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Author(s):

H. Yu, Y. Kou and H. Zhao

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DETECTION OF A NONTRIVIAL ELEMENT IN THE STABLE HOMOTOPY GROUPS OF SPHERES

H. YU, Y. KOU AND H. ZHAO*

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Abstract. Let p be a prime with $p \geq 7$ and $q = 2(p-1)$. In this paper we prove the existence of a nontrivial product of filtration $s+4$ in the stable homotopy groups of spheres. This nontrivial product is shown to be represented up to a nonzero scalar by the product element $\widetilde{\gamma}_s b_{n-1} g_0 \in \mathrm{Ext}_{\mathcal{A}}^{s+4,(p^n+sp^2+sp+s)q+s-3}(\mathbb{Z}/p, \mathbb{Z}/p)$ in the Adams spectral sequence where $n \geq 2$ and $3 \leq s \leq p-1$.
Keywords: Stable homotopy groups of sphere, Adams spectral sequence, May spectral sequence.
MSC(2010): Primary: 55Q45.

1. Introduction

Let p be an odd prime. Let $\mathcal{A}$ be the mod p Steenrod algebra and let S be the sphere spectrum localized at p. Throughout the paper we fix $q = 2(p-1)$. To determine the stable homotopy groups of sphere $\pi_* S$ is one of the central problems in homotopy theory. One of the main tools to approach it is the classical Adams spectral sequence (ASS) whose E_2-term is given by $E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p)$ which is the cohomology of $\mathcal{A}$. The Adams differential is given by

$$d_r \colon E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}.$$

From [6], we know that $\mathrm{Ext}_{\mathcal{A}}^{1,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ has $\mathbb{Z}/p$-basis consisting of $a_0 \in \mathrm{Ext}_{\mathcal{A}}^{1,1}(\mathbb{Z}/p, \mathbb{Z}/p)$, $h_i \in \mathrm{Ext}_{\mathcal{A}}^{1,p^i q}(\mathbb{Z}/p, \mathbb{Z}/p)$ for all $i \geq 0$ and we also know that $\mathrm{Ext}_{\mathcal{A}}^{2,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ has $\mathbb{Z}/p$-basis consisting of $\widetilde{\alpha}_2$, a_0^2, $a_0 h_i (i > 0)$, $g_i (i \geq 0)$, $k_i (i \geq 0)$, $b_i (i \geq 0)$, and $h_i h_j (j \geq i+2, i \geq 0)$ whose internal

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*Corresponding author.

degrees are $2q+1$, 2, p^iq+1, $(p^{i+1}+2p^i)q$, $(2p^{i+1}+p^i)q$, $p^{i+1}q$ and $(p^i+p^j)q$, respectively.

If a family of generators $x_i \in E_2^{s,*}$ converges nontrivially in the ASS, then we obtain a family of homotopy elements f_i in π_*S and we say that f_i has filtration s and is represented by $x_i \in E_2^{s,*}$ in the ASS. So far, very few families of homotopy elements in π_*S have been detected. The following are some known results. In [7] M. Mahowald detected an order 2 element $\eta_i \in_2 \pi_*^s$, which is represented by $h_1h_i \in \mathrm{Ext}_{\mathcal{A}}^{1,*}(\mathbb{Z}/2,\mathbb{Z}/2)$. By analogous argument at odd primes, R. Cohen [1] detected a family of homotopy elements $\zeta_n \in \pi_{p^nq+q-3}S$ which has filtration 3 and is represented by $h_0b_{n-1} \in \mathrm{Ext}_{\mathcal{A}}^{3,p^nq+q}(\mathbb{Z}/p,\mathbb{Z}/p)$ in the ASS. In [9] D. Ravenel proved that $b_n \in \mathrm{Ext}_{\mathcal{A}}^{2,p^nq}(\mathbb{Z}/p,\mathbb{Z}/p)$ does not converge in the ASS, which is known to be the odd prime Kervaire invariant element. Recently, Hill-Hopkins-Ravenel [2] proved that the mod 2 Kervaire invariant one elements $\theta_j \in \pi_{2^{j+2}-2}S$ exist only for $0 \le j \le 6$. This resolves a longstanding problem in algebraic topology.

Among the nontrivial elements of π_*S the periodic elements are especially important. The existence of the periodic elements is related to the existence of Toda-Smith spectra. Let BP be the Brown-Peterson spectrum localized p. It is a p-local ring spectrum with the coefficient ring

$$BP_* = BP_*S = \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$$

where v_i is the i-th Hazewinkel generator with degree $2(p^i-1)$. If X is a spectrum, then BP_*X is a comodule over the Hopf algebroid BP_*BP (refer to [10]). Toda [11] considered the existence of the finite spectra $V(n)$ with

$$BP_*V(n) \cong BP_*/I_{n+1} \text{ (as } BP_*\text{-module, hence as } BP_*BP\text{-comodule)}$$

where $I_{n+1} = (p, v_1, \cdots, v_n)$, the ideal generated by $p, v_1, \cdots, v_n$. In [11], Toda showed that $V(n)$ exists for $p > 2n$ with $n = 0,1,2,3$ and there exists Greek letter map

$$v_n\colon \Sigma^{2(p^n-1)}V(n-1) \longrightarrow V(n-1)$$

with $v_n = p, \alpha, \beta, \gamma$ for $n = 0,1,2,3$, respectively. Here we write $V(-1)$ for S. Moreover, the cofibre of v_n is $V(n)$ given by the cofibration

$$\Sigma^{2(p^n-1)}V(n-1) \xrightarrow{v_n} V(n-1) \xrightarrow{i_n} V(n) \xrightarrow{j_n} \Sigma^{2p^n-1}V(n-1).$$

If we write

$$\alpha_s = j_0(v_1^s)i_0, \quad \beta_s = j_0j_1(v_2^s)i_1i_0 \quad \text{and} \quad \gamma_s = j_0j_1j_2(v_3^s)i_2i_1i_0,$$

then α_s, β_s, γ_s are the well known first, second and third periodic elements in π_*S with filtration s (refer to [8]). It was shown in [12] that when $n < p$ and $s \not\equiv 0, 1, \cdots, n-1 \bmod p$, there is a non-zero cohomology class $\widetilde{\alpha}_s^{(n)} \in \mathrm{Ext}_{\mathcal{A}}^{s,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ which is called the n-th Greek letter element in Ext. When $n = 1, 2, 3$, the elements $\widetilde{\alpha}_s^{(n)}$ are written as $\widetilde{\alpha}_s$, $\widetilde{\beta}_s$ and $\widetilde{\gamma}_s$ which represent the homotopy elements α_s, β_s and γ_s respectively.

Given two elements $\widetilde{x}$ and $\widetilde{y}$ in $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$, suppose that $\widetilde{x}$ and $\widetilde{y}$ converge nontrivially to elements x and y in π_*S, respectively. We are wondering whether or not the product $\widetilde{x} \cdot \widetilde{y}$ in the ASS can also converge nontrivially to the product $x \cdot y$ in π_*S. In particularly, we are interested in considering the convergence of the product of $\widetilde{\beta}_s$ or $\widetilde{\gamma}_s$ with some other elements in $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$. For example, it was shown in [4] that the product $\widetilde{\gamma}_s h_0 b_{n-1} \in \mathrm{Ext}_{\mathcal{A}}^{s+3,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ is nontrivial in the ASS when $p \geq 7$, $n \geq 2$ and $3 \leq s \leq p-2$. It converges to a nontrivial element $\gamma_s \zeta_n \in \pi_*S$. By a similar method, Liu-Ma [5] verified the convergence of the product $h_n h_m \tilde{\beta}_s$ in the ASS when $p \geq 5$, $n \geq m+2 > 5$ and $2 \leq s \leq p-1$. In this paper, we will improve their method and use it to show that $\widetilde{\gamma}_s b_{n-1} g_0$ in the ASS converges to a nontrivial element of π_*S. The following statements are our main results.

Theorem 1.1. *Let $p \geq 7$ and $n \geq 2$. If $3 \leq s \leq p-1$ then the product*

$$\widetilde{\gamma}_s b_{n-1} g_0 \in \mathrm{Ext}_{\mathcal{A}}^{s+4,(s+sp+sp^2+p^n)q+s-3}(\mathbb{Z}/p, \mathbb{Z}/p)$$

is nontrivial in the Adams spectral sequence and converges to a homotopy nontrivial element $\xi_n \in \pi_*S$.

This paper is organized as follows. In Section 2, we will introduce a method to compute the generators of the E_1-term of the May spectral sequence (MSS). As an application of this method, in Section 3 we do an explicit computation for the sake of proof of Theorem 1.1. Then in Section 4, we give the proof of Theorem 1.1.

2. Preliminary knowledge on the May spectral sequence

In this section we will recall some elementary knowledge on the May spectral sequence (MSS). By reference [10], there is a 3-graded May spectral sequence $\{E_r^{s,t,*}, d_r \colon E_r^{s,t,M} \longrightarrow E_r^{s+1,t,M-r}\}$ which converges to $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{Z}/p,\mathbb{Z}/p)$. The E_1-term of MSS is given by

$$E_1^{*,*,*} = E[h_{m,i}|m>0, i\geq 0]\otimes P[b_{m,i}|m>0, i\geq 0]\otimes P[a_n|n\geq 0]$$

where $E[\,]$ denotes the exterior algebra and $P[\,]$ denotes the polynomial algebra. It is known that $h_{1,i}\in E_1^{1,p^iq,*}$ converges nontrivially to $h_i\in \mathrm{Ext}_{\mathcal{A}}^{1,p^iq}(\mathbb{Z}/p,\mathbb{Z}/p)$. Thus $d_r(h_{1,i})=0$ for any $r\geq 1$. We list the degrees of the E_1-term generators as follows:

$$h_{m,i}\in E_1^{1,2(p^m-1)p^i,2m-1}, b_{m,i}\in E_1^{2,2(p^m-1)p^{i+1},(2m-1)p}, a_n\in E_1^{1,2p^n-1,2n+1}.$$

For the r-th May differential $d_r\colon E_r^{s,t,M}\longrightarrow E_r^{s+1,t,M-r}$ with $r\geq 1$, if $x\in E_r^{s,t,*}$ and $y\in E_r^{s',t',*}$ then

$$d_r(xy)=d_r(x)y+(-1)^{s+t}xd_r(y).$$

The MSS satisfies the graded commutativity $xy=(-1)^{ss'+tt'}yx$ for $\{x,y\}\subset\{h_{m,i},b_{m,i},a_n\}$. On each generator the first May differential

$$d_1\colon E_1^{s,t,M}\longrightarrow E_1^{s+1,t,M-1}$$

has an explicit description as

$$d_1(h_{i,j})=\sum_{0<k<i}h_{i-k,k+j}h_{k,j},\quad d_1(a_i)=\sum_{0\leq k<i}h_{i-k,k}a_k,\quad d_1(b_{i,j})=0.$$

Given an element $x\in E_1^{s,t,M}$, we define $\dim(x)=s$, $\deg(x)=t$ and $M(x)=M$. Then we have

$$\begin{array}{ll}\dim(h_{i,j}) = \dim(a_i)=1, & \dim(b_{i,j})=2,\\ M(h_{i,j})=M(a_{i-1})=2i-1, & M(b_{i,j})=(2i-1)p,\\ \deg(h_{i,j})=2(p^i-1)p^j=(p^j+\cdots+p^{i+j-1})q, & \\ \deg(b_{i,j})=2(p^i-1)p^{j+1}=(p^{j+1}+\cdots+p^{i+j})q, & \\ \deg(a_i)=2p^i-1=(1+\cdots+p^{i-1})q+1, & \\ \deg(a_0)=1 & \end{array}$$

where $i\geq 1$ and $j\geq 0$.

A method of computing E_1-term of the MSS was introduced in [5], but the computation process in [5] is very obscure. Hence, we are about to introduce a new computation method and then in Section 3 we show how to use it in a more effective way for our target.

We denote a_i, $h_{i,j}$ and $b_{i,j}$ by x, y and z, respectively. By the graded commutativity of $E_1^{*,*,*}$, we can write a generator as

$$h=(x_1\cdots x_u)(y_1\cdots y_v)(z_1\cdots z_l)\in E_1^{s,t+b,*}$$

where $t = (\bar{c}_0 + \bar{c}_1 p + \cdots + \bar{c}_n p^n)q$ with $0 \le \bar{c}_i < p$ $(0 \le i < n)$, $\bar{c}_n > 0$, $s < b + q$ with $0 < b < q$. We claim that $u = b$. Otherwise, by the characteristics of $\deg(a_i)$, $\deg(h_{i,j})$, $\deg(b_{i,j})$ and t, there exists some $w > 0$ such that $u = b + wq$. It follows that $\dim(h) \ge b + wq > s = \dim(h)$ which is a contradiction. Thus

$$h = (x_1 \cdots x_b)(y_1 \cdots y_v)(z_1 \cdots z_l) \in E_1^{b+v+2l,t+b,*}.$$

Note that the degrees of x_i, y_i and z_i can be uniquely expressed as

$$\begin{aligned} \deg(x_i) &= (x_{i,0} + x_{i,1}p + \cdots + x_{i,n}p^n)q + 1, \\ \deg(y_i) &= (y_{i,0} + y_{i,1}p + \cdots + y_{i,n}p^n)q, \\ \deg(z_i) &= (0 + z_{i,1}p + \cdots + z_{i,n}p^n)q \end{aligned}$$

where the sequence $(x_{i,0}, x_{i,1}, \cdots, x_{i,n})$ is of the form $(1, \cdots, 1, 0, \cdots, 0)$, while $(y_{i,0}, y_{i,1}, \cdots, y_{i,n})$ and $(0, z_{i,1}, \cdots, z_{i,n})$ are both of the form

$$(0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0).$$

According to the graded commutativity of $E_1^{*,*,*}$, the generator

$$h = (x_1 \cdots x_b)(y_1 \cdots y_v)(z_1 \cdots z_l) \in E_1^{b+v+2l,t+b,*}$$

can be arranged in the following way:

(a) if $i > j$, we put a_i on the left side of a_j;
(b) if $j < k$, we put $h_{i,j}$ on the left side of $h_{w,k}$;
(c) if $i > w$, we put $h_{i,j}$ on the left side of $h_{w,j}$;
(d) apply the same rules (b) and (c) to $b_{i,j}$.

Hence the above $x_{i,j}$, $y_{i,j}$ and $z_{i,j}$ satisfy the following conditions (2.1):

(i) $x_{1,j} \ge x_{2,j} \ge \cdots \ge x_{b,j}$, $x_{i,0} \ge x_{i,1} \ge \cdots \ge x_{i,n}$ for $i \le b$ and $j \le n$;
(ii) if $y_{i,j-1} = 0$ and $y_{i,j} = 1$, then for all $k < j$ there is $y_{i,k} = 0$;
(iii) if $y_{i,j} = 1$ and $y_{i,j+1} = 0$, then for all $k > j$ there is $y_{i,k} = 0$;
(iv) $y_{1,0} \ge y_{2,0} \ge \cdots \ge y_{v,0}$;
(v) if $y_{i,0} = y_{i+1,0}$, $y_{i,1} = y_{i+1,1}, \cdots, y_{i,j} = y_{i+1,j}$, then $y_{i,j+1} \ge y_{i+1,j+1}$;
(vi) apply the same rules (ii)$\sim$(iv) to $z_{i,j}$.

According to the p-adic expression of the coefficient of q in second degree $\deg(x_i)$, $\deg(y_i)$ and $\deg(z_i)$ as above, by the properties of p-adic numbers we obtain the following group of equations (2.2)

$$\begin{cases} x_{1,0}+\cdots+x_{b,0}+y_{1,0}+\cdots+y_{v,0}=\bar{c}_0+k_1p=c_0 \\ x_{1,1}+\cdots+x_{b,1}+y_{1,1}+\cdots+y_{v,1}+z_{1,1}+\cdots+z_{l,1} \\ \qquad\qquad\qquad\qquad =\bar{c}_1-k_1+k_2p=c_1 \\ \cdots \qquad \cdots \qquad \cdots \\ x_{1,n-1}+\cdots+x_{b,n-1}+y_{1,n-1}+\cdots+y_{v,n-1}+z_{1,n-1}+\cdots+z_{l,n-1} \\ \qquad\qquad\qquad\qquad =\bar{c}_{n-1}-k_{n-1}+k_np=c_{n-1} \\ x_{1,n}+\cdots+x_{b,n}+y_{1,n}+\cdots+y_{v,n}+z_{1,n}+\cdots+z_{l,n}=\bar{c}_n-k_n=c_n. \end{cases}$$

From the above group of equations, we obtain two integer sequences

$$K=(k_1,\cdots,k_n) \quad \text{and} \quad S=(c_0,\cdots,c_n)$$

which are determined by $(k_1,\cdots,k_n)$ and $(\bar{c}_0,\cdots,\bar{c}_n)$, respectively. We say that the group of equations (2.2) has a solution if it has a solution satisfying the conditions (2.1).

Intuitively the above group of equations has the form of matrix as

$$(2.1) \qquad \begin{array}{c} \quad A \qquad\qquad\qquad B \qquad\qquad\qquad C \\ \left(\begin{array}{ccc|ccc|ccc} x_{1,0} & \cdots & x_{b,0} & y_{1,0} & \cdots & y_{m,0} & 0 & \cdots & 0 \\ x_{1,1} & \cdots & x_{b,1} & y_{1,1} & \cdots & y_{m,1} & z_{1,1} & \cdots & z_{l,1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{1,n} & \cdots & x_{b,n} & y_{1,n} & \cdots & y_{m,n} & z_{1,n} & \cdots & z_{l,n} \end{array}\right) \begin{array}{c} c_0 \\ c_1 \\ \vdots \\ c_n. \end{array} \end{array}$$

According to the conditions (2.1), the section A in the matrix is the form of trapezoid as

$$(2.2) \qquad \begin{pmatrix} 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 \\ 1 & \cdots & 1 & \cdots & \cdots & 1 & & \\ \vdots & \vdots & \vdots & \vdots & & & & \\ 1 & \cdots & 1 & & & & & \\ 1 & & & & & & & \end{pmatrix}$$

where the vacant place denotes zero. The section B has the form as

(2.3)

$$\begin{pmatrix}
1 & \cdots & 1 & \cdots & 1 & \cdots & 1 & & & & & & & & \\
1 & \cdots & 1 & \cdots & 1 & & & \ddots & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & & \ddots & & & & & & \\
1 & \cdots & 1 & & & & & & 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 \\
1 & & & & & & & & 1 & \cdots & 1 & \cdots & 1 & & \\
 & & & & & & & & \vdots & \vdots & \vdots & & & & \\
 & & & & & & & & 1 & \cdots & 1 & & & & \\
 & & & & & & & & 1 & & & & & &
\end{pmatrix}.$$

The section C has the similar form as B except that the first horizontal line are all zero.

Each column in section A determines some x_i. Each column in section B or C determines some y_i or z_i. Recall that each column in the matrix does not admit the form $(\cdots,1,0,\cdots,0,1,\cdots)^T$. In summary, for the $E_1^{s,t+b,*}$-term of MSS where $t=(\bar{c}_0+\bar{c}_1p+\cdots+\bar{c}_np^n)q$ with $0\le\bar{c}_i<p$ ($\bar{c}_n>0$), $s<b+q$ with $0\le b<q$, the determination of $E_1^{s,t+b,*}$ is reduced to the following steps:

Step 1. List all the possible (b,v,l) such that $b+v+2l=s$.

Step 2. For each (b,v,l), list up all the sequences $K=(k_1,\cdots,k_n)$ and $S=(c_0,\cdots,c_n)$ such that $\max\{c_0,c_1,\cdots,c_n\}\le b+v+l$.

Step 3. For each (b,v,l) and the sequence $K=(k_1,\cdots,k_n)$, solve the corresponding group of equations (2.2). As stated before, the solutions are of the forms $(1,\cdots,1,0,\cdots,0)$ or $(0,\cdots,0,1,\cdots,1,0,\cdots,0)$ which uniquely determine x_i, y_i or z_i which correspond to elements of form a_i, $h_{i,j}$ or $b_{i,j}$ respectively.

3. Computation of E_1-term of the MSS

In order to prove Theorem 1.1, in this section we will apply the method in Section 2 to compute the generators of $E_1^{s-r+4,tq+(s-r-2),*}$ for $1\le r\le p+2$, where $t=s+sp+sp^2+p^n$. Let M_i^n ($i\ge1$) denote the May filtration for different n. Our results are stated as follows:

Theorem 3.1. *For $1\le r\le p+2$ and $n\ge2$, let $t=s+sp+sp^2+p^n$ with $3\le s\le p-1$. Then the generators of $E_1^{s-r+4,tq+(s-r-2),*}$ are listed as follows:*

(1) *when* $3 \le r \le p+2$, *there is no generator;*
(2) *when* $r = 2$, *there is a generator* $a_3^{s-4}h_{4,0}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,2}$ *for* $n = 3$ *and no generator for* $n \ne 3$*;*
(3) *when* $r = 1$, *for different* n *the generators are as follows:*
 (a) *for* $n = 2$, *there is no generator.*
 (b) *for* $n = 3$, *there are 20 generators as*

$$\begin{cases} \left.\begin{array}{c} a_3^{s-4}a_1h_{4,0}h_{3,0}h_{2,0}h_{2,1}b_{1,1},\ a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{3,1}b_{1,1}, \\ a_3^{s-4}a_2h_{4,0}h_{3,0}h_{2,0}h_{1,2}b_{1,1},\ a_3^{s-3}h_{4,0}h_{2,0}h_{1,0}h_{2,1}b_{1,1}, \\ a_3^{s-3}h_{4,0}h_{3,0}h_{1,0}h_{1,1}b_{1,1},\ a_4a_3^{s-4}h_{3,0}h_{2,0}h_{1,0}h_{2,1}b_{1,1}, \\ a_3^{s-3}h_{4,0}h_{3,0}h_{1,0}h_{1,2}b_{1,0}, \end{array}\right\} (M_3^3 = 7s+p-8) \\ \left.\begin{array}{c} a_3^{s-4}a_1h_{4,0}h_{3,0}h_{2,0}h_{1,2}b_{2,0},\ a_3^{s-3}h_{4,0}h_{2,0}h_{1,0}h_{1,2}b_{2,0}, \\ a_3^{s-4}a_1h_{4,0}h_{3,0}h_{1,0}h_{2,1}b_{2,0},\ a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{2,2}b_{2,0}, \\ a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{2,1}b_{2,1},\ a_4a_3^{s-4}h_{3,0}h_{2,0}h_{1,0}h_{1,2}b_{2,0}, \end{array}\right\} (M_4^3 = 7s+3p-10) \\ a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{1,2}b_{3,0}, \quad (M_5^3 = 7s+5p-12) \\ \left.\begin{array}{l} a_3^{s-4}a_0h_{4,0}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,2},\ a_3^{s-4}a_1h_{3,0}h_{2,0}h_{1,0}h_{3,1}h_{2,1}h_{1,2}, \\ a_3^{s-4}a_1h_{4,0}h_{3,0}h_{1,0}h_{2,1}h_{1,1}h_{1,2},\ a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,2}h_{1,3}, \\ a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{1,1}h_{2,2}h_{1,2},\ a_3^{s-4}a_2h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{2,2}h_{1,2}; \end{array}\right\} (M_6^3 = 7s-8) \end{cases}$$

 (c) *for* $n > 3$, *there are eleven families of generators as*

$$\begin{cases} a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,2}h_{1,n} \quad (M_7^n = 7s-8) \\ \mathbf{h_1^{(i)}} = a_n^{s-3}h_{n,0}h_{i,0}h_{4,0}h_{n-3,3}b_{n-i,i-1} \quad (0<i<n; i\ne 4); \quad (M_8^n) \\ \mathbf{h_2^{(i)}} = a_n^{s-3}h_{n,0}h_{i,0}h_{n-i,i}h_{4,0}b_{n-3,2} \quad (0<i<n; i\ne 4); \quad (M_9^n) \\ \left.\begin{array}{l} \mathbf{h_3^{(i)}} = a_n^{s-4}a_4h_{n,0}h_{j,0}h_{n-j,j}h_{i,0}h_{n-i,i}h_{n-3,3}(0<i,j<n; i\ne j; i,j\ne 3) \\ \mathbf{h_4^{(i)}} = a_n^{s-4}a_jh_{n,0}h_{4,0}h_{i,0}h_{n-i,i}h_{n-j,j}h_{n-3,3}(0<i,j<n; i\ne j; i\ne 3,4; j\ne 3) \end{array}\right\} (M_{10}^n) \\ \left.\begin{array}{l} \mathbf{h_5^{(i)}} = a_n^{s-3}h_{n,0}h_{j,0}h_{i,0}h_{n-j,j}h_{n-i,i}h_{1,3}(0<i,j<n; i\ne j) \\ \mathbf{h_6^{(i)}} = a_n^{s-3}h_{j,0}h_{i,0}h_{4,0}h_{n-i,i}h_{n-j,j}h_{n-3,3}(0<i,j<n; i\ne j; i\ne 3,4; j\ne 3) \\ \mathbf{h_7^{(i)}} = a_n^{s-3}h_{n,0}h_{j,0}h_{i-j,j}h_{4,0}h_{n-i,i}h_{n-3,3}(0<i,j<n; i>j; j\ne 4; i\ne 3) \\ \mathbf{h_8^{(i)}} = a_n^{s-3}h_{n,0}h_{i,0}h_{j,0}h_{4-j,j}h_{n-i,i}h_{n-3,3}(0<i<n; 0<j<4; i\ne j; i\ne 3) \\ \mathbf{h_9^{(i)}} = a_n^{s-3}h_{n,0}h_{i,0}h_{4,0}h_{n-i-j,i}h_{j,n-j}h_{n-3,3}(0<i<n; j<n-i; i\ne 4) \\ \mathbf{h_{10}^{(i)}} = a_n^{s-3}h_{n,0}h_{i,0}h_{4,0}h_{n-i,i}h_{n-3-j,3}h_{j,n-j}(0<i<n; j<n-3; i\ne 4) \end{array}\right\} (M_{11}^n) \end{cases}$$

where $s = p-1$ *for* $\mathbf{h_1^{(i)}}, \cdots, \mathbf{h_{10}^{(i)}}$ *and* $M_8^n = (4n-2i+1)(p-1)-5$, $M_9^n = 2np-2n+p-5$, $M_{10}^n = 2np-4n+p-8$, $M_{11}^n = 2np-2n+p-8$.

Proof. For $n \le 3$, we have

$$\overline{S} = \begin{cases} (s,s,s+1) & n = 2, \\ (s,s,s,1) & n = 3. \end{cases}$$

For $n > 3$, we have

$$S = \begin{cases} S_1 = (s, s, s, 0, \cdots, 0, 1), \\ S_i = (s, s, s, 0, \cdots, 0, p^{(i)}, p-1, \cdots, p-1, 0), \quad i \geq 4. \end{cases}$$

By the reason of dimension, all the possibilities of h are listed as

$$\begin{cases} x_1 \cdots x_{s-r-2} z_1 z_2 z_3 \\ x_1 \cdots x_{s-r-2} y_1 y_2 z_1 z_2 \\ x_1 \cdots x_{s-r-2} y_1 y_2 y_3 y_4 z_1 \\ x_1 \cdots x_{s-r-2} y_1 y_2 y_3 y_4 y_5 y_6. \end{cases}$$

Let us consider the generators $h \in \mathrm{Ext}_r^{s-r+4, tq+s-r-2}(\mathbb{Z}/p, \mathbb{Z}/p)$ case by case.

Case 1. $h = x_1 \cdots x_{s-r-2} z_1 z_2 z_3$.

Note that $s \leq p-1$ and $r \geq 1$. Since $\sum\limits_{i=1}^{s-r-2} x_{i,0} = s - r - 2 < s = \bar{c}_0$, the first equation of (2.2) has no solution. It follows that such h is impossible to exist.

Case 2. $h = x_1 \cdots x_{s-r-2} y_1 y_2 z_1 z_2$.

Note that $s \leq p-1$ and $r \geq 1$. Since

$$\sum_{i=1}^{s-r-2} x_{i,0} + y_{1,0} + y_{2,0} = s - r - 2 + y_{1,0} + y_{2,0} < s = \bar{c}_0,$$

the first equation of (2.2) has no solution. It follows that such h is impossible to exist.

Case 3. $h = x_1 \cdots x_{s-r+1} y_1 y_2 y_3 y_4 z_1$.

Subcase 3.1. $s \leq p-1$, $n \geq 2$, $r > 2$.

Since $\sum\limits_{i=1}^{s-r-2} x_{i,0} + \sum\limits_{i=1}^{4} y_{i,0} \leq s - r - 2 + 4 < s = \bar{c}_0$, the first equation of (2.2) has no solution. It follows that such h is impossible to exist.

Subcase 3.2. $s \leq p-1$, $n \geq 2$, $r = 2$.

For $\overline{S} = (s, s, s+1)$, solve the corresponding group of equations (2.2) by virtue of (2.1), we get the generators which are all zero since they are all contain $h_{3,0}^2 = 0$.

Subcase 3.3. $s \leq p-1$, $r = 1$.

Subcase 3.3.1. $s \leq p-1$, $n = 2$, $r = 1$.

For $\overline{S} = (s, s, s+1)$, solve the corresponding group of equations (2.2) by virtue of (2.1), we get the generators

$$a_3^{s-3} h_{3,0}^2 h_{2,0} h_{1,2} b_{1,1}, \quad a_3^{s-3} h_{3,0}^2 h_{1,0} h_{1,2} b_{2,0}, \quad a_3^{s-3} h_{3,0}^2 h_{1,0} h_{2,1} b_{1,1}$$

which are all zero due to $h_{3,0}^2 = 0$.

Subcase 3.3.2. $s \le p-1$, $n = 3$, $r = 1$.

This will be a hard case. Our strategy of computation is as follows. For the unique sequence in this case $\overline{S} = (s, s, s, 1)$, we see that the maximal number in $\overline{S}$ is s. This leads us to first compute all the generators (may be zero) of $E_1^{s,tq+(s-3),*}$ with the form $x_1 \cdots x_{s-3} y_1 y_2 y_3$. For each a_i and $h_{i,j}$ in $x_1 \cdots x_{p-4} y_1 y_2 y_3$, we resolve

$$h_{i,j} \to h_{k,j} h_{i-k,j+k} \qquad \text{or} \qquad a_i \to a_j h_{i-j,j}$$

and then repeat this step once. Finally we do the replacement $h_{i,j} \to b_{i,j-1}$. If the obtained elements are nonzero, then we get all of the desired generators of $E_1^{s+3,tq+(s-3),*}$ with the form $x_1 \cdots x_{p-4} y_1 y_2 y_3 z_1$. We do this computation by the following steps:

Step 1. For $\overline{S} = (s, s, s, 1)$, solve the corresponding group of equations (2.2) by virtue of (2.1), we obtain two generators of $E_1^{s,tq+s-3,*}$

$$a_3^{s-3} h_{4,0} \underline{h_{3,0} h_{3,0}}, \qquad a_4 a_3^{s-4} \underline{h_{3,0} h_{3,0} h_{3,0}}$$

Step 2. Resolve a_i or $h_{i,j}$.

$$a_3^{s-3} h_{4,0} \underline{h_{3,0} h_{3,0}} \xrightarrow{h_{3,0}} \begin{cases} a_3^{s-3} h_{4,0} h_{3,0} h_{2,0} h_{1,2} \\ a_3^{s-3} h_{4,0} h_{3,0} h_{1,0} h_{2,1} \end{cases}$$

$$a_4 a_3^{s-4} \underline{h_{3,0} h_{3,0} h_{3,0}} \xrightarrow{h_{3,0}} \begin{cases} a_4 a_3^{s-4} \underline{h_{3,0} h_{3,0}} h_{2,0} h_{1,2} \\ a_4 a_3^{s-4} \underline{h_{3,0} h_{3,0}} h_{1,0} h_{2,1} \end{cases}$$

Step 3. Repeat the Step 2 and replace some $h_{i',j'}$ by $b_{i',j'-1}$. Since there are many identical generators appearing in the obtained first May differentials, we will only write out the different nonzero generators for simplicity.

(3.1)

$$a_3^{s-3}h_{4,0}h_{3,0}h_{2,0}h_{1,2} \xrightarrow{\text{one by one}} \left\{ \begin{array}{l} a_3^{s-4}a_1h_{4,0}h_{3,0}h_{2,0}h_{2,1}h_{1,2} \xrightarrow{h_{2,1} \text{ or } h_{1,2}} \\ \qquad\qquad \left\{ \begin{array}{l} a_3^{s-4}a_1h_{4,0}h_{3,0}h_{2,0}h_{2,1}b_{1,1} \\ a_3^{s-4}a_1h_{4,0}h_{3,0}h_{2,0}h_{1,2}b_{2,0} \end{array} \right. \\ a_3^{s-4}a_2h_{4,0}h_{3,0}h_{2,0}\underline{h_{1,2}h_{1,2}} \xrightarrow{h_{1,2}} \\ \qquad\qquad a_3^{s-4}\underline{a_2h_{4,0}h_{3,0}}h_{2,0}h_{1,2}b_{1,1} \\ a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{3,1}h_{1,2} \xrightarrow{h_{1,2} \text{ or } h_{3,1}} \\ \qquad\qquad \left\{ \begin{array}{l} a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{3,1}b_{1,1} \\ a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{1,2}b_{3,0} \end{array} \right. \\ a_3^{s-3}h_{4,0}h_{2,0}h_{1,0}h_{2,1}h_{1,2} \xrightarrow{h_{1,2} \text{ or } h_{2,1}} \\ \qquad\qquad \left\{ \begin{array}{l} a_3^{s-3}h_{4,0}h_{2,0}h_{1,0}h_{2,1}b_{1,1} \\ a_3^{s-3}h_{4,0}h_{2,0}h_{1,0}h_{1,2}b_{2,0} \end{array} \right. \\ a_3^{s-3}h_{4,0}h_{3,0}h_{1,0}h_{1,1}h_{1,2} \xrightarrow{h_{1,1} \text{ or } h_{1,2}} \\ \qquad\qquad \left\{ \begin{array}{l} a_3^{s-3}h_{4,0}h_{3,0}h_{1,0}h_{1,1}b_{1,1} \\ a_3^{s-3}h_{4,0}h_{3,0}h_{1,0}h_{1,2}b_{1,0} \end{array} \right. \end{array} \right.$$

(3.2)

$$a_3^{s-3}h_{4,0}h_{3,0}h_{1,0}h_{2,1} \xrightarrow{a_3 \text{ or } h_{4,0}} \left\{ \begin{array}{l} a_3^{s-4}a_1h_{4,0}h_{3,0}h_{1,0}\underline{h_{2,1}h_{2,1}} \xrightarrow{h_{2,1}} \\ \qquad\qquad a_3^{s-4}\underline{a_1h_{4,0}h_{3,0}}h_{1,0}h_{2,1}b_{2,0} \\ a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{2,2}h_{2,1} \xrightarrow{h_{2,1} \text{ or } h_{2,2}} \\ \qquad\qquad \left\{ \begin{array}{l} a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{2,2}b_{2,0} \\ a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{2,1}b_{2,1} \end{array} \right. \end{array} \right.$$

$$(3.3) \qquad a_4a_3^{s-4}\underline{h_{3,0}h_{3,0}}h_{2,0}h_{1,2} \xrightarrow{h_{3,0}} a_4a_3^{s-4}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,2} \xrightarrow{h_{2,1} \text{ or } h_{1,2}}$$

$$\left\{ \begin{array}{l} a_4a_3^{s-4}h_{3,0}h_{2,0}h_{1,0}h_{2,1}b_{1,1} \\ a_4a_3^{s-4}h_{3,0}h_{2,0}h_{1,0}h_{1,2}b_{2,0} \end{array} \right.$$

In the above diagrams the elements over the first left arrow and the second right arrow mean their resolution and replacement, respectively.

Subcase 3.3.3. $s > p-2$, $n > 3$, $r = 1$.

Since $\sum_{j=1}^{s-r-2} x_{j,i} + \sum_{k=1}^{4} y_{k,i} + z_{1,i} \le s+2 < p = \bar{c}_i$, the i-th equation of (2.2) has no solution. It follows that such h is impossible to exist.

Subcase 3.3.4. $s = p-2, n > 3, r = 1$.

For S_1, solve the corresponding group of equations (2.2) by virtue of (2.1), we get the generators listed as

$$a_3^{p-5}h_{3,0}^2h_{2,0}h_{1,2}b_{1,n}, \quad a_3^{p-5}h_{3,0}^2h_{1,0}h_{2,1}b_{1,n},$$
$$a_3^{p-5}h_{3,0}^2h_{2,0}h_{1,n+1}b_{1,1}, \quad a_3^{p-5}h_{3,0}^2h_{1,0}h_{1,n+1}b_{2,0}$$

which are all zero due to $h_{3,0}^2 = 0$.

For $S_4 = (s, s, s, p, p-1, \cdots, p-1, 0)$, solve the corresponding group of equations (2.2) by virtue of (2.1), we get the generators listed as following

$$a_n^{p-6}a_4h_{n,0}^3h_{n-3,3}b_{n-3,2}, \quad a_n^{p-5}h_{n,0}^2h_{4,0}h_{n-3,3}b_{n-3,2},$$
$$a_n^{p-5}h_{n,0}^3h_{n-3,3}b_{1,2}, \quad a_n^{p-5}h_{n,0}^3h_{1,3}b_{n-3,2}$$

which are all zero due to $h_{n,0}^2 = 0$.

For $S_i (i \geq 5)$, there is no solution.

Subcase 3.3.5. $s = p-1, n > 3, r = 1$.

For S_1, we get the similar generators as Subcase 3.3.4 which are all zero.

For $S_4 = (p-1, p-1, p-1, p, p-1, \cdots, p-1, 0)$, we obtain a set of generators

$$(3.4) \qquad a_n^{p-4}h_{n,0}^2h_{4,0}h_{n-3,3} \xrightarrow{h_{n,0}} a_n^{p-4}h_{n,0}h_{i,0}h_{n-i,i}h_{4,0}h_{n-3,3}$$

$$\xrightarrow{h_{n-3,3} \text{ or } h_{n-i,i}} \begin{cases} a_n^{p-4}h_{n,0}h_{i,0}h_{4,0}h_{n-3,3}b_{n-i,i-1} \\ a_n^{p-4}h_{n,0}h_{i,0}h_{n-i,i}h_{4,0}b_{n-3,2} \end{cases}$$

where $0 < i < n$ and $i \neq 4$.

For S_i $(i \geq 5)$, there is no solution.

Case 4. $h = x_1 \cdots x_{s-r-2}y_1y_2y_3y_4y_5y_6$.

Subcase 4.1. $s \leq p-1$, $n \geq 2$, $r > 4$.

There is no solution since

$$\sum_{i=1}^{s-r-2} x_{i,0} + \sum_{j=1}^{6} y_{j,0} \leq s - r + 4 < s = \bar{c}_0.$$

Subcase 4.2. $s \leq p-1$, $r = 4$.

Subcase 4.2.1. $s \leq p-1$, $r = 4$, $n = 2$.

There is no solution since

$$\sum_{i=1}^{s-r-2} x_{i,0} + \sum_{j=1}^{6} y_{j,0} \leq s - r + 4 < s + 1 = \bar{c}_2.$$

Subcase 4.2.2. $s \le p-1$, $r=4$, $n=3$.

Solve the corresponding group of equations (2.2) by virtue of (2.1), we get two generators $a_3^{s-3}h_{4,0}h_{3,0}^5$ and $a_4a_3^{s-4}h_{3,0}^6$ which are both zero due to $h_{3,0}^2=0$.

Subcase 4.2.3. $s \le p-1$, $r=4$, $n>3$.

For $S_1=(s,s,s,0,\cdots,0,1)$, there is no solution.

For $S_i(i\ge 4)$, $S=(s,s,s,0,\cdots,0,p^{(i)},p-1,\cdots,p-1,0)$, there is no solution due to

$$\sum_{j=1}^{s-r-2} x_{j,i} + \sum_{k=1}^{6} y_{k,i} < s+1 \le p = \bar{c}_i.$$

Subcase 4.3. $s \le p-1$, $r=3$.

Subcase 4.3.1. $s \le p-1$, $2\le n\le 3$, $r=3$.

Solve the corresponding group of equations (2.2) by virtue of (2.1), we see that all the generators are zero since they all contain $h_{3,0}^2=0$.

Subcase 4.3.2. $s \le p-1$, $n>3$, $r=3$.

For S_1, we get one generator $a_3^{s-5}h_{3,0}^5h_{1,n}$ which is zero as $h_{3,0}^2=0$.

For $S_i(i\ge 4)$, $S=(s,s,s,0,\cdots,0,p^{(i)},p-1,\cdots,p-1,0)$, there is no solution since

$$\sum_{j=1}^{s-r-2} x_{j,i} + \sum_{k=1}^{6} y_{k,i} < s+1 < p = \bar{c}_i.$$

Subcase 4.4. $s \le p-1$, $r=2$.

Subcase 4.4.1. $s \le p-1$, $n=2$, $r=2$.

Solve the corresponding group of equations (2.2) by virtue of (2.1), we get one generator
$a_3^{s-3}h_{3,0}^3h_{1,0}h_{2,1}h_{1,2}$ which is zero due to $h_{3,0}^3=0$.

Subcase 4.4.2. $s \le p-1$, $n=3$, $r=2$.

There is one nonzero generator

$$a_3^{s-4}h_{4,0}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,2}. \tag{3.5}$$

Subcase 4.4.3. $s < p-2$, $n>3$, $r=2$.

For S_1, solve the corresponding group of equations (2.2) by virtue of (2.1), we get two generators

$$a_3^{s-4}h_{3,0}^3h_{2,0}h_{1,2}h_{1,n} \quad \text{and} \quad a_3^{s-4}h_{3,0}^3h_{1,0}h_{2,1}h_{1,n}$$

which are both zero since $h_{3,0}^2=0$.

For $S_i (i \geq 4)$, $S = (s, s, s, 0, \cdots, 0, p^{(i)}, p-1, \cdots, p-1, 0)$, there is no solution since

$$\sum_{j=1}^{s-r-2} x_{j,i} + \sum_{k=1}^{6} y_{k,i} < s+2 < p = \bar{c}_i.$$

Subcase 4.4.4. $s = p-2$, $n > 3$, $r = 2$.

For S_1, we get two similar generators as Subcase 4.4.3 by replacing s with $p-2$.

For S_4, we get two generators

$$a_n^{s-4} h_{n,0}^3 h_{4,0} h_{n-3,3}^2 \quad \text{and} \quad a_n^{s-4} h_{n,0}^4 h_{n-3,3} h_{1,3}$$

which are both zero since $h_{n,0}^2 = 0$.

For $S_i (i \geq 5)$, there is no solution.

Subcase 4.4.5. $s = p-1$, $n > 3$, $r = 2$.

For S_1, we get two similar generators as Subcase 4.4.3 by replacing s with $p-1$.

For S_4, solve the corresponding group of equations (2.2) by virtue of (2.1), we get the generators they are all zero because they all contain $h_{n,0}^2 = 0$.

For $S_i (i \geq 5)$, there is no solution.

Subcase 4.5. $s \leq p-1$, $r = 1$.

Subcase 4.5.1. $s \leq p-1$, $n = 2$, $r = 1$.

Solve the corresponding group of equations (2.2) by virtue of (2.1), we get the generators listed as

$$a_3^{s-3} h_{3,0} h_{2,0} h_{1,0} h_{2,1} h_{1,2}^2, \quad a_3^{s-3} h_{3,0} h_{2,0}^2 h_{1,2}^3,$$

$$a_3^{s-3} h_{3,0}^2 h_{1,0} h_{1,1} h_{1,2}^2, \quad a_3^{s-3} h_{3,0} h_{1,0}^2 h_{2,1}^2 h_{1,2}.$$

They are all zero due to $h_{1,2}^2 = 0$ or $h_{1,0}^2 = 0$.

Subcase 4.5.3. $s \leq p-1$, $n = 3$, $r = 1$.

We adopt the methods similar to Subcase 3.3.2. We still compute all the generators (may be zero) of $E_1^{s,tq+s-3,*}$ with the form $x_1 \cdots x_{s-3} y_1 y_2 y_3$. First we resolve a_i or $h_{i,j}$ in $x_1 \cdots x_{s-3} y_1 y_2 y_3$, and then for the obtained elements we repeat the first step twice. If the obtained element is nonzero, then it is our desired generator of $E_1^{s+3,t+s-3,*}$.

Step 1. For $\overline{S} = (s, s, s, 1) = (3, 3, 3, 1)$, solve the corresponding group of equations (2.2) by virtue of (2.1), we obtain two generators of $E_1^{s,t+s-3,*}$ as

$$a_3^{s-3} h_{4,0} \underline{h_{3,0} h_{3,0}} \quad \text{and} \quad a_4 a_3^{s-4} \underline{h_{3,0} h_{3,0} h_{3,0}}$$

Step 2. Resolve a_i or $h_{i,j}$.

$$a_3^{s-3}h_{4,0}\underline{h_{3,0}h_{3,0}} \xrightarrow{h_{3,0}} \begin{cases} a_3^{s-3}h_{4,0}h_{3,0}h_{2,0}h_{1,2}, \\ a_3^{s-3}h_{4,0}h_{3,0}h_{1,0}h_{2,1} \end{cases}$$

$$a_4a_3^{s-4}\underline{h_{3,0}h_{3,0}h_{3,0}} \xrightarrow{h_{3,0}} \begin{cases} a_4a_3^{s-4}\underline{h_{3,0}h_{3,0}}h_{2,0}h_{1,2}, \\ a_4a_3^{s-4}\underline{h_{3,0}h_{3,0}}h_{1,0}h_{2,1} \end{cases}$$

Step 3. Repeat the Step 2 twice. We still only write out the different nonzero generators for simplicity.

(3.6)

$$a_3^{s-3}h_{4,0}h_{3,0}h_{2,0}h_{1,2} \xrightarrow{\text{one by one}} \begin{cases} a_3^{s-4}a_0h_{4,0}\underline{h_{3,0}h_{3,0}}h_{2,0}h_{1,2} \xrightarrow{h_{3,0}} \\ \qquad a_3^{s-4}\overline{a_0h_{4,0}}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,2} \\ a_3^{s-4}a_1h_{4,0}h_{3,0}h_{2,0}h_{2,1}h_{1,2} \xrightarrow{h_{4,0} \text{ or } h_{2,0}} \\ \qquad \begin{cases} a_3^{s-4}a_1h_{3,0}h_{2,0}h_{1,0}h_{3,1}h_{2,1}h_{1,2} \\ a_3^{s-4}a_1h_{4,0}h_{3,0}h_{1,0}h_{2,1}h_{1,1}h_{1,2} \end{cases} \\ a_3^{s-3}\underline{h_{3,0}h_{3,0}}h_{2,0}h_{1,2}h_{1,3} \xrightarrow{h_{3,0}} \\ \qquad a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,2}h_{1,3} \\ a_3^{s-3}h_{3,0}\underline{h_{2,0}h_{2,0}}h_{2,2}h_{1,2} \xrightarrow{h_{2,0}} \\ \qquad a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{1,1}h_{2,2}h_{1,2} \end{cases}$$

$$\text{(3.7)} \qquad a_3^{s-3}h_{4,0}h_{3,0}h_{1,0}h_{2,1} \xrightarrow{a_3} a_3^{s-4}a_2h_{4,0}h_{3,0}h_{1,0}h_{2,1}h_{1,2} \xrightarrow{h_{4,0}} a_3^{s-4}a_2h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{2,2}h_{1,2}$$

Subcase 4.5.3. $s < p-3$, $n > 3$, $r = 1$.

For S_1, we get one generator as follows:

$$\text{(3.8)} \qquad a_3^{s-3}\underline{h_{3,0}h_{3,0}h_{3,0}}h_{1,n} \xrightarrow{h_{3,0}} \left\{ \begin{array}{l} a_3^{s-3}\underline{h_{3,0}h_{3,0}}h_{2,0}h_{1,2}h_{1,n} \\ a_3^{s-3}\underline{h_{3,0}h_{3,0}}h_{1,0}h_{2,1}h_{1,n} \end{array} \right\} \longrightarrow \xrightarrow{h_{3,0}} a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,2}h_{1,n}.$$

For $S_i, (i \geq 4)$, there is no solution since $\sum_{j=1}^{s-3} x_{j,i} + \sum_{k=1}^{6} y_{k,i} \leq s+3 < p = \bar{c}_i$.

Subcase 4.5.4. $s = p-3, n > 3, r = 1$.

For S_1, we get the similar generator as Subcase 4.5.3 by replacing s with $p-3$.

For S_4, solve the corresponding group of equations (2.2) by virtue of (2.1), the generators we get are all zero since they all contain $h_{n,0}^2 = 0$.

For $S_i(i > 4)$, there is no solution.

Subcase 4.5.5. $s = p-2, n > 3, r = 1$.

For S_1, we get the similar generator as Subcase 4.5.3 by replacing s with $p-2$.

For S_4, solve the corresponding group of equations (2.2) by virtue of (2.1), we find that all the generators are zero.

For $S_i(i > 4)$, there is no solution.

Subcase 4.5.6. $s = p-1, n > 3, r = 1$.

For S_1, we get the similar generator as Subcase 4.5.3 by replacing s with $p-1$.

For S_4, solve the corresponding group of equations (2.2) by virtue of (2.1), we get the generators as follows.

(3.9)
$$\begin{cases} a_n^{p-4}h_{n,0}^3h_{1,3} \xrightarrow{h_{n,0}} a_n^{p-4}h_{n,0}^2h_{i,0}h_{n-i,i}h_{1,3} \xrightarrow{h_{n,0}} a_n^{p-4}h_{n,0}h_{j,0}h_{i,0}h_{n-j,j}h_{n-i,i}h_{1,3} \\ (0 < i; j < n; i \neq j) \\ a_n^{p-5}a_4h_{n,0}^3h_{n-3,3} \xrightarrow{h_{n,0}} a_n^{p-5}a_4h_{n,0}^2h_{i,0}h_{n-i,i}h_{n-3,3} \xrightarrow{h_{n,0}} \\ \qquad a_n^{p-5}a_4h_{n,0}h_{j,0}h_{n-j,j}h_{i,0}h_{n-i,i}h_{n-3,3}(0 < i, j < n; i \neq j; i, j \neq 3) \end{cases}$$

$$(3.10) \qquad a_n^{p-4}h_{n,0}^2h_{4,0}h_{n-3,3} \xrightarrow{h_{n,0}} a_n^{p-4}h_{n,0}h_{i,0}h_{4,0}h_{n-i,i}h_{n-3,3}$$

$$\xrightarrow{\text{one by one}} \begin{cases} a_n^{p-5}a_jh_{n,0}h_{4,0}h_{i,0}h_{n-i,i}h_{n-j,j}h_{n-3,3} \ (0 < i, j < n; i \neq j; i \neq 3, 4; j \neq 3) \\ a_n^{p-4}h_{j,0}h_{i,0}h_{4,0}h_{n-i,i}h_{n-j,j}h_{n-3,3} \ (0 < i, j < n; i \neq j; i \neq 3, 4; j \neq 3) \\ a_n^{p-4}h_{n,0}h_{j,0}h_{i-j,j}h_{4,0}h_{n-i,i}h_{n-3,3} \ (0 < i; j < n, i > j; j \neq 4; i \neq 3) \\ a_n^{p-4}h_{n,0}h_{i,0}h_{j,0}h_{4-j,j}h_{n-i,i}h_{n-3,3} \ (0 < i < n; 0 < j < 4; i \neq j; i \neq 3) \\ a_n^{p-4}h_{n,0}h_{i,0}h_{4,0}h_{n-i-j,i}h_{j,n-j}h_{n-3,3} \ (0 < i < n, j < n-i, i \neq 4) \\ a_n^{p-4}h_{n,0}h_{i,0}h_{4,0}h_{n-i,i}h_{n-3-j,3}h_{j,n-j} \ (0 < i < n; j < n-3; i \neq 4) \end{cases}$$

For S_i $(i \geq 5)$, there is no solution. □

4. **Proof of Theorem 1.1.**

In this section we will give the proof of Theorem 1.1. First we need the following lemma:

Lemma 4.1. *Let $p \geq 7$ and $n \geq 2$. Let $t = s + sp + sp^2 + p^n$. Then the following two properties hold:*

(a) *for $2 \leq r \leq s+4$, $\mathrm{Ext}_{\mathcal{A}}^{s-r+4, tq+s-r-2}(\mathbb{Z}/p, \mathbb{Z}/p) = 0$.*

(b) *the product $\widetilde{\gamma}_s b_{n-1} g_0 \in \mathrm{Ext}_{\mathcal{A}}^{s+4, tq+s-3}(\mathbb{Z}/p, \mathbb{Z}/p)$ is non-zero.*

Proof. Since the elements $b_{1,n-1}$, $h_{1,0}h_{2,0}$, $a_3^{s-3}h_{3,0}h_{2,1}h_{1,2} \in E_1^{*,*,*}$ are all permanent cycles in the MSS and converge nontrivially to b_{n-1}, g_0, $\widetilde{\gamma}_s \in \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}/p,\mathbb{Z}/p)$ respectively, it follows that the product element $\widetilde{\gamma}_s b_{n-1}g_0 \in \mathrm{Ext}_{\mathcal{A}}^{s+4,tq+s-3}(\mathbb{Z}/p,\mathbb{Z}/p)$ is represented by

$$a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{2,0}h_{1,0}b_{1,n-1} \in E_1^{s+4,tq+s-3,7s+p-8}$$

in the MSS.

(a) According to Theorem 3.1(1)(2), $E_1^{s-r+4,tq+s-r-2,*} = 0$ for $r \geq 2$ and $n \neq 3$. The only nontrivial case is

$$E_1^{s-r+4,tq+s-r-2,*} = \mathbb{Z}/p\{\underline{a_3^{s-4}h_{3,0}h_{2,1}h_{1,2}}h_{4,0}\underline{h_{2,0}h_{1,0}}\}$$

for $r = 2$ and $n = 3$. Since $d_1(h_{4,0}) = h_{1,0}h_{3,1} + h_{2,0}h_{2,2} + h_{3,0}h_{1,3} \neq 0$, then $h_{4,0}$ becomes zero in $E_k^{*,*,*}$ for $k \geq 2$. According to the above statement $a_3^{s-4}h_{3,0}h_{2,1}h_{1,2}$ and $h_{2,0}h_{1,0}$ are permanent cycles in the May spectral sequence, thus they are always nontrivial in $E_k^{*,*,*}$ for any $k \geq 1$. It follows that $\underline{a_3^{s-4}h_{3,0}h_{2,1}h_{1,2}}h_{4,0}\underline{h_{2,0}h_{1,0}}$ becomes trivial in $E_k^{s+2,tq+s-4,*}$ for $k \geq 2$.

Since

$$E_k^{s-r+4,tq+s-r-2,*} \Longrightarrow \mathrm{Ext}_{\mathcal{A}}^{s-r+4,tq+s-r-2}(\mathbb{Z}/p,\mathbb{Z}/p)$$

for $k \geq 2$, it follows that $\mathrm{Ext}_{\mathcal{A}}^{s-r+4,tq+s-r-2,}(\mathbb{Z}/p,\mathbb{Z}/p) = 0$ for $r \geq 2$. Thus we have proven part (a).

(b) In what follows we need to show that $a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{2,0}h_{1,0}b_{1,n-1}$ can not be hit by the May differential $d_r\colon E_r^{s+3,tq+s-3,7s+p+r-8} \longrightarrow E_r^{s+4,tq+s-3,7s+p-8}$ for any $r \geq 1$.

According to the above analysis we only need to consider the generators of $E_1^{s+3,tq+s-3,M}$ with $M > 7s+p-8$. Hence by Theorem 3.1 we do not need to consider the generators with May filtrations M_3^3, M_6^3, M_7^n as their May differentials will not touch $a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{1,0}h_{2,0}b_{1,n-1}$. For other generators we list them up with their first May differentials in the following table:

	$E_1^{s+3,tq+s-3,M_i^n}$	M_i^n	first May differential
$1st$	$a_3^{s-4}a_1h_{4,0}h_{3,0}h_{2,0}h_{1,2}b_{2,0}$	M_4^3	$a_3^{s-4}a_1\underline{h_{1,0}h_{3,1}}h_{3,0}h_{2,0}h_{1,2}b_{2,0}+\cdots$
$2nd$	$a_3^{s-3}h_{4,0}h_{2,0}h_{1,0}h_{1,2}b_{2,0}$	M_4^3	$a_3^{s-3}\underline{h_{3,0}h_{1,3}}h_{2,0}h_{1,0}h_{1,2}b_{2,0}+\cdots$
$3rd$	$a_3^{s-4}a_1h_{4,0}h_{3,0}h_{1,0}h_{2,1}b_{2,0}$	M_4^3	$a_3^{s-4}a_1\underline{h_{2,0}h_{2,2}}h_{3,0}h_{1,0}h_{2,1}b_{2,0}+\cdots$
$4th$	$a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{2,2}b_{2,0}$	M_4^3	$a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}\underline{h_{1,2}h_{1,3}}b_{2,0}+\cdots$
$5th$	$a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{2,1}b_{2,1}$	M_4^3	$a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}\underline{h_{1,1}h_{1,2}}b_{2,1}+\cdots$
$6th$	$a_4a_3^{s-4}h_{3,0}h_{2,0}h_{1,0}h_{1,2}b_{2,0}$	M_4^3	$\underline{a_1h_{3,1}}a_3^{s-4}h_{3,0}h_{2,0}h_{1,0}h_{1,2}b_{2,0}+\cdots$
$7th$	$a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{1,2}b_{3,0}$	M_5^3	zero
$8th$	$\mathbf{h_1^{(i)}}$ $(s=p-1)$ $(0<i<n, i\neq 4)$	M_8^n	$a_n^{s-3}\underline{h_{j,0}h_{n-j,j}}h_{i,0}h_{4,0}h_{n-3,3}b_{n-i,i-1}+\cdots$ $(j\neq i,3,4)$
$9th$	$\mathbf{h_2^{(i)}}$ $(s=p-1)$ $(0<i<n, i\neq 4)$	M_9^n	$a_n^{s-3}\underline{h_{j,0}h_{n-j,j}}h_{i,0}h_{n-i,i}h_{4,0}b_{n-3,2}+\cdots$ $(j\neq i,4)$
$10th$	$\mathbf{h_3^{(i)}}$ $(s=p-1)$ $(0<i,j<n; i\neq j; i,j\neq 3)$	M_{10}^n	$a_n^{s-4}a_4\underline{h_{k,0}h_{n-k,k}}h_{j,0}h_{n-j,j}h_{i,0}h_{n-i,i}h_{n-3,3}+\cdots$ $(k\neq i,j,3)$
$11th$	$\mathbf{h_4^{(i)}}$ $(s=p-1)$ $(0<i,j<n; i\neq j; i\neq 3,4; j\neq 3)$	M_{10}^n	$a_n^{s-4}a_j\underline{h_{k,0}h_{n-k,k}}h_{4,0}h_{i,0}h_{n-i,i}h_{n-j,j}h_{n-3,3}+\cdots$ $(k\neq i,j,3,4)$
$12th$	$\mathbf{h_5^{(i)}}$ $(s=p-1)$ $(0<i,j<n; i\neq j)$	M_{11}^n	$a_n^{s-3}\underline{h_{k,0}h_{n-k,k}}h_{i,0}h_{n-i,i}h_{j,0}h_{n-j,j}h_{1,3}+\cdots$ $(k\neq i,j)$
$13th$	$\mathbf{h_6^{(i)}}$ $(s=p-1)$ $(0<i,j<n; i\neq j; i\neq 3,4; j\neq 3)$	M_{11}^n	$a_n^{s-4}\underline{a_kh_{n-k,k}}h_{4,0}h_{i,0}h_{n-i,i}h_{j,0}h_{n-j,j}h_{n-3,3}+\cdots$ $(k\neq i,j,3)$
$14th$	$\mathbf{h_7^{(i)}}$ $(s=p-1)$ $(0<i,j<n; i>j; j\neq 4; i\neq 3)$	M_{11}^n	$a_n^{s-4}\underline{a_kh_{n-k,k}}h_{n,0}h_{j,0}h_{i-j,j}h_{n-i,i}h_{4,0}h_{n-3,3}+\cdots$ $(k\neq 0,3)$
$15th$	$\mathbf{h_8^{(i)}}$ $(s=p-1)$ $(0<i<n; 0<j<4; i\neq j; i\neq 3)$	M_{11}^n	$a_n^{s-4}\underline{a_kh_{n-k,k}}h_{n,0}h_{i,0}h_{n-i,i}h_{j,0}h_{4-j,j}h_{n-3,3}+\cdots$ $(k\neq 0,i,3)$
$16th$	$\mathbf{h_9^{(i)}}$ $(s=p-1)$ $(0<i<n; j<n-i; i\neq 4)$	M_{11}^n	$a_n^{s-4}\underline{a_kh_{n-k,k}}h_{i,0}h_{n-i-j,i}h_{j,n-j}h_{4,0}h_{n-3,3}+\cdots$ $(k\neq 3)$
$17th$	$\mathbf{h_{10}^{(i)}}$ $(s=p-1)$ $(0<i<n; j<n-3; i\neq 4)$	M_{11}^n	$a_n^{s-4}\underline{a_kh_{n-k,k}}h_{4,0}h_{i,0}h_{n-i,i}h_{n-3-j,3}h_{j,n-j}+\cdots$ $(k\neq i)$

In the above diagram the first May differential of the seventh generator is zero since the first May differentials of $a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}$, $h_{1,2}$ and $b_{3,0}$ are all zero. For the first six generators with May filtration M_4^3, we see that the first May differential of each generator contains at least a term which is not in the first May differential of the other generators. It follows that the first May differentials of the generators are linearly independent and thus the cycle of $E_1^{s+3,tq+s-3,M_4^3}$ must be zero. This implies that $E_r^{s+3,tq+s-3,M_4^3}=0$ for $r\geq 2$ and hence

$$a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{1,0}h_{2,0}b_{1,n-1}\notin d_r(E_r^{s+3,tq+s-3,M_4^3}) \text{ for } r\geq 1.$$

By the same method we can similarly show that for the generators counting from 7 to 11, there is

$$a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{1,0}h_{2,0}b_{1,n-1} \notin d_r(E_r^{s+3,tq+s-3,M_i^n}) \text{ for } r \geq 1$$

with corresponding i and n for each mentioned generator.

We deal with the last six families of generators differently. By the table (10) in the last part of the proof of Theorem 3.1, we see that $\mathbf{h_m^{(i)}}$ $(5 \leq m \leq 10)$ all come from the first May differential of

$$a_n^{s-3}h_{n,0}h_{i,0}h_{4,0}h_{n-i,i}h_{n-3,3}.$$

Thus $d_1(\mathbf{h_m^{(i)}})$ $(5 \leq m \leq 10)$ will possibly be linearly dependent in this case. In order to avoid this, we first just consider $\mathbf{h_5^{(i)}}$. Since $d_1(\mathbf{h_5^{(i)}}) \neq 0$, it follows that $\mathbf{h_5^{(i)}}$ vanishes in $E_2^{s+3,tq+s-3,M_{11}^n}$. Now for the remaining five families of generators, according to the above table we see that the first May differential of each generator contains at least a term which is not in the first May differential of the other generators. It follows that the first May differentials of these generators are linearly independent. Thus $\mathbf{h_m^{(i)}}$ $(6 \leq m \leq 10)$ also vanish in $E_2^{s+3,tq+s-3,M_{11}^n}$. Hence $E_k^{s+3,tq+s-3,M_{11}^n} = 0$ for $k \geq 2$ and then

$$a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{1,0}h_{2,0}b_{1,n-1} \notin d_r(E_r^{s+3,tq+s-3,M_{11}^n)} \text{ for } r \geq 1.$$

According to the above discussion, we see that

$$a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{1,0}h_{2,0}b_{1,n-1}$$

cannot be hit by any May differential for $n \geq 2$. Thus it is a permanent cycle in the MSS and converges nontrivially to $\widetilde{\gamma}_s b_{n-1}g_0 \in \mathrm{Ext}_{\mathcal{A}}^{s+4,tq+s-3}(\mathbb{Z}/p,\mathbb{Z}/p)$. Thus we have proven the part (b). □

We need also the following lemma:

Lemma 4.2. *[3] Let $p \geq 5$ and $n \geq 2$. Then*

$$(i_1i_0)_*(b_{n-1}g_0) \in \mathrm{Ext}_{\mathcal{A}}^{4,(2+p+p^n)q}(H^*(V(1)),\mathbb{Z}/p)$$

converges to a nontrivial homotopy element $\zeta_n \in \pi_{(2+p+p^n)q-4}V(1)$.

In what follows we give our proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 4.2 we see that

$$(i_1i_0)_*(b_{n-1}g_0) \in \mathrm{Ext}_{\mathcal{A}}^{4,(2+p+p^n)q}(H^*(V(1)),\mathbb{Z}/p)$$

converges to a nontrivial homotopy element $\zeta_n \in \pi_{(2+p+p^n)q-4}V(1)$. Consider the following composite of maps:

$$\widetilde{f}\colon \Sigma^{(2+p+p^n)q-4}S \xrightarrow{\xi_n} V(1) \xrightarrow{i_2} V(2) \xrightarrow{\gamma^s} \Sigma^{-s(1+p+p^2)q}V(2) \xrightarrow{j_0j_1j_2} \Sigma^{-s(1+p+p^2)q+(p+2)q+3}S$$

where γ^s denotes the n-times of composites of γ. Thus $\widetilde{f}$ is represented by

$$\widetilde{c} = (j_0j_1j_2)_*(\gamma^s)_*(i_2i_1i_0)_*(b_{n-1}g_0) = (j_0j_1j_2\gamma^s i_2i_1i_0)_*(b_{n-1}g_0)$$

in the ASS. Since $\gamma_s \in \pi_{(s-3+(s-1)p+sp^2)q-3}S$ is represented by

$$\widetilde{\gamma}_s \in \mathrm{Ext}_A^{s,(s-3+(s-1)p+sp^2)q+s-3}(\mathbb{Z}/p,\mathbb{Z}/p)$$

due to [12], by the knowledge of Yoneda products we know that the composite

$$\mathrm{Ext}_{\mathcal{A}}^{0,0}(\mathbb{Z}/p,\mathbb{Z}/p) \xrightarrow{(i_2i_1i_0)_*} \mathrm{Ext}_{\mathcal{A}}^{0,0}(H^*V(2),\mathbb{Z}/p) \xrightarrow{(j_0j_1j_2)_*(\gamma_*)^s} \mathrm{Ext}_{\mathcal{A}}^{s,(s-3+(s-1)p+sp^2)q+s-3}(\mathbb{Z}/p,\mathbb{Z}/p)$$

is multiplication (up to a nonzero scalar) by

$$\widetilde{\gamma}_s \in \mathrm{Ext}_{\mathcal{A}}^{s,(s-3+(s-1)p+sp^2)q+s-3}(\mathbb{Z}/p,\mathbb{Z}/p)).$$

Hence $\widetilde{f}$ is represented up to a nonzero scalar by a non-zero element

$$\widetilde{\gamma}_s b_{n-1}g_0 \in \mathrm{Ext}_{\mathcal{A}}^{s+4,(s+sp+sp^2+p^n)q+s-3}(\mathbb{Z}/p,\mathbb{Z}/p))$$

in the ASS (see Lemma 4.1(b)).

Moreover, by Lemma 4.1(a), $\widetilde{\gamma}_s b_{n-1}g_0$ cannot be hit by the Adams differential

$$d_r\colon E_r^{s-r+4,(s+sp+sp^2+p^n)q+s-r-2} \longrightarrow E_r^{s+4,(s+sp+sp^2+p^n)q+s-3},$$

for $r \geq 2$, hence the corresponding homotopy element $\widetilde{f} \in \pi_*S$ is nontrivial. Thus we have finished the proof of Theorem 1.1. □

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(Haibo Yu) School of Mathematical Sciences, South China Normal University, 510631, Guangzhou, P. R. China
E-mail address: yuhb@scnu.edu.cn

(Yanlei Kou) School of Mathematical Sciences, South China Normal University, 510631, Guangzhou, P. R. China
E-mail address: kouyanlei@tom.com

(Hao Zhao) School of Mathematical Sciences, South China Normal University, 510631, Guangzhou, P. R. China
E-mail address: zhaohao120@hotmail.com