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SOME PROPERTIES OF A CLASS OF POLYHEDRAL SEMIGROUPS BASED UPON THE SUBWORD REVERSING METHOD

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Abstract. In this paper a certain class of polyhedral semigroups which has a presentation $< a, b \mid a^{2n} = b^{2n} = (ab)^n >$ is examined. The completeness of the presentation and solvability of word problem of this class of semigroups is determined. Moreover the combinatorial distance between two words is determined.
Keywords: Polyhedral semigroup, complete, presentation, subword reversing.
MSC(2010): Primary: 20M05; Secondary: 08A50, 11Y50.

1. Introduction

The polyhedral groups ($(l,\ m,\ n)$ for $l,\ m,\ n > 1$) have been studied for a long time in the past and in recent years (See [3–6].) There are two types of presentations which define polyhedral groups. The presentations are:

$$< a, b, c \mid a^l = b^m = c^n = (abc) = 1 > \text{ and}$$
$$< a, b \mid a^l = b^m = (ab)^n = 1 >$$

The group defined by the presentation $< a, b, c \mid a^l = b^m = c^n = (abc) >$ is a larger group $< l,\ m,\ n >$ and was considered by Threlfall(1932). He called them the *binary polyhedral groups*. Another presentation is given by $< a, b \mid a^l = b^m = (ab)^n >$ and it is a generalization of polyhedral groups which is a generalization of the second presentation of $(l,\ m,\ n)$. In [1] a new generator $t = ab$ is added to this presentation and some Tietze transformations are applied. The presentations

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$\pi_1 =< a, b \mid a^l = b^m = (ab)^n >$ and $\pi_2 =< a, b \mid a^l = b^m,\ a(ab)^n = a >$ are obtained. In this paper we consider a class of polyhedral semigroups which are defined by the presentation $< a, b \mid a^{2n} = b^{2n} = (ab)^n >$ $(n \in N)$. We use the subword reversing method which has been developed in various studies and the results are obtained in different sources(for example [8–11]). In [7] the main aspects of the method, its range, uses and efficiency are discussed. In this paper we use some applications of this method. We show that this type of presentation is complete, and the semigroups defined by these presentations have solvable word problem. We also find the distance between two words in $\{a,\ b\}^+$.

2. A combinatorial method for investigating a presented semigroup: subword reversing

In this section we define a combinatorial method for investigating a presented semigroup which is called subword reversing. This method can be also used for investigating a group. If one wants to declare whether a semigroup has a solvable word problem, one may reduce a pair of given words to some normal form. The most known approaches are the Knuth-Bendix algorithm and Grbner-Shirshov bases. However, by the help of subword reversing method, one directly compares the words one to the other.

Let $< X \mid R >$ be a semigroup presentation. X is a generating set and R is a set of relations. Let $w,\ w' \in X^*$. Assume that there is a finite sequence of words $(w_0,\ w_1, ...,\ w_n)$ such that $w_0 = w,\ w_n = w'$ and for each i there exists $\{r,\ s\} \in R$ and $u,\ v \in X^*$ such that $\{w_i,\ w_{i+1}\} = \{urv,\ usv\}$. Then w and w' are called R-equivalent words. It is denoted by $w \equiv w'$.

The reversing method can be described by using words in an alphabet $X \cup X^{-1}$. One may think that X^{-1} consists of the inverses of the elements in X but X^{-1} is a formal copy of X consisting of a copy x^{-1} for each letter $x \in X$. Words in the alphabet $X \cup X^{-1}$ are called signed words. (see [7]).

Definition 2.1. ***(reversing)****(see [7]) Let $< X \mid R >$ be a semigroup presentation and let w and w' be signed words in the alphabet $X \cup X^{-1}$. w reverses to w' in one step if there exists a relation $xv' = x'v$ in R and signed words $u,\ u'$ satisfying $w = ux^{-1}x'u'$ and $w' = uv'v^{-1}u'$ and is denoted by $w \curvearrowright R^1 w'$. If there exist words $w_0,\ w_1, ...,\ w_k$ satisfying $w_0 = w,\ w_k = w'$ and $w_i \curvearrowright R^1 w_{i+1}$ for each i, the sequence*

$(w_0,\ w_1, ...,\ w_k)$ is called an R-reversing sequence from w to w'. It is denoted by $w \curvearrowright R^k w'$ or simply $w \curvearrowright Rw'$.

Remark 2.2. *When we apply the reversing method, we can either delete a subword $u^{-1}u$ with $u \in X^+$ or replace a subword $u^{-1}v$, $(u,\ v \neq \varepsilon)$ with a word $v'u'^{-1}$ such that $uv' = vu'$ is a relation of R. These arguments can be applied step by step.*

Let $< X \mid R >$ be a semigroup presentation and $w,\ w'$ be words in the alphabet X. If $w^{-1}w' \curvearrowright \varepsilon$ then $w \equiv w'$.

Definition 2.3. *(complete presentation) Let $w,\ w'$ be words in the alphabet X. If $w \equiv w'$ implies $w^{-1}w' \curvearrowright \varepsilon$ the presentation $< X \mid R >$ is called a complete presentation.*

In the following definition we aim to decide about homogeneity of a given presentation.

Definition 2.4. *(see [2]) Let $< X \mid R >$ be a semigroup presentation. Let $\lambda : X^* \to Z^+$ which satisfies $\lambda(xu) > \lambda(u)$ for each $x \in X$ and $u \in X^*$. Moreover λ is invariant under $\equiv$. In other words, the presentation admits a pseudolength.*

If all pairs of words satisfying the relations in R have the same length, then the length function is a *pseudolength* for $< X \mid R >$. Now we give an algorithm which can be used to check completeness of a homogenous presentation (see [2]). The same algorithm is called the *cube condition* in [7, Definition 2.6].

Algorithm 1. *(see [2]) Let $< X \mid R >$ be a semigroup presentation. Let $x,\ y,\ z \in X$ be a triple of letters:*
1) Reverse $x^{-1}zz^{-1}y$ to all possible words of the form uv^{-1} $(u,\ v \in X^*)$.
2) For each uv^{-1} obtained, check $(xu)^{-1}(yv) \curvearrowright \varepsilon$.

Lemma 2.5. *Assume that $< X \mid R >$ is a homogenous semigroup presentation. Then $< X \mid R >$ is complete if and only if the conditions 1 and 2 of Algorithm 1 are satisfied. (See [2, Proposition 2.8].)*

3. Word problem

In this section we consider a class of polyhedral semigroups which are defined by the presentation $\wp_n =< a, b \mid a^{2n} = b^{2n} = (ab)^n >$ $(n \in N)$. with the help of the previous section we try to determine solvability of word problem for the polyhedral semigroups defined by the presentation $\wp_n =< a, b \mid a^{2n} = b^{2n} = (ab)^n >$ $(n \in N)$.

Theorem 3.1. *If* $\wp_n =< a, b \mid a^{2n} = b^{2n} = (ab)^n > \quad (n \in N)$ *is a semigroup presentation which defines a polyhedral semigroup for each* $n \in N$*, then* $\wp_n$ *satisfies the conditions of Algorithm 1.*

Proof. The presentation $\wp_n =< a, b \mid a^{2n} = b^{2n} = (ab)^n > \quad (n \in N)$ is homogenous so we can apply Lemma 2.4. We will check whether the given presentation satisfies the conditions of Algorithm 1. We must think about all possible triples of the set $X = \{a,\ b\}$. All triples $(x,\ y,\ z)$ occur as follows:

$$(a,\ a,\ a),\ (a,\ a,\ b),\ (a,\ b,\ a),\ (a,\ b,\ b),\ (b,\ a,\ a),\ (b,\ a,\ b),$$
$$(b,\ b,\ a),\ (b,\ b,\ b)$$

We will try to reverse all these triples to all possible words of the form uv^{-1} and then we will check whether $(xu)^{-1}(yv) \curvearrowright \varepsilon$ is satisfied. If $(x,\ y,\ z) = (a,\ a,\ a)$, then $x^{-1}zz^{-1}y = a^{-1}aa^{-1}a = \varepsilon$. Let $(x,\ y,\ z) = (a,\ a,\ b)$. Then $x^{-1}zz^{-1}y = (a^{-1}b)(b^{-1}a)$. Now we will reverse $(a^{-1}b)(b^{-1}a)$ to a possible word of the type $uv^{-1} \quad (u,\ v \in X^*)$. We can reverse the word step by step by applying the reversing method. Since the relation $a^{2n} = b^{2n}$ is a relation in R, we have $a.a^{2n-1} = b.b^{2n-1}$. So $a^{-1}b \curvearrowright a^{2n-1}(b^{2n-1})^{-1}$. By the same argument $b^{-1}a \curvearrowright b^{2n-1}(a^{2n-1})^{-1}$ by using the relation $b^{2n} = a^{2n}$. Thus we obtain $(a^{-1}b)(b^{-1}a) \curvearrowright a^{2n-1}(a^{2n-1})^{-1}$. Now we check whether $(xu)^{-1}(yv) \curvearrowright \varepsilon$ or not. $(xu)^{-1}(yv) = (a.a^{2n-1})^{-1}(a.a^{2n-1}) = (a^{2n})^{-1}.a^{2n} = \varepsilon$. So the 2^{nd} condition of Algorithm 1 is satisfied. Let $(x,\ y,\ z) = (a,\ b,\ a)$. Then $x^{-1}zz^{-1}y = a^{-1}aa^{-1}b = a^{-1}b$. Since $a^{-1}b \curvearrowright a^{2n-1}(b^{2n-1})^{-1}$ we have $(xu)^{-1}(yv) = (a.a^{2n-1})^{-1}.b.b^{2n-1} = a^{-2n}.b^{2n} = b^{-2n}.b^{2n} = \varepsilon$. Let $(x,\ y,\ z) = (a,\ b,\ b)$. Then $x^{-1}zz^{-1}y = (a^{-1}b)(b^{-1}b) = a^{-1}b\varepsilon$. and $(a^{-1}b) \curvearrowright a^{2n-1}(b^{2n-1})^{-1}$. So $(xu)^{-1}(yv) = (a.a^{2n-1})^{-1}.(b.b^{2n-1}) = ((a^{2n})^{-1}).(b^{2n})$. Since $a^{2n} = b^{2n}$ then $(a^{2n})^{-1} = (b^{2n})^{-1}.b^{2n} = \varepsilon$. Let $(x,\ y,\ z) = (b,\ a,\ a)$. Then $x^{-1}zz^{-1}y = b^{-1}aa^{-1}a = b^{-1}a$. Since $b^{-1}a \curvearrowright b^{2n-1}(a^{2n-1})^{-1}$ we have $(xu)^{-1}(yv) = (b.b^{2n-1})^{-1}.a.a^{2n-1} = b^{-2n}.a^{2n} = a^{-2n}.a^{2n} = \varepsilon$. Let $(x,\ y,\ z) = (b,\ a,\ b)$. Then $x^{-1}zz^{-1}y = b^{-1}bb^{-1}a = b^{-1}a$. Since
$b^{-1}a \curvearrowright b^{2n-1}(a^{2n-1})^{-1}$ we have $(xu)^{-1}(yv) = (b.b^{2n-1})^{-1}.a.a^{2n-1} = (b^{2n})^{-1}.a^{2n} = a^{-2n}.a^{2n} = \varepsilon$. Let $(x,\ y,\ z) = (b,\ b,\ a)$. Then $x^{-1}zz^{-1}y = b^{-1}aa^{-1}b = b^{-1}a$. Since $b^{-1}a \curvearrowright b^{2n-1}(a^{2n-1})^{-1}$ and
$a^{-1}b \curvearrowright a^{2n-1}(b^{2n-1})^{-1}$ we have

$$b^{-1}aa^{-1}b \curvearrowright b^{2n-1}(a^{2n-1})^{-1}a^{2n-1}(b^{2n-1})^{-1} = b^{2n-1}(b^{2n-1})^{-1}.$$

So $(xu)^{-1}(yv) = (b.b^{2n-1})^{-1}.b.b^{2n-1} = (b^{2n})^{-1}.b^{2n} = b^{-2n}.b^{2n} = \varepsilon$. Let $(x,\ y,\ z) = (b,\ b,\ b)$. Then $x^{-1}zz^{-1}y = b^{-1}bb^{-1}b = \varepsilon$. □

By a combination of Theorem 3.1 and Lemma 2.4 we give the following corollary.

Corollary 3.2. *The presentation* $\wp_n =< a, b \mid a^{2n} = b^{2n} = (ab)^n >$ $(n \in N)$ *is complete for all* $n \in N$.

Remark 3.3. *If a semigroup presentation is complete, it does not mean that the semigroup defined by the presentation has a solvable word problem. In order to check the solvability of the word problem, we must show that all the R-reversing sequences are finite. With this aim, we give the following lemma. (See [7, Section 3.3].)*

Lemma 3.4. *(see [7]) Let* $\wp =< X \mid R >$ *be a complete presentation. If every R-reversing sequence is finite then the semigroup defined by* $\wp$ *has a solvable word problem.*

Theorem 3.5. *Every polyhedral semigroup defined by the presentation* $\wp_n =< a, b \mid a^{2n} = b^{2n} = (ab)^n >$ $(n \in N)$ *has a solvable word problem.*

Proof. Let w be a signed word in the alphabet $X \cup X^{-1}$. Let $w = ux^{-1}x'u'$. We must show that w reverses to a word w' of the form $w' = uv'v^{-1}u'$ where $xv' = x'v$ is a relation in R and after a finite number of reversing steps we must obtain a word of the form st^{-1} $(s,\ t \in X^*)$. Since $X = \{a,\ b\}$ either $x \in \{a,\ b\}$ or $x' \in \{a,\ b\}$. If $x = a = x'$ then $w = ua^{-1}au' = uu'$. Therefore, we do not need to apply word reversing. The same condition is satisfied for $x = b = x'$. Now, let $x = a$ and $x' = b$. Then $w = ua^{-1}bu'$. In the proof of Theorem 3.1 above, we have shown that $a^{-1}b \curvearrowright a^{2n-1}(b^{2n-1})^{-1}$. So $w \curvearrowright ua^{2n-1}(b^{2n-1})^{-1}u'$. Let $u' = x_1x_2...x_k$ $(x_i \in \{a,\ b\})$. We have $x_1 = a$ or $x_1 = b$. We may assume that $x_1 = a$. (If $x_1 = b$ the same argument can be evaluated). Then consider $ua^{2n-1}(b^{2n-1})^{-1}ax_2...x_k$. Since $b^{2n} = a^{2n}$ is a relation in R we obtain $(b^{2n-1})^{-1}a \curvearrowright b(a^{2n-1})^{-1}$. Thus $ua^{2n-1}(b^{2n-1})^{-1}ax_2...x_k \curvearrowright ua^{2n-1}b(a^{2n-1})^{-1}x_2...x_k$. The same argument can be evaluated step by step. Without loss of generality let $x_2 = b$. We have $ua^{2n-1}b(a^{2n-1})^{-1}b...$
x_k. When we use word reversing again $ua^{2n-1}b(a^{2n-1})^{-1}b...x_k \curvearrowright$
$ua^{2n-1}ba(b^{2n-1})^{-1}...x_k$. One can immediately see that after a finite number of steps we obtain a word of the form st^{-1} $(s,\ t \in X^*)$. So the polyhedral semigroup defined by $\wp_n =< a, b \mid a^{2n} = b^{2n} = (ab)^n >$ $(n \in N)$ has a solvable word problem. □

4. **Combinatorial distance**

In this section we define the *combinatorial distance* for given equivalent words.

Definition 4.1. *Let $< X \mid R >$ be a semigroup presentation and $w,\ w'$ be equivalent words of X^*. The minimal number of relations of R which are needed to transform w to w' is defined as combinatorial distance. It is denoted $dist(w,\ w')$. (see [7, Definition 4.1])*

Now we try to find a formula for $dist(w,\ w')$ for the given presentation $\wp_n =< a, b \mid a^{2n} = b^{2n} = (ab)^n > \ (n \in N)$.

Lemma 4.2. *Let $w = a^s$ $(s \geq 2n)$. Let $w \equiv w'$ and let w' be the word of the form $a^{r_1}b^{t_1}a^{r_2}b^{t_2}...a^{r_h}b^{t_h}$ $(0 \leq r_i < 2n,\ 0 \leq t_j < 2n)$. Then $dist(w,\ w') = 1$.*

Proof. Since $w = a^s$ $(s \geq 2n)$ we have $s = 2n.s_1 + s_2$, $(0 \leq s_2 < 2n)$. So $a^s = a^{2n.s_1+s_2} = (a^{2n})^{s_1}.a^{s_2} = ((ab)^n)^{s_1}.a^{s_2} = (ab)(ab)...(ab).a^{s_2}$. There is no other relation that can be applied from R, so $w' = (ab)(ab)...(ab).a^{s_2}$. Only the relation $a^{2n} = (ab)^n$ is used, so $dist(w,\ w') = 1$. □

By the help of the Lemma 4.2 we give the next Lemma.

Lemma 4.3. *Let $w = b^p$ $(p \geq 2n)$. Let $w \equiv w'$ and let w' be the word of the form $a^{r_1}b^{t_1}a^{r_2}b^{t_2}...a^{r_h}b^{t_h}$ $(0 \leq r_i < 2n,\ 0 \leq t_j < 2n)$. Then $dist(w,\ w') = 1$.*

Proof. We immediately follow the instructions given in the proof of the previous Lemma and obtain the result. □

In the following Theorem we examine an arbitrary word $w \in X^*$.

Theorem 4.4. *Let $w = a^{f_1}b^{g_1}a^{f_2}b^{g_2}...a^{f_m}b^{g_m}$ $(f_i \in N \cup \{0\},\ g_j \in N \cup \{0\})$. Let $w \equiv w'$ and let w' be the word of the form $a^{r_1}b^{t_1}a^{r_2}b^{t_2}...a^{r_h}b^{t_h}$ $(0 \leq r_i < 2n,\ 0 \leq t_j < 2n)$. If $A = \{f_{i_1},\ f_{i_2}, ...,\ f_{i_k}\} \cup \{g_{j_1},\ g_{j_2}, ...,\ g_{j_l}\}$ $(f_{i_r} \geq 2n,\ g_{j_r} \geq 2n)$, then $dist(w,\ w') = k + l$.*

Proof. Let $w = a^{f_1}b^{g_1}a^{f_2}b^{g_2}...a^{f_m}b^{g_m}$ $(f_i \in N \cup \{0\},\ g_j \in N \cup \{0\})$. $A = \{f_{i_1},\ f_{i_2}, ...,\ f_{i_k}\} \cup \{g_{j_1},\ g_{j_2}, ...,\ g_{j_l}\}$ $(f_{i_r} \geq 2n,\ g_{j_r} \geq 2n)$. If we apply division algorithm for each f_{i_r} and g_{j_r} we have $f_{i_r} = 2n.f'_{i_r} + f''_{i_r}$ $(0 \leq f''_{i_r} < 2n)$ and $g_{j_r} = 2n.g'_{j_r} + g''_{j_r}$ $(0 \leq g''_{j_r} < 2n)$. If we use the relations $a^{2n} = (ab)^n$ and $b^{2n} = (ab)^n$ we obtain $(a)^{2n.f'_{i_r}} = (ab)^{n.f'_{i_r}}$ and $(b)^{2n.g'_{j_r}} = (ab)^{n.g'_{j_r}}$. If we apply these relations step by step we obtain $w' = a^{r_1}b^{t_1}a^{r_2}b^{t_2}...a^{r_h}b^{t_h}$ $(0 \leq r_i < 2n,\ 0 \leq t_j < 2n)$. The relations which we have applied are $a^{2n} = (ab)^n$ and $b^{2n} = (ab)^n$. Since $|A| = k+l$, we apply these relations $k+l$ times. So $dist(w,\ w') = k+l$. $\square$

5. Conclusion

In this paper, we have considered a class of polyhedral semigroups defined by the presentation $\wp_n =< a, b \mid a^{2n} = b^{2n} = (ab)^n > \ (n \in N)$ and we have shown that every semigroup which has a presentation of this type is complete. We have also shown that these semigroups have solvable word problem. For two arbitrary words w and w' such that $w \equiv w'$ we have determined $dist(w,\ w')$. For future work, different types of semigroup and monoid presentations may be considered.

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