(X)$, for each $\epsilon > 0$, $A_f^\epsilon$ is a finite set of isolated points and hence $X \setminus Z(f) = \bigcup_{n=1}^\infty A_f^{\frac{1}{n}}$ is a countable set of isolated points which should be closed by our hypothesis. Hence $Z(f)$ is open and we are through. □

We recall that $\infty$-compact spaces are those spaces $X$ for which $C_\infty(X) = C_K(X)$. In [1], it is shown that $C_\infty(X)$ is a regular ring if and only if $X$ is an $\infty$-compact, $P_\infty$-space.

**Corollary 2.9.** *The following statements are equivalent.*
*(i) $C_\infty(X)$ is a regular ring.*
*(ii) Every open locally compact $\sigma$-compact subset of $X$ is finite.*
*(iii) $C_\infty(X)$ is finite direct sum of minimal ideals of $C(X)$.*

*Proof.* (i)⇒(ii) By Theorem 4.1 in [1], whenever $C_\infty(X)$ is regular, then every open locally compact $\sigma$-compact subset of $X$ is compact and by the same theorem, $X$ is also a $P_\infty$-space. Now using our Proposition 2.8, every open locally compact $\sigma$-compact subset of $X$ is finite.

(ii)⇒(iii) By Theorem 4.1 in [1], part (ii) implies that $X$ is $\infty$-compact, i.e., $C_K(X) = C_\infty(X)$ and $X$ is also a $P_\infty$-space which implies that $X$ is pseudodiscrete by our Proposition 2.8, i.e., $C_K(X) = C_F(X)$. Therefore, it is enough to show that $\mathbb{I}(X)$ is finite. If $\mathbb{I}(X)$ is infinite, then every countable subset of $\mathbb{I}(X)$ is an open locally compact $\sigma$-compact set and hence it must be finite by (ii), a contradiction. This implies that $C_F(X) = C_\infty(X)$ is a finite direct sum of minimal ideals of $C(X)$.

(iii)⇒(i) Part (iii) implies that $C_\infty(X) = C_K(X) = C_F(X)$, i.e., $X$ is an $\infty$-compact pseudodiscrete space. On the other hand, Since $C_F(X)$ is finite direct sum of minimal ideals of $C(X)$, the set of isolated points of $X$ is finite and hence $X$ should be a $P_\infty$-space by Proposition 2.8. Now Theorem 4.1 in [1] implies that $C_\infty(X)$ is a regular ring. We note that Theorem 4.1 in [1] states that $C_\infty(X)$ is a regular ring if and only $X$is an $\infty$-compact $P_\infty$-space. □

## 3. $z$-ideals, prime ideals and semiprime ideals of ideals

It is well-known that the sum of every two prime (semiprime) ideals of $C(X)$ is either a prime (semiprime) ideal or all of $C(X)$, see [15]. This fact may not happen for prime ideals of an ideal. In this section we show that in each ideal of $C(X)$, the sum of every two prime (semiprime) ideals is a prime (semiprime) ideal if and only if $X$ is an $F$-space. First we need the following lemma.

**Lemma 3.1.** *Let $P, Q, T$ and $I$ be ideals in $C(X)$. If $P, Q$ and $T$ are prime ideals in $C(X)$ and $P \cap I + Q \cap I = T \cap I$, then $T \cap I = (P + Q) \cap I$.*

*Proof.* Clearly $P\cap I+Q\cap I\subseteq (P+Q)\cap I$. For the reverse inclusion, first suppose that $I\subseteq T$, then $I=P\cap I+Q\cap I\subseteq P+Q$ and trivially $(P+Q)\cap I=I=P\cap I+Q\cap I$. Next suppose that $I\nsubseteq T$. Pick $i\in I\setminus T$ and $p\in P$. Hence $pi\in I\cap P\subseteq I\cap T$ implies that $p\in T$, i.e., $P\subseteq T$. Similarly we have $Q\subseteq T$, hence $P+Q\subseteq T$ and we are through. □

**Theorem 3.2.** *The following statements are equivalent.*
*(i) $X$ is an $F$-space.*
*(ii) In each ideal of $C(X)$, the sum of every two semiprime ideals is a semiprime ideal or all of the ideal.*
*(iii) In each ideal of $C(X)$, the sum of every two prime ideals is a prime ideal or all of the ideal.*

*Proof.* If $X$ is an $F$-space, then every ideal in $C(X)$ is absolutely convex and the equality $I\cap(J+K)=I\cap J+I\cap K$ holds for every ideals $I,J$ and $K$ in $C(X)$. Hence, it is enough to show that part (iii) implies part (i). To see this, it suffices to show that given $p\in\beta X$, prime ideals containing $O^p$ form a chain, see Theorem 14.25 in [9]. Let $P$ and $Q$ be two prime ideals in $C(X)$ containing $O^p$, and neither contains the other. Take $f\in P+Q$, but $f\notin P$ and $f\notin Q$. Now by our hypothesis and Lemma 3.1, we have $(f)\cap P+(f)\cap Q=(f)\cap(P+Q)$. But $f\in P+Q$ implies that $(f)\cap P+(f)\cap Q=(f)$. Hence $f\in(f)\cap P+(f)\cap Q$, i.e., $f=uf+vf$ for some $u\in P$ and $v\in Q$. Now $f(1-u-v)=0$ implies that $1-u-v\in P$, so $1-v\in P$ and hence $1\in P+Q$, a contradiction, for $P+Q\subseteq M^p$ is a proper ideal. □

It is also well-known that prime ideals in $C(X)$ containing a given prime ideal form a chain. In the following proposition we observe that prime ideals of an ideal in $C(X)$ also have this property.

**Proposition 3.3.** *Let $P,Q$ and $T$ be prime ideals of an ideal $I$ in $C(X)$. If $T\subseteq P$ and $T\subseteq Q$, then $P$ and $Q$ are comparable.*

*Proof.* By Theorem 5.1 of [12], there are prime ideals $P^*,Q^*$ and $T^*$ in $C(X)$ such that $P=I\cap P^*,Q=I\cap Q^*$ and $T=I\cap T^*$. There are only four cases:

Case 1. $I\subseteq P^*$ or $I\subseteq Q^*$.
Case 2. $T^*\subseteq P^*$ and $T^*\subseteq Q^*$.
Case 3. $I\nsubseteq P^*$, $I\nsubseteq Q^*$ and $T^*\nsubseteq P^*$.
Case 4. $I\nsubseteq P^*$, $I\nsubseteq Q^*$ and $T^*\nsubseteq Q^*$.

Case 3 does not happen, for we can take $a\in T^*\setminus P^*$ and $b\in I\setminus P^*$. Therefore $ab\in T^*\cap I\subseteq P^*\cap I$ implies that $ab\in P^*$, a contradiction.

Similarly case 4 also does not happen. Hence we have only cases 1 and 2. In case 1, clearly $P$ and $Q$ are comparable and in case 2, $P^*$ and $Q^*$, and consequently $P$ and $Q$ are comparable. □

Whenever $I$ and $J$ are two semiprime ideals in $C(X)$, then we have $IJ = I \cap J$. The following proposition shows that this is not true even for two maximal ideals of an ideal in $C(X)$.

**Proposition 3.4.** *For each ideal $I$ in $C(X)$ and every two maximal ideals $M_1$ and $M_2$ of $I$, we have $M_1 \cap M_2 = M_1 M_2$ if and only if $X$ is a $P$-space.*

*Proof.* Let $f \in C(X)$, we will show that $Z(f)$ is an open set. If $X \setminus Z(f)$ is empty or singleton, then clearly $Z(f)$ is open. Now suppose that $x, y \in X \setminus Z(f)$, consider $g \in C(X)$ such that $Z(f) \cap Z(g) = \emptyset$ and $x, y \in Z(g)$. Since $x, y \notin Z(f)$, $(f) \cap M_x$ and $(f) \cap M_y$ are two maximal ideals in $(f)$, by Theorem 1.2. Now by our hypothesis, $((f) \cap M_x)((f) \cap M_y) = (f) \cap M_x \cap M_y$. But $fg \in (f) \cap M_x \cap M_y = ((f) \cap M_x)((f) \cap M_y)$, then $fg = f^2 ts$ for some $t \in M_x$ and $s \in M_y$. Now we have $f(g - fts) = 0$ and hence $Z(f) \cup Z(g - fts) = X$. Moreover, if $f(u) = 0$, then $(g - fts)(u) = g(u) \neq 0$, for $Z(f) \cap Z(g) = \emptyset$. This means that $Z(f) \cap Z(g - fts) = \emptyset$ and therefore $Z(f)$ is open. Whenever $X$ is a $P$-space, the proof is clear. □

Motivated by the definition of $z$-ideals in $C(X)$, whenever $J$ is an ideal of $C(X)$ and $I$ is an ideal of $J$, we call $I$ a $z$-ideal of $J$ if $Z(f) \subseteq Z(g)$, $f \in I$ and $g \in J$ imply that $g \in I$. If for each $f \in C(X)$, we denote the intersection of all maximal ideals of $C(X)$ containing $f$, by $M_f$, then we have $M_f = \{g \in C(X) : Z(f) \subseteq Z(g)\}$, see [6]. Using this notation, it is evident that $I$ is a $z$-ideal of $J$ if and only if $M_f \cap J \subseteq I, \ \forall f \in I$. Whenever $I$ is an ideal of an ideal $J$ in $C(X)$ and $I$ is also a $z$-ideal of $J$, then $I$ is also an ideal of $C(X)$. In fact, if $f \in I$ and $g \in C(X)$, then $fg \in J$ and $Z(f) \subseteq Z(fg)$ implies that $fg \in I$, i.e., $I$ is an ideal of $C(X)$. Hence our definition coincides with the definition mentioned in [7], i.e., if $I$ is an ideal of $J$, then $I$ is a $z$-ideal of $J$ if and only if $I$ is a $z_J$-ideal, see [7] for definition of $z_J$-ideal and some properties of such ideals. Now by this coincidence, we immediately have the following result from [7].

**Proposition 3.5.** *For each ideal $J$ of $C(X)$, the sum of every two $z$-ideals of $J$ is a $z$-ideal of $J$ if and only if $X$ is an $F$-space.*

It is well-known that every $z$-ideal in $C(X)$ containing a prime ideal is prime, see Theorem 2.9 in [9]. This fact is also true if we consider an ideal of $C(X)$ instead of $C(X)$. The following proposition is a counterpart of Theorem 2.9 in [9] and its proof is more or less the same as that of Theorem 2.9 in [9].

**Proposition 3.6.** *Let $J$ be an ideal of $C(X)$ and $I$ be a $z$-ideal of $J$. Then, the following statements are equivalent.*
*(a) $I$ is a prime ideal of $J$.*
*(b) $I$ contains a prime ideal of $J$.*
*(c) For all $g, h \in J$, if $gh = 0$, then $g \in I$ or $h \in I$.*
*(d) For each $f \in J$, there is a zero-set in $Z[I]$ on which $f$ does not change sign.*

*Proof.* Clearly part (a) implies part (b). If (b) holds, then there exists a prime ideal $P$ in $C(X)$ such that $P \cap J \subseteq I$. Now if $gh = 0$ and $g, h \in J$, then $g \in P \cap J$ or $h \in P \cap J$ and hence (c) holds. Part (c) also implies part (d), in fact $f \in J$ implies that $f(f \vee 0), f(f \wedge 0) \in J$ and $f^2(f \vee 0)(f \wedge 0) = 0$. Hence, either $g = f(f \vee 0) \in I$ or $h = f(f \wedge 0) \in I$, so $f$ does not change sign on $Z(g)$ or on $Z(h)$. Finally, to prove (d) implies (a), let $g, h \in J$ and $gh \in I$. Consider the function $g^2 - h^2 \in J$. By hypothesis, there is a zero-set $Z(i)$, $i \in I$ on which $g^2 - h^2$ is nonnegative, say. Then $Z(g) \cap Z(i) \subseteq Z(h)$ and hence $Z(h) \supseteq Z(g) \cap Z(i) = Z(gh) \cap Z(i) = Z(g^2h^2 + i^2)$, so that $h \in I$, for $g^2h^2 + i^2 \in I$ and $I$ is a $z$-ideal of $J$. Thus $I$ is a prime ideal of $J$. □

**Corollary 3.7.** *Let $J$ be an ideal of $C(X)$ and $I$ be a $z$-ideal of $J$. If $P$ is a prime ideal in $C(X)$ and $P \cap J \subseteq I$, then there exists a prime ideal $Q$ in $C(X)$ such that $I = Q \cap J$.*

## 4. Separability of ideals in $C(X)$ vs. dense separability of subspaces of $X$

An ideal $I$ of a ring $R$ is called separable if for each family $\{M_s\}_{s \in S}$ of maximal ideals of $I$ with $\bigcap_{s \in S} M_s = (0)$, there exists a countable subset $F$ of $S$ such that $\bigcap_{s \in F} M_s = (0)$. This concept is first introduced and studied in [16]. An ideal $I$ is said to be strongly separable if for each family $\{I_s\}_{s \in S}$ of ideals of $I$ with $\bigcap_{s \in S} I_s = (0)$, there is a countable subset $F$ of $S$ such that $\bigcap_{s \in F} I_s = (0)$. Clearly every strongly separable ideal is separable but not conversely, see the introduction of [16]. In the following proposition, we show that these two concepts for an ideal $I$ of $C(X)$ coincide with dense separability of $\beta X \setminus \theta(I)$.

By a dense separable space, we call a space in which every dense subset is separable. Dense separable spaces are introduced and studied in [14]. It is clear that every dense separable space is separable, but not conversely, see [14]. The sorgenfery line, $\beta\mathbb{Q}$ and $\beta\mathbb{Q}\setminus\mathbb{Q}$ are examples of dense separable spaces, see also [14]. Before giving the proposition, we need the following lemma which states that every ideal of an ideal in $C(X)$ contains an ideal of $C(X)$.

**Lemma 4.1.** *Let $J$ be an ideal of $C(X)$. For each ideal $I$ of $J$, we have $O^A \subseteq I \subseteq M^A \cap J$, where $A = \bigcap_{f\in I} cl_{\beta X} Z(f)$.*

*Proof.* Suppose that $g \in O^A$, i.e., $A \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(g)$. The compactness of $A$ in $\beta X$ implies that there are $f_1, ..., f_n \in I$ such that

$$\bigcap_{k=1}^{n} \mathrm{cl}_{\beta X} Z(f_i) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(g).$$

Now using 7.14 in [9], there exists $h \in C(X)$ such that

$$Z(f_1^2 + ... + f_n^2) \subseteq X \setminus Z(h) \subseteq Z(g).$$

This implies that $g$ is a multiple of $f_1^2 + ... + f_n^2$, by 1D in [9], i.e., there exists $k \in C(X)$ such that $g = k(f_1^2 + ... + f_n^2)$. For each $i = 1, 2, ..., n$, we have $kf_i^2 = (kf_i)f_i \in I$, for $kf_i \in J$. Therefore $g \in I$, i.e., $O^A \subseteq I$. The inclusion $I \subseteq M^A \cap J$ is evident by Theorem 1.3 in [7]. □

**Proposition 4.2.** *Let $I$ be an ideal of $C(X)$. Then the following statements are equivalent.*
*(i) $I$ is a separable ideal.*
*(ii) $I$ is a strongly separable ideal.*
*(iii) $\beta X \setminus \theta(I)$ is a dense separable subspace of $\beta X$*

*Proof.* (i)⇒(ii) Let $\{I_s\}_{s\in S}$ be an arbitrary family of ideals of $I$ with $\bigcap_{s\in S} I_s = (0)$. By Lemma 4.1, for each $s \in S$, $O^{A_s} \subseteq I_s \subseteq M^{A_s} \cap I$, where $A_s = \bigcap_{f\in I_s} \mathrm{cl}_{\beta X} Z(f)$. Hence we have

$$\bigcap_{s\in S} O^{A_s} \subseteq \bigcap_{s\in S} I_s \subseteq \bigcap_{s\in S} (M^{A_s} \cap I).$$

Now $\bigcap_{s\in S} I_s = (0)$ implies that $\bigcap_{s\in S} O^{A_s} = (0)$ and hence $O^{\cup_{s\in S} A_s} = (0)$. This means that $\cup_{s\in S} A_s$ is dense in $\beta X$, so $M^{\cup_{s\in S} A_s} = (0)$ and therefore

$$\bigcap_{a\in \cup_{s\in S} A_s} (M^a \cap I) = \bigcap_{s\in S} (M^{A_s} \cap I) = M^{\cup_{s\in S} A_s} \cap I = (0).$$

Now by part (i), there exists a countable set $F \subseteq \bigcup_{s\in S} A_s$ such that $\bigcap_{a\in F} M^a \cap I = (0)$. Letting $F = \{a_1, ..., a_n, ...\}$, for each $n \in \mathbb{N}$, there exists $s_n \in S$ such that $a_n \in A_{s_n}$. Hence $I_{s_n} \subseteq M^{A_{s_n}} \cap I \subseteq M^{a_n} \cap I$ which implies that $\bigcap_{n=1}^{\infty} I_{s_n} = \bigcap_{n=1}^{\infty}(M^{a_n} \cap I) = (0)$, i.e., $I$ is strongly separable.

(ii)$\Rightarrow$(i) Evident.

(i)$\Rightarrow$(iii) Let $A$ be dense in $\beta X \setminus \theta(I)$. We show that $A$ has a countable subset which is dense in $\beta X \setminus \theta(I)$. Since $S = A \cup \theta(I)$ is dense in $\beta X$, we have $M^S = (0)$ and hence $M^S \cap I = (0)$. But $I$ is separable, hence there exists a countable subset $F$ of $S$ such that $M^F \cap I = (0)$. Now $O^{F\cup\theta(I)} \subseteq M^F \cap I = (0)$ implies that $F \cup \theta(I)$ is dense in $\beta X$. This means that $F$ is dense in $\beta X \setminus \theta(I)$, for $\beta X \setminus \theta(I)$ is open in $\beta X$. Therefore $\beta X \setminus \theta(I)$ is dense separable.

(iii)$\Rightarrow$(i) Let $M^A \cap I = (0)$, where $A \subseteq \beta X \setminus \theta(I)$. It follows that $A \cup \theta(I)$ is dense in $\beta X$ and $A$ will be dense in $\beta X \setminus \theta(I)$, for $\beta X \setminus \theta(I)$ is open in $\beta X$. Now using dense separability of $\beta X \setminus \theta(I)$, there is a countable subset $F$ of $A$ which is dense in $\beta X \setminus \theta(I)$. Hence $M^{F\cup\theta(I)} = (0)$ implies that $M^F \cap I = (0)$, i.e., $I$ is separable. □

**Corollary 4.3.** *$C(X)$ has a separable non maximal prime ideal if and only if $X$ is not a countable discrete space and $\beta X$ is dense separable.*

*Proof.* If $P$ is a separable non-maximal prime ideal, then $P \subseteq M^x$ for some non isolated point $x$. This implies that $X$ is not a countable discrete space. Now by Proposition 4.2, $P$ is separable if and only if $\beta X \setminus \{x\}$ is dense separable and this is equivalent to saying that $\beta X$ is dense separable. The converse is obvious by Proposition 4.2. □

For an ideal $I$ in $C(X)$, we have $\Delta(I) = \theta(I) \cap X$ and hence $X \setminus \Delta(I) = (\beta X \setminus \theta(I)) \cap X$. Since $X$ is dense and $\beta X \setminus \theta(I)$ is open in $\beta X$, $\mathrm{cl}_{\beta X}(\beta X \setminus \theta(I)) = \mathrm{cl}_{\beta X}(X \setminus \Delta(I))$. This immediately shows that whenever $\beta X \setminus \theta(I)$ is dense separable, then $X \setminus \Delta(I)$ is too. Now using this argument, we prove the following result.

**Proposition 4.4.** *A $P$-ideal $I$ in $C(X)$ is separable if and only if $X \setminus \Delta(I)$ is a countable set consisting entirely of isolated points.*

*Proof.* Let $I$ be a separable $P$-ideal in $C(X)$. By the argument above, $X \setminus \Delta(I)$ is dense in $\beta X \setminus \theta(I)$ and by Proposition 4.2, $\beta X \setminus \theta(I)$ is dense separable. This yields that $X \setminus \Delta(I)$ has a countable dense subset. But $I$ is a $P$-ideal, hence $X \setminus \Delta(I)$ is a $P$-space. Since countable subsets of a $P$-space are closed and discrete, see 4K in [9], $X \setminus \Delta(I)$ will be a countable discrete subspace. On the other hand, $X \setminus \Delta(I)$ is open, so every point of $X \setminus \Delta(I)$ is an isolated point. Conversely, let $X \setminus \Delta(I)$ be a countable set consisting entirely of isolated points. If $A$ is a dense subset of $\beta X \setminus \theta(I)$, then it must contain $X \setminus \Delta(I)$, i.e., $\beta X \setminus \theta(I)$ is dense separable. Now by Proposition 4.2. $I$ is separable. $\square$

Using Proposition 2.1 in [2] and our Proposition 4.4, the following result is evident.

**Corollary 4.5.** *$C(X)$ has an essential separable $P$-ideal if and only if $X$ contains a countable set of isolated points dense in $X$.*

**Proposition 4.6.** *Every ideal of $C(X)$ is a separable $P$-ideal if and only if $X$ is a countable discrete space.*

*Proof.* If $X$ is a countable discrete space and $I$ is an ideal of $C(X)$, then $X \setminus \Delta(I)$ is a countable set consisting entirely of isolated points and hence $I$ is a separable $P$-ideal, by Proposition 4.4. Conversely, suppose that $x \in X$. By our hypothesis, $M_x$ is a separable $P$-ideal and hence $X \setminus \Delta(I) = X \setminus \{x\}$ is countable and consisting entirely of isolated points, by Proposition 4.4. This implies that $\{x\}$ is a zero set in $Z[M_x]$. Similarly, since $M_y$ for $y \neq x$ is also a $P$-ideal, $X \setminus \{y\}$ is a $P$-space. But $\{x\} \subseteq X \setminus \{y\}$ is a zeroset, then $\{x\}$ is open, i.e., $x$ should be an isolated point. Therefore $X$ is a countable discrete space. $\square$

We know that $C_F(X)$ is a $P$-ideal and $X \setminus \Delta(C_F(X)) = \mathbb{I}(X)$. Thus, using Proposition 4.4, $C_F(X)$ is separable if and only if $|\mathbb{I}(X)| \leq \aleph_0$. This proves the first part of the following result.

**Proposition 4.7.** *Every $P$-ideal of $C(X)$ is separable if and only if $|\mathbb{I}(X)| \leq \aleph_0$ and the set of non-$P$-points of $X$ is dense in $X \setminus \mathbb{I}(X)$.*

*Proof.* Let every $P$-ideal of $C(X)$ be separable. Then $C_F(X)$ is separable and hence $|\mathbb{I}(X)| \leq \aleph_0$ by the preceding argument. Now suppose that $U$ is an open set in $X$ all of whose elements are $P$-points and contains at least one non-isolated point. Hence $\mathrm{cl}_{\beta X}(X \setminus U) \neq \beta X$, so the ideal $I = O^{\mathrm{cl}_{\beta X}(X \setminus U)}$ is a proper $P$-ideal. In fact $I$ is pure and $X \setminus \Delta(I) = U$ is a $P$-space. This contradicts Proposition 4.4. Conversely suppose that $|\mathbb{I}(X)| \leq \aleph_0$ and the set of non-$P$-points of $X$ is dense in $X \setminus \mathbb{I}(X)$. Let $I$ be a $P$-ideal in $C(X)$. Since $X \setminus \Delta(I)$ is an open set consisting entirely of $P$-points, then $(X \setminus \Delta(I)) \cap (X \setminus \mathbb{I}(X)) = \emptyset$, by our hypothesis. This implies that $X \setminus \Delta(I) \subseteq \mathbb{I}(X)$ and hence $I$ is separable by Proposition 4.4. $\square$

The following corollary is an immediate consequence of Proposition 4.7.

**Corollary 4.8.** *Let $X$ be a space without any isolated point. Then every $P$-ideal of $C(X)$ is separable if and only if the set of non-$P$-points of $X$ is dense in $X$.*

To prove the final result of this section which topologically characterizes the separability of the ideals $C_K(X)$ and $C_\psi(X)$, we need the following lemmas. First we denote $\mathcal{L}(X)$ by the set of all points of $X$ which have compact neighborhoods.

**Lemma 4.9.** *Let $I$ be an ideal of $C(X)$. Whenever $X \setminus \Delta(I)$ is locally compact dense separable, then $\beta X \setminus \theta(I)$ is also dense separable. Note that the dense separability of $\beta X \setminus \theta(I)$ always implies the dense separability of $X \setminus \Delta(I)$.*

*Proof.* By the comment preceding Proposition 4.4, we have $\mathrm{cl}_{\beta X}(X \setminus \Delta(I)) = \mathrm{cl}_{\beta X}(\beta X \setminus \theta(I))$. Now if $A$ is dense in $\beta X \setminus \theta(I)$, then $\mathrm{cl}_{\beta X \setminus \theta(I)} A = \beta X \setminus \theta(I)$. Since $X \setminus \Delta(I)$ is open in $\beta X \setminus \theta(I)$, $\mathrm{cl}_{X \setminus \Delta(I)}(A \cap (X \setminus \Delta(I)) = X \setminus \Delta(I)$, hence there exists a countable subset $F$ of $A$ such that $\mathrm{cl}_{X \setminus \Delta(I)} F = X \setminus \Delta(I)$. Therefore $\mathrm{cl}_{\beta X \setminus \theta(I)} F = \beta X \setminus \theta(I)$, i.e., $\beta X \setminus \theta(I)$ is dense separable. □

**Lemma 4.10.** *For a space $X$, $\beta X \setminus cl_{\beta X}(\beta X \setminus X) = \mathcal{L}(X) = X \setminus \Delta(C_K(X))$.*

*Proof.* The second equality is obvious and hence we need to show the first one. Since $\beta X \setminus \mathrm{cl}_{\beta X}(\beta X \setminus X) \subseteq X$ is open in $\beta X$, it is locally compact and evidently it is a subset of $\mathcal{L}(X)$. Conversely suppose that $x \in \mathcal{L}(X)$, hence there exists an open subset $U \subseteq X$ such that $x \in U$ and $\mathrm{cl}_X U$ is compact. Now take an open subset $V \subseteq \beta X$ with $V \cap X = U$. By density of $X$, we have $\mathrm{cl}_{\beta X} V = \mathrm{cl}_X U$. This shows that $V \cap (\beta X \setminus X) = \emptyset$ and so $x \notin \mathrm{cl}_{\beta X}(\beta X \setminus X)$. This ends our proof. □

**Proposition 4.11.** *(i) $C_K(X)$ is separable if and only if $\mathcal{L}(X)$ is dense separable.*

*(ii) $C_\psi(X)$ is separable if and only if $\mathcal{L}(\upsilon X)$ is dense separable.*

## 5. **Goldie dimension of ideals**

In this section we generalize the notion of Goldie dimension to an arbitrary ideal. A collection $\{I_\alpha\}_{\alpha \in S}$ of nonzero subideals (ideals) of an ideal $I$ is said to be independent if $I_\beta \cap \sum_{\beta \neq \alpha \in S} I_\alpha = (0)$, i.e.,

$\sum_{\alpha\in S} I_\alpha = \bigoplus_{\alpha\in S} I_\alpha$. We denote the Goldie dimension (generalized Goldie dimension) of $I$ by Gdim$I$ (G$_g$dim$I$) and define it to be the smallest cardinal number $\mathfrak{a}$ such that every independent set of nonzero subideals (ideals) of $I$ has cardinality less than or equal to $\mathfrak{a}$. It is clear that Gdim$I \leq$ G$_g$dim$I$.

In $C(X)$ we observe that Gdim$I =$ G$_g$dim$I$ for each ideal $I$ of $C(X)$ and they coincide with the cellularity of $X \setminus \Delta(I)$. The smallest cardinal number $\mathfrak{b}$ such that every family of pairwise disjoint nonempty open subsets of a space $Y$ has cardinality less than or equal to $\mathfrak{b}$ is called the cellularity or souslin number of $Y$ and is denoted by $c(Y)$ or $\mathcal{S}(Y)$, see [22] and [8] for more details. It is well-known that Gdim$C(X) = c(X)$, see [2].

**Theorem 5.1.** *Let $I$ be an ideal of $C(X)$. Then GdimI $=$ $G_g$dimI $=$ $c(X \setminus \Delta(I))$.*

*Proof.* To prove the first equality, it is enough to show that Gdim$I \geq$ G$_g$dim$I$. Let $\{I_\alpha\}_{\alpha\in S}$ be an independent collection of nonzero ideals of $I$, then by Lemma 4.1, we have $O^{A_\alpha} \subseteq I_\alpha$, where $A_\alpha = \theta(I_\alpha)$. Clearly $\{O^{A_\alpha}\}_{\alpha\in S}$ is also an independent set of subideals of $I$ which means that Gdim$I \geq$ G$_g$dim$I$.

Now suppose that Gdim$I = \mathfrak{a}$ and $\{G_\alpha : \alpha \in S\}$ is a family of pairwise disjoint open subsets of $X \setminus \Delta(I)$. For each $\alpha \in S$, there exists $0 \neq f_\alpha \in C(X)$ such that $f(X \setminus G_\alpha) = \{0\}$. Note that each $G_\alpha$ is also open in $X$. It is easy to see that $(f_\alpha) \cap (f_\beta) = (0)$ for all $\alpha, \beta \in S$ with $\alpha \neq \beta$. $\{(f_\alpha)I : \alpha \in S\}$ is an independent collection of nonzero subideals of $I$ which means that Gdim$I \geq c(X \setminus \Delta(I))$.

Conversely suppose that $c(X \setminus \Delta(I)) = \mathfrak{a}$ and $\{I_\alpha : \alpha \in S\}$ is an independent collection of nonzero subideals of $I$. For each $\alpha \in S$, take $0 \neq f_\alpha \in I_\alpha$. Clearly $X \setminus Z(f_\alpha) \subseteq X \setminus \Delta(I)$, hence $\{X \setminus Z(f_\alpha) : \alpha \in S\}$ is a collection of pairwise disjoint open subsets of $X \setminus \Delta(I)$ which means that $c(X \setminus \Delta(I)) \geq$ Gdim$I$. □

**Theorem 5.2.** *Let $I$ be an ideal of $C(X)$. Then GdimI is finite if and only if $I$ is a finite direct sum of minimal ideals of $I$.*

*Proof.* By Theorem 5.1, $c(X \setminus \Delta(I))$ is finite and since $X \setminus \Delta(I)$ is open, $X \setminus \Delta(I) = \{x_1, ..., x_n\}$, where each $x_i$ is an isolated point. This implies that $I$ is a $P$-ideal and so $I = O^{\beta X\setminus\{x_1,...,x_n\}}$ or $I = \sum_{i=1}^n O^{\beta X\setminus\{x_i\}}$. It is not hard to see that each $O^{\beta X\setminus\{x_i\}}$ is a minimal ideal of $C(X)$ and also a minimal ideal of $I$. The converse is obvious. □

As a final consequence of Theorem 5.1, we need the following lemma which is Exercise 6L.8 of [17].

**Lemma 5.3.** *If $c(X) = \aleph_0$, then $X$ is an $F$-space if and only if $X$ is extremally disconnected.*

Now using this lemma, together with Propositions 4.2 and 4.4, the equivalence of parts (ii) and (iii) of the following corollary is evident.

**Corollary 5.4.** *Let $X$ be a space with nonmeasurable cardinal and $I$ be a $P$-ideal of $C(X)$. Then the following statements are equivalent.*
*(i) $GdimI = \aleph_0$.*
*(ii) $X \setminus \Delta(I)$ is countable and it consists entirely of isolated points.*
*(iii) $I$ is separable.*

*Proof.* It is enough to show that parts (i) and (ii) are equivalent. Clearly part (ii) implies part (i), so it suffices to prove that (i) implies (ii). By Lemma 5.3, $X \setminus \Delta(I)$ is an extremally disconnected $P$-space with nonmeasurable cardinal. But using 12G(6) in [9], $X \setminus \Delta(I)$ should be discrete and hence it must be countable by (i). Since $X \setminus \Delta(I)$ is open, then its points are isolated. □

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