Detection of a nontrivial element in the stable homotopy groups of spheres

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DETECTION OF A NONTRIVIAL ELEMENT IN THE STABLE HOMOTOPY GROUPS OF SPHERES

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Abstract. Let \( p \) be a prime with \( p \geq 7 \) and \( q = 2(p - 1) \). In this paper we prove the existence of a nontrivial product of filtration \( s + 4 \) in the stable homotopy groups of spheres. This nontrivial product is shown to be represented up to a nonzero scalar by the product element \( \gamma_s b_{n-1}g_0 \) in the Adams spectral sequence where \( n \geq 2 \) and \( 3 \leq s \leq p - 1 \).

Keywords: Stable homotopy groups of sphere, Adams spectral sequence, May spectral sequence.


1. Introduction

Let \( p \) be an odd prime. Let \( A \) be the mod \( p \) Steenrod algebra and let \( S \) be the sphere spectrum localized at \( p \). Throughout the paper we fix \( q = 2(p - 1) \). To determine the stable homotopy groups of sphere \( \pi_*S \) is one of the central problems in homotopy theory. One of the main tools to approach it is the classical Adams spectral sequence (ASS) whose \( E_2 \)-term is given by \( E_2^{s,t} = \text{Ext}_{A}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p) \) which is the cohomology of \( A \). The Adams differential is given by \( d_r: E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1} \).

From [6], we know that \( \text{Ext}_{A}^{1,1}(\mathbb{Z}/p, \mathbb{Z}/p) \) has \( \mathbb{Z}/p \)-basis consisting of \( a_0 \in \text{Ext}_{A}^{1,1}(\mathbb{Z}/p, \mathbb{Z}/p) \), \( h_i \in \text{Ext}_{A}^{1, (p^i+1)(p^{i+1}+1)}(\mathbb{Z}/p, \mathbb{Z}/p) \) for all \( i \geq 0 \) and we also know that \( \text{Ext}_{A}^{2,1}(\mathbb{Z}/p, \mathbb{Z}/p) \) has \( \mathbb{Z}/p \)-basis consisting of \( \tilde{a}_2 \), \( a_0^2 \), \( a_0h_i(i > 0) \), \( g_i(i \geq 0) \), \( k_i(i \geq 0) \), \( b_i(i \geq 0) \), and \( h_ih_j(j \geq i + 2, i \geq 0) \) whose internal
degrees are $2q+1$, $2$, $p^i q + 1$, $(p^{i+1} + 2p^i)q$, $(2p^{i+1} + p^i)q$, $p^{i+1}q$ and 
$(p^i + p^j)q$, respectively.

If a family of generators $x_i \in E^{s,*}_2$ converges nontrivially in the ASS, then we obtain a family of homotopy elements $f_i$ in $\pi_* S$ and we say that $f_i$ has filtration $s$ and is represented by $x_i \in E^{s,*}_2$ in the ASS. So far, very few families of homotopy elements in $\pi_* S$ have been detected. The following are some known results. In [7] M. Mahowald detected an order 2 element $\eta_i \in_2 \pi_*^s$, which is represented by $h_1 h_i \in \text{Ext}^1_{\mathbb{A}}(\mathbb{Z}/2, \mathbb{Z}/2)$. By analogous argument at odd primes, R. Cohen [1] detected a family of homotopy elements $\zeta_n \in p^{n+q+q-3}S$ which has filtration 3 and is represented by $h_0 h_{n-1} \in \text{Ext}^2_{\mathbb{A}}(\mathbb{Z}/p, \mathbb{Z}/p)$ in the ASS. In [9] D. Ravenel proved that $b_n \in \text{Ext}^2_{\mathbb{A}}(\mathbb{Z}/p, \mathbb{Z}/p)$ does not converge in the ASS, which is known to be the odd prime Kervaire invariant element. Recently, Hill-Hopkins-Ravenel [2] proved that the mod 2 Kervaire invariant one elements $\theta_j \in \pi_{2j+2-2}S$ exist only for $0 \leq j \leq 6$. This resolves a longstanding problem in algebraic topology.

Among the nontrivial elements of $\pi_* S$ the periodic elements are especially important. The existence of the periodic elements is related to the existence of Toda-Smith spectra. Let $BP$ be the Brown-Peterson spectrum localized at $p$. It is a $p$-local ring spectrum with the coefficient ring

$$BP_* = BP_* S = \mathbb{Z}(p)[v_1, v_2, \cdots]$$

where $v_i$ is the $i$-th Hazewinkel generator with degree $2(p^i - 1)$. If $X$ is a spectrum, then $BP_* X$ is a comodule over the Hopf algebroid $BP_* BP$ (refer to [10]). Toda [11] considered the existence of the finite spectra $V(n)$ with

$$BP_* V(n) \cong BP_* / I_{n+1}$$

as $BP_*$-module, hence as $BP_* BP$-comodule

where $I_{n+1} = (p, v_1, \cdots, v_n)$, the ideal generated by $p, v_1, \cdots, v_n$. In [11], Toda showed that $V(n)$ exists for $p > 2n$ with $n = 0, 1, 2, 3$ and there exists Greek letter map

$$v_n: \Sigma^{2(p^n-1)} V(n-1) \to V(n-1)$$

with $v_n = p, \alpha, \beta, \gamma$ for $n = 0, 1, 2, 3$, respectively. Here we write $V(-1)$ for $S$. Moreover, the cofibre of $v_n$ is $V(n)$ given by the cofibration

$$\Sigma^{2(p^n-1)} V(n-1) \xrightarrow{v_n} V(n-1) \xrightarrow{i_n} V(n) \xrightarrow{j_n} \Sigma^{2p^n-1} V(n-1).$$

If we write

$$\alpha_s = j_0(v_1^s)i_0, \quad \beta_s = j_0j_1(v_2^s)i_1i_0 \quad \text{and} \quad \gamma_s = j_0j_1j_2(v_3^s)i_2i_1i_0,$$
then $\alpha_s$, $\beta_s$, $\gamma_s$ are the well known first, second and third periodic elements in $\pi_s S$ with filtration $s$ (refer to [8]). It was shown in [12] that when $n < p$ and $s \not\equiv 0, 1, \cdots, n - 1 \mod p$, there is a non-zero cohomology class $\tilde{\alpha_s}^{(n)} \in \text{Ext}^s_\mathbb{A}(\mathbb{Z}/p, \mathbb{Z}/p)$ which is called the $n$-th Greek letter element in Ext. When $n = 1, 2, 3$, the elements $\tilde{\alpha_s}^{(n)}$ are written as $\tilde{\alpha_s}$, $\tilde{\beta_s}$ and $\tilde{\gamma_s}$ which represent the homotopy elements $\alpha_s$, $\beta_s$ and $\gamma_s$ respectively.

Given two elements $\tilde{x}$ and $\tilde{y}$ in $\text{Ext}^*_\mathbb{A}(\mathbb{Z}/p, \mathbb{Z}/p)$, suppose that $\tilde{x}$ and $\tilde{y}$ converge nontrivially to elements $x$ and $y$ in $\pi_s S$, respectively. We are wondering whether or not the product $\tilde{x} \cdot \tilde{y}$ in the ASS can also converge nontrivially to the product $x \cdot y$ in $\pi_s S$. In particularly, we are interested in considering the convergence of the product of $\tilde{\beta_s}$ or $\tilde{\gamma_s}$ with some other elements in $\text{Ext}^*_\mathbb{A}(\mathbb{Z}/p, \mathbb{Z}/p)$. For example, it was shown in [4] that the product $\tilde{\gamma_s} h_0 b_{n-1} \in \text{Ext}^{s+3}_\mathbb{A}(\mathbb{Z}/p, \mathbb{Z}/p)$ is nontrivial in the ASS when $p \geq 7$, $n \geq 2$ and $3 \leq s \leq p - 2$. It converges to a nontrivial element $\gamma_s \zeta_n \in \pi_s S$. By a similar method, Liu-Ma [5] verified the convergence of the product $h_n h_m \tilde{\beta_s}$ in the ASS when $p \geq 5$, $n \geq m + 2 > 5$ and $2 \leq s \leq p - 1$. In this paper, we will improve their method and use it to show that $\tilde{\gamma_s} b_{n-1} g_0$ in the ASS converges to a nontrivial element of $\pi_s S$. The following statements are our main results.

**Theorem 1.1.** Let $p \geq 7$ and $n \geq 2$. If $3 \leq s \leq p - 1$ then the product

$$\tilde{\gamma_s} b_{n-1} g_0 \in \text{Ext}^{s+4, (s+sp+sp^2+p^n)q+s-3}_\mathbb{A}(\mathbb{Z}/p, \mathbb{Z}/p)$$

is nontrivial in the Adams spectral sequence and converges to a homotopy nontrivial element $\xi_n \in \pi_s S$.

This paper is organized as follows. In Section 2, we will introduce a method to compute the generators of the $E_1$-term of the May spectral sequence (MSS). As an application of this method, in Section 3 we do an explicit computation for the sake of proof of Theorem 1.1. Then in Section 4, we give the proof of Theorem 1.1.

### 2. Preliminary knowledge on the May spectral sequence

In this section we will recall some elementary knowledge on the May spectral sequence (MSS). By reference [10], there is a 3-graded May spectral sequence $\{E_r^{s,t,M}, d_r: E_r^{s,t,M} \to E_r^{s+1,t,M-r}\}$ which converges...
to Ext\(^{s,t}\)(\(\mathbb{Z}/p, \mathbb{Z}/p\)). The \(E_1\)-term of MSS is given by
\[
E^{s,t,*}_1 = E[h_{m,i}|m > 0, i \geq 0] \otimes P[b_{m,i}|m > 0, i \geq 0] \otimes P[a_n|n \geq 0]
\]
where \(E[\ ]\) denotes the exterior algebra and \(P[\ ]\) denotes the polynomial algebra. It is known that \(h_{1,i} \in E^{1,p^iq,*}_1\) converges nontrivially to \(h_i \in \text{Ext}^{1,p^i q}(\mathbb{Z}/p, \mathbb{Z}/p)\). Thus \(d_r(h_{1,i}) = 0\) for any \(r \geq 1\). We list the degrees of the \(E_1\)-term generators as follows:
\[
h_{m,i} \in E_1^{1,2(p^m-1)p^i,2m-1}, b_{m,i} \in E_1^{2,2(p^m-1)p^{i+1},(2m-1)p}, a_n \in E_1^{1,2p^n-1,2n+1}.
\]
For the \(r\)-th May differential \(d_r: E^{s,t,M}_r \to E^{s+1,t,M-r}_r\) with \(r \geq 1\), if \(x \in E^{s,t,*}_r\) and \(y \in E^{s',t',*}_r\) then
\[
d_r(xy) = d_r(x)y + (-1)^{s+t}xd_r(y).
\]
The MSS satisfies the graded commutativity \(xy = (-1)^{ss'+tt'}yx\) for \(\{x, y\} \subset \{h_{m,i}, b_{m,i}, a_n\}\). On each generator the first May differential
\[
d_1: E^{s,t,M}_1 \to E^{s+1,t,M-1}_1
\]
has an explicit description as
\[
d_1(h_{i,j}) = \sum_{0<k<i} h_{i-k,k+j}h_{k,j}, \quad d_1(a_i) = \sum_{0<k<i} h_{i-k,k}a_k, \quad d_1(b_{i,j}) = 0.
\]
Given an element \(x \in E^{s,t,M}_1\), we define \(\dim(x) = s\), \(\deg(x) = t\) and \(M(x) = M\). Then we have
\[
\dim(h_{i,j}) = \dim(a_i) = 1, \quad \dim(b_{i,j}) = 2,
\]
\[
M(h_{i,j}) = M(a_{i-1}) = 2i - 1, \quad M(b_{i,j}) = (2i - 1)p,
\]
\[
\deg(h_{i,j}) = 2(p^i - 1)p^j = (p^i + \cdots + p^{i+j-1})q,
\]
\[
\deg(b_{i,j}) = 2(p^i - 1)p^{i+1} = (p^{i+1} + \cdots + p^{i+j})q,
\]
\[
\deg(a_i) = 2p^i - 1 = (1 + \cdots + p^{i-1})q + 1,
\]
\[
\deg(a_0) = 1
\]
where \(i \geq 1\) and \(j \geq 0\).

A method of computing \(E_1\)-term of the MSS was introduced in [5], but the computation process in [5] is very obscure. Hence, we are about to introduce a new computation method and then in Section 3 we show how to use it in a more effective way for our target.

We denote \(a_i, h_{i,j}\) and \(b_{i,j}\) by \(x, y\) and \(z\), respectively. By the graded commutativity of \(E^{s,t,*}_1\), we can write a generator as
\[
h = (x_1 \cdots x_n)(y_1 \cdots y_o)(z_1 \cdots z_l) \in E^{s,t+b,*}_1
\]
where \( t = (c_0 + c_1p + \cdots + c_np^n)q \) with \( 0 \leq c_i < p \ (0 \leq i < n) \), 
\( c_0 > 0, s < b + q \) with \( 0 < b < q \). We claim that \( u = b \). Otherwise, by the characteristics of \( \deg(a_i), \deg(h_{i,j}), \deg(b_{i,j}) \) and \( t \), there exists some \( w > 0 \) such that \( u = b + wq \). It follows that \( \dim(h) \geq b + wq > s = \dim(h) \) which is a contradiction. Thus

\[ h = (x_1 \cdots x_b)(y_1 \cdots y_v)(z_1 \cdots z_l) \in E_1^{b+v+2l,t+b,*}. \]

Note that the degrees of \( x_i, y_i \) and \( z_i \) can be uniquely expressed as

\[
\begin{align*}
\deg(x_i) &= (x_{i,0} + x_{i,1}p + \cdots + x_{i,n}p^n)q + 1, \\
\deg(y_i) &= (y_{i,0} + y_{i,1}p + \cdots + y_{i,n}p^n)q, \\
\deg(z_i) &= (0 + z_{i,1}p + \cdots + z_{i,n}p^n)q
\end{align*}
\]

where the sequence \((x_{i,0}, x_{i,1}, \cdots, x_{i,n})\) is of the form \((1, \cdots, 1, 0, \cdots, 0)\),
while \((y_{i,0}, y_{i,1}, \cdots, y_{i,n})\) and \((0, z_{i,1}, \cdots, z_{i,n})\) are both of the form

\[(0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0).\]

According to the graded commutativity of \( E_1^{*,*,*} \), the generator

\[ h = (x_1 \cdots x_b)(y_1 \cdots y_v)(z_1 \cdots z_l) \in E_1^{b+v+2l,t+b,*} \]

can be arranged in the following way:

(a) if \( i > j \), we put \( a_i \) on the left side of \( a_j \);
(b) if \( j < k \), we put \( h_{i,j} \) on the left side of \( h_{w,k} \);
(c) if \( i > w \), we put \( h_{i,j} \) on the left side of \( h_{w,j} \);
(d) apply the same rules (b) and (c) to \( b_{i,j} \).

Hence the above \( x_{i,j}, y_{i,j} \) and \( z_{i,j} \) satisfy the following conditions (2.1):

(i) \( x_{1,j} \geq x_{2,j} \geq \cdots \geq x_{b,j}, \ x_{i,0} \geq x_{i,1} \geq \cdots \geq x_{i,n} \) for \( i \leq b \) and \( j \leq n \);
(ii) if \( y_{i,j-1} = 0 \) and \( y_{i,j} = 1 \), then for all \( k < j \) there is \( y_{i,k} = 0 \);
(iii) if \( y_{i,j} = 1 \) and \( y_{i,j+1} = 0 \), then for all \( k > j \) there is \( y_{i,k} = 0 \);
(iv) \( y_{i,0} \geq y_{2,0} \geq \cdots \geq y_{v,0} \);
(v) if \( y_{i,0} = y_{i+1,0}, y_{i,1} = y_{i+1,1}, \cdots, y_{i,j} = y_{i+1,j} \), then \( y_{i,j+1} \geq y_{i+1,j+1} \);
(vi) apply the same rules (ii)~(iv) to \( z_{i,j} \).

According to the \( p \)-adic expression of the coefficient of \( q \) in second degree \( \deg(x_i), \deg(y_i) \) and \( \deg(z_i) \) as above, by the properties of \( p \)-adic
numbers we obtain the following group of equations (2.2)

\[
\begin{align*}
    x_{1,0} + \cdots + x_{b,0} + y_{1,0} + \cdots + y_{v,0} &= \tilde{c}_0 + k_1p = c_0 \\
    x_{1,1} + \cdots + x_{b,1} + y_{1,1} + \cdots + y_{v,1} + z_{1,1} + \cdots + z_{l,1} &= \tilde{c}_1 - k_1 + k_2p = c_1 \\
    \cdots & \\
    x_{1,n-1} + \cdots + x_{b,n-1} + y_{1,n-1} + \cdots + y_{v,n-1} + z_{1,n-1} + \cdots + z_{l,n-1} &= \tilde{c}_{n-1} - k_{n-1} + k_np = c_{n-1} \\
    x_{1,n} + \cdots + x_{b,n} + y_{1,n} + \cdots + y_{v,n} + z_{1,n} + \cdots + z_{l,n} &= \tilde{c}_n - k_n = c_n.
\end{align*}
\]

From the above group of equations, we obtain two integer sequences

\[
K = (k_1, \ldots, k_n) \quad \text{and} \quad S = (c_0, \ldots, c_n)
\]

which are determined by \((k_1, \ldots, k_n)\) and \((\tilde{c}_0, \ldots, \tilde{c}_n)\), respectively. We say that the group of equations (2.2) has a solution if it has a solution satisfying the conditions (2.1).

Intuitively the above group of equations has the form of matrix as

\[
\begin{pmatrix}
    x_{1,0} & \cdots & x_{b,0} & y_{1,0} & \cdots & y_{v,0} & 0 & \cdots & 0 \\
    x_{1,1} & \cdots & x_{b,1} & y_{1,1} & \cdots & y_{v,1} & z_{1,1} & \cdots & z_{l,1} \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    x_{1,n} & \cdots & x_{b,n} & y_{1,n} & \cdots & y_{v,n} & z_{1,n} & \cdots & z_{l,n}
\end{pmatrix}
\begin{pmatrix}
    A \\
    B \\
    C
\end{pmatrix}
\]

According to the conditions (2.1), the section A in the matrix is the form of trapezoid as

\[
\begin{pmatrix}
    1 & \cdots & 1 & \cdots & 1 & \cdots & 1 \\
    1 & \cdots & 1 & \cdots & 1 & \cdots & 1 \\
    \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    1 & \cdots & 1 & \cdots & 1 & \cdots & 1 \\
    1 & \cdots & 1 & \cdots & 1 & \cdots & 1
\end{pmatrix}
\]

\[(2.2)\]
where the vacant place denotes zero. The section B has the form as

\[
\begin{pmatrix}
1 & \cdots & 1 & \cdots & 1 \\
1 & \cdots & 1 & \cdots & 1 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
1 & \cdots & 1 & \cdots & 1 \\
1 & \cdots & 1 & \cdots & 1 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
1 & \cdots & 1 & \cdots & 1 \\
\end{pmatrix}
\]

The section C has the similar form as B except that the first horizontal line are all zero.

Each column in section A determines some \( x_i \). Each column in section B or C determines some \( y_i \) or \( z_i \). Recall that each column in the matrix does not admit the form \((\cdots, 1, 0, \cdots, 0, 1, \cdots)^T\). In summary, for the \( E_{s;t}^{s,t+b,*} \)-term of MSS where \( t = (\tilde{c}_0 + \tilde{c}_1 p + \cdots + \tilde{c}_n p^n) q \) with \( 0 \leq \tilde{c}_i < p \) \( (\tilde{c}_n > 0) \), \( s < b + q \) with \( 0 \leq b < q \), the determination of \( E_{s;t}^{s,t+b,*} \) is reduced to the following steps:

1. List all the possible \((b, v, l)\) such that \( b + v + 2l = s \).
2. For each \((b, v, l)\), list up all the sequences \( K = (k_1, \cdots, k_n) \) and \( S = (c_0, \cdots, c_n) \) such that \( \max\{c_0, c_1, \cdots, c_n\} \leq b + v + l \).
3. For each \((b, v, l)\) and the sequence \( K = (k_1, \cdots, k_n) \), solve the corresponding group of equations (2.2). As stated before, the solutions are of the forms \((1, \cdots, 1, 0, \cdots, 0)\) or \((0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0)\) which uniquely determine \( x_i \), \( y_i \) or \( z_i \) which correspond to elements of form \( a_i \), \( h_{i;j} \) or \( b_{i;j} \) respectively.

3. Computation of \( E_1 \)-term of the MSS

In order to prove Theorem 1.1, in this section we will apply the method in Section 2 to compute the generators of \( E_{s-r+4,tq+(s-r-2),s}^{s-r+4,tq+(s-r-2),s} \) for \( 1 \leq r \leq p + 2 \), where \( t = s + sp + sp^2 + p^n \). Let \( M_{i}^n \) \((i \geq 1)\) denote the May filtration for different \( n \). Our results are stated as follows:

**Theorem 3.1.** For \( 1 \leq r \leq p + 2 \) and \( n \geq 2 \), let \( t = s + sp + sp^2 + p^n \) with \( 3 \leq s \leq p - 1 \). Then the generators of \( E_{s-r+4,tq+(s-r-2),s}^{s-r+4,tq+(s-r-2),s} \) are listed as follows:
(1) when $3 \leq r \leq p + 2$, there is no generator;
(2) when $r = 2$, there is a generator $a_{3}^{-4}h_{4,0}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,2}$ for $n = 3$ and no generator for $n \neq 3$;
(3) when $r = 1$, for different $n$ the generators are as follows:
   (a) for $n = 2$, there is no generator.
   (b) for $n = 3$, there are 20 generators as

\[
\begin{align*}
\{ \ & a_{3}^{-4}a_{1}h_{4,0}h_{3,0}h_{2,0}h_{1,0}h_{1,2}, a_{3}^{-3}h_{3,0}h_{2,0}h_{1,0}h_{1,1}, \\
\ & a_{3}^{-4}a_{2}h_{4,0}h_{3,0}h_{2,0}h_{1,2}b_{1,1}, a_{3}^{-3}h_{4,0}h_{3,0}h_{1,0}h_{1,2}b_{1,1}, \\
\ & a_{3}^{-3}h_{4,0}h_{3,0}h_{1,0}h_{1,1}b_{1,1}, a_{4}a_{3}^{-4}h_{3,0}h_{2,0}h_{1,0}h_{2,0}b_{1,1}, \\
\ & a_{3}^{-4}a_{1}h_{4,0}h_{3,0}h_{2,0}h_{1,2}b_{2,0}, a_{3}^{-3}h_{4,0}h_{3,0}h_{1,0}h_{1,2}b_{2,0}, \\
\ & a_{3}^{-3}h_{3,0}h_{2,0}h_{1,0}h_{2,1}b_{2,1}, a_{4}a_{3}^{-4}h_{3,0}h_{2,0}h_{1,0}h_{2,2}b_{2,0}, \\
\ & a_{3}^{-3}h_{3,0}h_{2,0}h_{1,0}h_{1,2}b_{3,0} \} \quad (M_{3}^{3} = 7s + p - 8) \\
\{ \ & a_{3}^{-4}a_{0}h_{4,0}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,2}, a_{3}^{-4}a_{1}h_{3,0}h_{2,0}h_{1,0}h_{3,1}h_{2,1}h_{1,2}, \\
\ & a_{3}^{-3}a_{1}h_{4,0}h_{3,0}h_{1,0}h_{1,2}h_{1,1}h_{1,2}, a_{3}^{-3}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,2}, \\
\ & a_{3}^{-3}h_{3,0}h_{2,0}h_{1,0}h_{1,2}h_{2,1}, a_{3}^{-3}a_{2}h_{3,0}h_{2,0}h_{1,0}h_{2,2}h_{1,2} \} \quad (M_{6}^{3} = 7s - 8)
\end{align*}
\]

(c) for $n > 3$, there are eleven families of generators as

\[
\begin{align*}
\{ \ & a_{3}^{-3}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,n} \} \quad (M_{7}^{2} = 7s - 8) \\
\{ \ & h_{1}^{(i)} = a_{n}^{-3}h_{n,0}h_{i,0}h_{4,0}h_{n-3}b_{n-i-1} \quad (0 < i < n; i \neq 4); \quad (M_{8}^{n}) \\
\ & h_{2}^{(i)} = a_{n}^{-3}h_{n,0}h_{i,0}h_{n-1,i}, h_{4,0}h_{n-3,2} \quad (0 < i < n; i \neq 4); \quad (M_{9}^{n}) \\
\ & h_{3}^{(i)} = a_{n}^{-4}a_{4}h_{n,0}h_{i,j}h_{n-3,j}, h_{n-1,i}, h_{n-3,3} \quad (0 < i, j < n; i \neq j; i, j \neq 3) \} \quad (M_{10}^{n}) \\
\{ \ & h_{5}^{(i)} = h_{n,0}h_{i,0}h_{4,0}h_{n-1,i}, h_{n-3,j}, h_{n-1,i}, h_{3,3} \quad (0 < i, j < n; i \neq j) \} \quad (M_{11}^{n}) \\
\{ \ & h_{10}^{(i)} = a_{n}^{-3}h_{n,0}h_{i,0}h_{4,0}h_{n-1,i}, h_{n-3,j}, h_{n-3,j}, h_{n-3,j}, h_{n-3,j} \quad (0 < i, j < n; j \neq i; i \neq 3) \}
\end{align*}
\]

where $s = p - 1$ for $h_{1}^{(i)}, \ldots, h_{10}^{(i)}$ and $M_{8}^{n} = (4n - 2i + 1)(p - 1) - 5$,
$M_{9}^{n} = 2np - 2n + p - 5$, $M_{10}^{n} = 2np - 4n + p - 8$, $M_{11}^{n} = 2np - 2n + p - 8$.

Proof. For $n \leq 3$, we have

\[
\mathcal{S} = \begin{cases} 
(s, s, s + 1) & n = 2, \\
(s, s, s + 1) & n = 3.
\end{cases}
\]
For $n > 3$, we have
\[ S = \{ S_1 = (s, s, s, 0, \ldots, 0, 1), \]
\[ S_i = (s, s, s, 0, \ldots, 0, p^{(i)}, p - 1, \ldots, p - 1, 0), \quad i \geq 4. \]

By the reason of dimension, all the possibilities of $h$ are listed as
\[
\{ x_1 \cdots x_{s-r-2} z_1 z_2 z_3, \quad x_1 \cdots x_{s-r-2} y_1 y_2 z_1 z_2, \quad x_1 \cdots x_{s-r-2} y_1 y_2 y_3 y_4 z_1, \quad x_1 \cdots x_{s-r-2} y_1 y_2 y_3 y_4 y_5 y_6. \}
\]

Let us consider the generators $h \in \text{Ext}_{s-r}^{s-r+4, t_q + s-r-2}(\mathbb{Z}/p, \mathbb{Z}/p)$ case by case.

Case 1. $h = x_1 \cdots x_{s-r-2} z_1 z_2 z_3$.

Note that $s \leq p - 1$ and $r \geq 1$. Since $\sum_{i=1}^{s-r-2} x_i, 0 = s - r - 2 < s = c_0$, the first equation of (2.2) has no solution. It follows that such $h$ is impossible to exist.

Case 2. $h = x_1 \cdots x_{s-r-2} y_1 y_2 z_1 z_2$.

Note that $s \leq p - 1$ and $r \geq 1$. Since $\sum_{i=1}^{s-r-2} x_i, 0 + y_1, 0 + y_2, 0 = s - r - 2 + y_1, 0 + y_2, 0 < s = c_0$, the first equation of (2.2) has no solution. It follows that such $h$ is impossible to exist.

Case 3. $h = x_1 \cdots x_{s-r-1} y_1 y_2 y_3 y_4 z_1$.

Subcase 3.1. $s \leq p - 1$, $n \geq 2$, $r > 2$.

Since $\sum_{i=1}^{s-r-2} x_i, 0 + \sum_{i=1}^{4} y_i, 0 \leq s - r - 2 + 4 < s = c_0$, the first equation of (2.2) has no solution. It follows that such $h$ is impossible to exist.

Subcase 3.2. $s \leq p - 1$, $n \geq 2$, $r = 2$.

For $\mathcal{S} = (s, s, s + 1)$, solve the corresponding group of equations (2.2) by virtue of (2.1), we get the generators which are all zero since they are all contain $h^2_{3,0} = 0$.

Subcase 3.3. $s \leq p - 1$, $r = 1$.

Subcase 3.3.1. $s \leq p - 1$, $n = 2$, $r = 1$.

For $\mathcal{S} = (s, s, s + 1)$, solve the corresponding group of equations (2.2) by virtue of (2.1), we get the generators
\[
a^s_{3} h^2_{3,0} h_{2,0, h_{1,0, b_{1,1}}, \quad a^s_{3} h^2_{3,0} h_{1,0, b_{2,0}}, \quad a^s_{3} h^2_{3,0} h_{1,0, b_{2,1, b_{1,1}}}.
\]
which are all zero due to $h_{2,0}^2 = 0$.

Subcase 3.3.2. $s \leq p - 1$, $n = 3$, $r = 1$.

This will be a hard case. Our strategy of computation is as follows. For the unique sequence in this case $\overline{S} = (s, s, s, 1)$, we see that the maximal number in $\overline{S}$ is $s$. This leads us to first compute all the generators (may be zero) of $E_1^{s,tq+(s-3),*}$ with the form $x_1 \cdots x_{s-3}y_1y_2y_3$. For each $a_i$ and $h_{i,j}$ in $x_1 \cdots x_{p-4}y_1y_2y_3$, we resolve

$$h_{i,j} \to h_{k,j}h_{i-k,j+k} \quad \text{or} \quad a_i \to a_jh_{i-j,j}$$

and then repeat this step once. Finally we do the replacement $h_{i,j} \to b_{i,j-1}$. If the obtained elements are nonzero, then we get all of the desired generators of $E_1^{s,tq+(s-3),*}$ with the form $x_1 \cdots x_{p-4}y_1y_2y_3z_1$. We do this computation by the following steps:

Step 1. For $\overline{S} = (s, s, s, 1)$, solve the corresponding group of equations (2.2) by virtue of (2.1), we obtain two generators of $E_1^{s,tq+(s-3),*}$

$$a_3^{s-3}h_{4,0}h_{3,0}h_{3,0}, \quad a_4a_3^{s-4}h_{3,0}h_{3,0}h_{3,0}$$

Step 2. Resolve $a_i$ or $h_{i,j}$.

$$a_3^{s-3}h_{4,0}h_{3,0}h_{3,0} \xrightarrow{h_{3,0}} \begin{cases} a_3^{s-3}h_{4,0}h_{3,0}h_{2,0}h_{1,2} \\ a_3^{s-3}h_{4,0}h_{3,0}h_{1,0}h_{2,1} \end{cases}$$

$$a_4a_3^{s-4}h_{3,0}h_{3,0}h_{3,0} \xrightarrow{h_{3,0}} \begin{cases} a_4a_3^{s-4}h_{3,0}h_{3,0}h_{2,0}h_{1,2} \\ a_4a_3^{s-4}h_{3,0}h_{3,0}h_{1,0}h_{2,1} \end{cases}$$

Step 3. Repeat the Step 2 and replace some $h_{i',j'}$ by $b_{i',j'-1}$. Since there are many identical generators appearing in the obtained first May differentials, we will only write out the different nonzero generators for
simplicity.
(3.1)
\[
a_3^{s-3}h_4,0h_3,0h_2,0h_1,2 \xrightarrow{\text{one by one}} \begin{cases} 
a_3^{s-4}a_1h_4,0h_3,0h_2,0h_1,2h_2,1 \rightarrow h_1,2 \\
a_3^{s-4}a_1h_4,0h_3,0h_2,0h_1,1, \rightarrow h_1,2 \\
a_3^{s-4}a_1h_4,0h_3,0h_2,0h_1,2b_2,0 \\
a_3^{s-4}a_2h_4,0h_3,0h_2,0h_1,2h_1,2 \rightarrow h_1,2 \\
a_3^{s-4}a_2h_4,0h_3,0h_2,0h_1,2b_1,1 \\
a_3^{s-3}h_3,0h_2,0h_1,0h_3,1h_1,2 \rightarrow h_1,2 \text{ or } h_3,1 \\
a_3^{s-3}h_3,0h_2,0h_1,0h_3,1b_1,1 \\
a_3^{s-3}h_3,0h_2,0h_1,0h_1,2b_3,0 \\
a_3^{s-3}h_4,0h_2,0h_1,0h_2,1h_1,2 \rightarrow h_1,2 \text{ or } h_2,1 \\
a_3^{s-3}h_4,0h_2,0h_1,0h_2,1b_1,1 \\
a_3^{s-3}h_4,0h_2,0h_1,0h_1,2b_2,0 \\
a_3^{s-3}h_4,0h_3,0h_1,0h_1,1h_1,2 \rightarrow h_1,2 \text{ or } h_1,0 \\
a_3^{s-3}h_4,0h_3,0h_1,0h_1,1b_1,1 \\
a_3^{s-3}h_4,0h_3,0h_1,0h_1,2b_1,0 \\
\end{cases}
\]

(3.2)
\[
a_3^{s-3}h_4,0h_3,0h_1,0h_2,1 \xrightarrow{a_3 \text{ or } h_4,0} \begin{cases} 
a_3^{s-4}a_1h_4,0h_3,0h_1,0h_2,1h_2,1 \rightarrow h_2,3 \\
a_3^{s-4}a_1h_4,0h_3,0h_1,0h_2,1b_2,0 \\
a_3^{s-3}h_3,0h_2,0h_1,0h_2,2h_2,1 \rightarrow h_2,1 \text{ or } h_2,2 \\
a_3^{s-3}h_3,0h_2,0h_1,0h_2,2b_2,0 \\
a_3^{s-3}h_3,0h_2,0h_1,0h_2,1b_2,1 \\
\end{cases}
\]

(3.3)
\[
a_4a_3^{s-4}h_3,0h_3,0h_2,0h_1,2 \xrightarrow{h_3,0} \begin{cases} 
a_4a_3^{s-4}h_3,0h_2,0h_1,0h_2,1h_1,2 \rightarrow h_2,1 \text{ or } h_1,2 \\
\end{cases}
\]

\[
\begin{cases} 
a_4a_3^{s-4}h_3,0h_2,0h_1,0h_2,1b_1,1 \\
a_4a_3^{s-4}h_3,0h_2,0h_1,0h_2,1b_2,0 \\
\end{cases}
\]

In the above diagrams the elements over the first left arrow and the second right arrow mean their resolution and replacement, respectively.

Subcase 3.3.3. $s > p - 2$, $n > 3$, $r = 1$.

Since $\sum_{j=1}^{s-r-2} x_{j,i} + \sum_{k=1}^{4} y_{k,i} + z_{1,i} \leq s + 2 < p = c_i$, the $i$-th equation of (2.2) has no solution. It follows that such $h$ is impossible to exist.

Subcase 3.3.4. $s = p - 2$, $n > 3$, $r = 1$. 

For $S_1$, solve the corresponding group of equations (2.2) by virtue of (2.1), we get the generators listed as
\[
\begin{align*}
& a_3^{p-5}h_{3,0}^2h_{2,0}h_{1,2}b_{1,n}, \quad a_3^{p-5}h_{3,0}^2h_{1,0}h_{2,1}b_{1,n}, \\
& a_3^{p-5}h_{3,0}^2h_{2,0}h_{1,n+1}b_{1,1}, \quad a_3^{p-5}h_{3,0}^2h_{1,0}h_{1,n+1}b_{2,0}
\end{align*}
\]
which are all zero due to $h_{3,0}^2 = 0$.

For $S_4 = (s, s, s, p, p - 1, \cdots, p - 1, 0)$, solve the corresponding group of equations (2.2) by virtue of (2.1), we get the generators listed as following
\[
\begin{align*}
& a_n^{p-6}a_4h_{n,0}^3h_{n-3,3}b_{n-3,2}, \quad a_n^{p-5}h_{n,0}h_{4,0}h_{n-3,3}b_{n-3,2}, \\
& a_n^{p-5}h_{n,0}^2h_{n-3,3}b_{1,2}, \quad a_n^{p-5}h_{n,0}^2h_{1,3}b_{n-3,2}
\end{align*}
\]
which are all zero due to $h_{n,0}^2 = 0$.

For $S_i (i \geq 5)$, there is no solution.

Subcase 3.3.5. $s = p - 1, n > 3, r = 1$.

For $S_1$, we get the similar generators as Subcase 3.3.4 which are all zero.

For $S_4 = (p - 1, p - 1, p - 1, p - 1, \cdots, p - 1, 0)$, we obtain a set of generators
\[
(3.4) \quad a_n^{p-4}h_{n,0}^2h_{1,0}h_{n-3,3} \xrightarrow{h_{n-3,3} \text{ or } h_{n-1,i}} \begin{cases} a_n^{p-4}h_{n,0}h_{i,0}h_{n-i,1}h_{4,0}h_{n-3,3} \\ a_n^{p-4}h_{n,0}h_{i,0}h_{n-i,1}h_{4,0}b_{n-3,2} \end{cases}
\]
where $0 < i < n$ and $i \neq 4$.

For $S_i (i \geq 5)$, there is no solution.

Case 4. $h = x_1 \cdots x_{s-r-2}y_1y_2y_3y_4y_5y_6$.

Subcase 4.1. $s \leq p - 1, n \geq 2, r > 4$.

There is no solution since
\[
\sum_{i=1}^{s-r-2} x_{i,0} + \sum_{j=1}^{6} y_{j,0} \leq s - r + 4 < s = c_0.
\]

Subcase 4.2. $s \leq p - 1, r = 4$.

Subcase 4.2.1. $s \leq p - 1, r = 4, n = 2$.

There is no solution since
\[
\sum_{i=1}^{s-r-2} x_{i,0} + \sum_{j=1}^{6} y_{j,0} \leq s - r + 4 < s + 1 = c_2.
\]
Subcase 4.2.2. $s \leq p - 1, \, r = 4, \, n = 3$.
Solve the corresponding group of equations (2.2) by virtue of (2.1), we get two generators $a_3^{s-3}h_{4,0}h_{3,0}^5$ and $a_4a_3^{s-4}h_{3,0}^6$ which are both zero due to $h_{3,0}^2 = 0$.

Subcase 4.2.3. $s \leq p - 1, \, r = 4, \, n > 3$.
For $S_1 = (s, s, s, 0, \cdots, 0, 1)$, there is no solution.
For $S_i(i \geq 4), \, S = (s, s, s, 0, \cdots, 0, p^{(i)}, p - 1, \cdots, p - 1, 0)$, there is no solution due to $h_{2,3}^3; 0 = 0$.

Subcase 4.3. $s \leq p - 1, \, r = 3$.
Subcase 4.3.1. $s \leq p - 1, \, 2 \leq n \leq 3, \, r = 3$.
Solve the corresponding group of equations (2.2) by virtue of (2.1), we see that all the generators are zero since they all contain $h_{3,0}^2 = 0$.

Subcase 4.3.2. $s \leq p - 1, \, n > 3, \, r = 3$.
For $S_1$, we get one generator $a_3^{s-5}h_{3,0}^5h_{1,n}$ which is zero as $h_{3,0}^2 = 0$.
For $S_i(i \geq 4), \, S = (s, s, s, 0, \cdots, 0, p^{(i)}, p - 1, \cdots, p - 1, 0)$, there is no solution since $s \sum_{j=1}^{s-r-2} x_{j,i} + \sum_{k=1}^{6} y_{k,i} < s + 1 \leq p = c_i$.

Subcase 4.4. $s \leq p - 1, \, r = 2$.
Subcase 4.4.1. $s \leq p - 1, \, n = 2, \, r = 2$.
Solve the corresponding group of equations (2.2) by virtue of (2.1), we get one generator $a_3^{s-3}h_{3,0}^3h_{1,0}h_{2,1}h_{1,2}$ which is zero due to $h_{3,0}^2 = 0$.

Subcase 4.4.2. $s \leq p - 1, \, n = 3, \, r = 2$.
There is one nonzero generator
\begin{equation}
a_3^{s-4}h_{4,0}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,2}.\tag{3.5}
\end{equation}

Subcase 4.4.3. $s < p - 2, \, n > 3, \, r = 2$.
For $S_1$, solve the corresponding group of equations (2.2) by virtue of (2.1), we get two generators $a_3^{s-4}h_{3,0}^3h_{2,0}h_{1,2}h_{1,n}$ and $a_3^{s-4}h_{3,0}^3h_{1,0}h_{2,1}h_{1,n}$ which are both zero since $h_{3,0}^2 = 0$. 
For $S_i(i \geq 4)$, $S = (s, s, s, 0, \cdots, 0, p^{(i)}, p - 1, \cdots, p - 1, 0)$, there is no solution since
\[s-r-2 \sum_{j=1}^{s-r} x_{j,i} + \sum_{k=1}^{6} y_{k,i} < s + 2 < p = \bar{c}_i.

Subcase 4.4.4. $s = p - 2$, $n > 3$, $r = 2$.
For $S_1$, we get two similar generators as Subcase 4.4.3 by replacing $s$ with $p - 2$.
For $S_4$, we get two generators
\[a_n^{s-4}h_{n,0}^3h_{4,0}^2h_{n-3,3}^2 \text{ and } a_n^{s-4}h_{n,0}^4h_{n-3,3}^3h_{1,3}^3\]
which are both zero since $h_{n,0}^2 = 0$.
For $S_i(i \geq 5)$, there is no solution.

Subcase 4.4.5. $s = p - 1$, $n > 3$, $r = 2$.
For $S_1$, we get two similar generators as Subcase 4.4.3 by replacing $s$ with $p - 1$.
For $S_4$, solve the corresponding group of equations (2.2) by virtue of (2.1), we get the generators they are all zero because they all contain $h_{n,0}^2 = 0$.

Subcase 4.5. $s = p - 1$, $r = 1$.
Subcase 4.5.1. $s = p - 1$, $n = 2$, $r = 1$.
Solve the corresponding group of equations (2.2) by virtue of (2.1), we get the generators listed as
\[a_3^{s-3}h_{3,0}^2h_{1,1}^2h_{2,1}^2h_{1,2}^2, \quad a_3^{s-3}h_{3,0}^3h_{1,0}^2h_{1,2}^3, \quad a_3^{s-3}h_{3,0}^2h_{1,1}^2h_{3,0}^2h_{2,1}^2h_{1,2}^2.\]
They are all zero due to $h_{1,2}^2 = 0$ or $h_{1,0}^2 = 0$.
Subcase 4.5.3. $s = p - 1$, $n = 3$, $r = 1$.
We adopt the methods similar to Subcase 3.3.2. We still compute all the generators (may be zero) of $E^{s,tq+s-3,*}_1$ with the form $x_1 \cdots x_{s-3}y_1y_2y_3$.
First we resolve $a_i$ or $h_{i,j}$ in $x_1 \cdots x_{s-3}y_1y_2y_3$, and then for the obtained elements we repeat the first step twice. If the obtained element is nonzero, then it is our desired generator of $E^{s,t+3,t+s-3,*}_1$.

Step 1. For $S_1 = (s, s, s, 1) = (3, 3, 3, 1)$, solve the corresponding group of equations (2.2) by virtue of (2.1), we obtain two generators of $E^{s,t+3,t+s-3,*}_1$ as
\[a_3^{s-3}h_{4,0}^2h_{3,0}^3h_{3,0}^3 \text{ and } a_4^{s-4}h_{3,0}^3h_{3,0}^3h_{3,0}^3\]
Step 2. Resolve \( a_i \) or \( h_{i,j} \).

\[
a_3^{s-3}h_{4,0}h_{3,0}h_{3,0} \xrightarrow{h_{3,0}} \begin{cases} a_3^{s-3}h_{4,0}h_{3,0}h_{2,0}h_{1,2}, \\ a_3^{s-3}h_{4,0}h_{3,0}h_{1,0}h_{2,1} \end{cases}
\]

\[
a_4a_3^{s-4}h_{3,0}h_{3,0}h_{3,0} \xrightarrow{h_{3,0}} \begin{cases} a_4a_3^{s-4}h_{3,0}h_{3,0}h_{2,0}h_{1,2}, \\ a_4a_3^{s-4}h_{3,0}h_{3,0}h_{1,0}h_{2,1} \end{cases}
\]

Step 3. Repeat the Step 2 twice. We still only write out the different nonzero generators for simplicity.

(3.6)

\[
a_3^{s-3}h_{4,0}h_{3,0}h_{2,0}h_{1,2} \xrightarrow{\text{one by one}} \begin{cases} a_3^{s-4}a_0h_{4,0}h_{3,0}h_{3,0}h_{2,0}h_{1,2} \xrightarrow{h_{3,0}} \\ a_3^{s-4}a_0h_{4,0}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,2} \xrightarrow{h_{4,0}} \text{ or } h_{2,0} \\ a_3^{s-4}a_1h_{4,0}h_{3,0}h_{2,0}h_{2,1}h_{1,2} \xrightarrow{h_{4,0}} \\ a_3^{s-4}a_1h_{4,0}h_{3,0}h_{1,0}h_{2,1}h_{1,1}h_{1,2} \xrightarrow{h_{4,0}} \\ a_3^{s-3}h_{3,0}h_{3,0}h_{2,0}h_{1,2}h_{1,3} \xrightarrow{h_{3,0}} \\ a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,2}h_{1,3} \xrightarrow{h_{2,0}} \\ a_3^{s-3}h_{3,0}h_{2,0}h_{2,2}h_{1,2} \xrightarrow{h_{2,0}} \\ a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{1,1}h_{2,2}h_{1,2} \xrightarrow{h_{2,0}} \
\end{cases}
\]

(3.7) \[a_3^{s-3}h_{4,0}h_{3,0}h_{1,0}h_{2,1} \xrightarrow{a_1} a_3^{s-4}a_2h_{4,0}h_{3,0}h_{1,0}h_{2,1}h_{1,2} \xrightarrow{h_{4,0}} a_3^{s-4}a_2h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{2,2}h_{1,2}\]

Subcase 4.5.3. \( s < p - 3, n > 3, r = 1 \).

For \( S_1 \), we get one generator as follows:

(3.8) \[a_3^{s-3}h_{3,0}h_{3,0}h_{3,0}h_{1,n} \xrightarrow{h_{3,0}} \begin{cases} a_3^{s-3}h_{3,0}h_{3,0}h_{2,0}h_{1,2}h_{1,n} \\ a_3^{s-3}h_{3,0}h_{3,0}h_{1,0}h_{2,1}h_{1,n} \end{cases} \xrightarrow{h_{3,0}} a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,2}h_{1,n}.
\]

For \( S_i, (i \geq 4) \), there is no solution since \( \sum_{j=1}^{s-3} x_{j,i} + \sum_{k=1}^{6} y_{k,i} \leq s + 3 < p = c_i \).

Subcase 4.5.4. \( s = p - 3, n > 3, r = 1 \).

For \( S_1 \), we get the similar generator as Subcase 4.5.3 by replacing \( s \) with \( p - 3 \).
For $S_4$, solve the corresponding group of equations (2.2) by virtue of (2.1), the generators we get are all zero since they all contain $h_{n,0}^2 = 0$.

For $S_i(i > 4)$, there is no solution.

Subcase 4.5.5. $s = p - 2, n > 3, r = 1$.

For $S_1$, we get the similar generator as Subcase 4.5.3 by replacing $s$ with $p - 2$.

For $S_4$, solve the corresponding group of equations (2.2) by virtue of (2.1), we find that all the generators are zero.

For $S_i(i > 4)$, there is no solution.

Subcase 4.5.6. $s = p - 1, n > 3, r = 1$.

For $S_1$, we get the similar generator as Subcase 4.5.3 by replacing $s$ with $p - 1$.

For $S_4$, solve the corresponding group of equations (2.2) by virtue of (2.1), we get the generators as follows.

\[(3.9) \begin{align*}
&\begin{cases}
\eta_n^p - 4 h_{n,0}^3 h_{n,1,3}^1, n = 0 \\
\eta_n^p - 4 h_{n,0}^2 h_{n,-i,i}^1 h_{1,3}^1, n = 0 \\
\eta_n^p - 4 h_{n,0}^1 h_{n,-i,i}^1 h_{1,3}^1, n = 0 \\
\eta_n^p - 4 h_{n,0}^0 h_{n,-i,i}^1 h_{1,3}^1, n = 0
\end{cases}
\end{align*}
\]

\begin{align*}
&\begin{cases}
\eta_n^p - 4 h_{n,0}^3 h_{n,1,3}^1, n = 0 \\
\eta_n^p - 4 h_{n,0}^2 h_{n,-i,i}^1 h_{1,3}^1, n = 0 \\
\eta_n^p - 4 h_{n,0}^1 h_{n,-i,i}^1 h_{1,3}^1, n = 0 \\
\eta_n^p - 4 h_{n,0}^0 h_{n,-i,i}^1 h_{1,3}^1, n = 0
\end{cases}
\end{align*}

one by one

\begin{align*}
&\begin{cases}
\eta_n^p - 4 a_{n,0}^3 a_{4,0} h_{4,0} h_{n,-i,i} h_{n,-j,j} h_{n,-3,3} (0 < i, j < n; i \neq i; j \neq 3; 4; j \neq 3) \\
\eta_n^p - 4 a_{n,0}^3 a_{4,0} h_{4,0} h_{n,-i,i} h_{n,-j,j} h_{n,-3,3} (0 < i, j < n; i \neq i; j \neq 3; 4; j \neq 3)
\end{cases}
\end{align*}

one by one

\begin{align*}
&\begin{cases}
\eta_n^p - 4 a_{n,0}^3 a_{4,0} h_{4,0} h_{n,-i,i} h_{n,-j,j} h_{n,-3,3} (0 < i, j < n; i \neq i; j \neq 3; 4; j \neq 3) \\
\eta_n^p - 4 a_{n,0}^3 a_{4,0} h_{4,0} h_{n,-i,i} h_{n,-j,j} h_{n,-3,3} (0 < i, j < n; i \neq i; j \neq 3; 4; j \neq 3)
\end{cases}
\end{align*}

one by one

\begin{align*}
&\begin{cases}
\eta_n^p - 4 a_{n,0}^3 a_{4,0} h_{4,0} h_{n,-i,i} h_{n,-j,j} h_{n,-3,3} (0 < i, j < n; i \neq i; j \neq 3; 4; j \neq 3) \\
\eta_n^p - 4 a_{n,0}^3 a_{4,0} h_{4,0} h_{n,-i,i} h_{n,-j,j} h_{n,-3,3} (0 < i, j < n; i \neq i; j \neq 3; 4; j \neq 3)
\end{cases}
\end{align*}

For $S_i (i \geq 5)$, there is no solution.

\[
\Box
\]

4. Proof of Theorem 1.1.

In this section we will give the proof of Theorem 1.1. First we need the following lemma:

**Lemma 4.1.** Let $p \geq 7$ and $n \geq 2$. Let $t = s + sp + sp^2 + p^n$. Then the following two properties hold:

(a) for $2 \leq r \leq s + 4$, $\text{Ext}_A^{s-r+4,tq+s-r-2}(\mathbb{Z}/p, \mathbb{Z}/p) = 0$.

(b) the product $\tilde{t}_s h_{n-1} b_0 \in \text{Ext}_A^{s+4,tq+s-3}(\mathbb{Z}/p, \mathbb{Z}/p)$ is non-zero.
Proof. Since the elements $b_{n-1}$, $h_{1,0}h_{2,0}$, $a_3^{s-3}h_{3,0}h_{2,1}h_{1,2} \in \Ext_{\mathcal{A}}^{*,*,*}$ are all permanent cycles in the MSS and converge nontrivially to $b_{n-1}$, $g_0$, $\tilde{\gamma}_s \in \Ext_{\mathcal{A}}^*(\mathbb{Z}/p, \mathbb{Z}/p)$ respectively, it follows that the product element $\tilde{\gamma}_sb_{n-1}g_0 \in \Ext_{\mathcal{A}}^{*,*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ is represented by

$$a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{2,0}h_{1,0}b_{1,n-1} \in E_1^{s+4,tq+s-3,7s+p-8}$$

in the MSS.

(a) According to Theorem 3.1(1)(2), $E_1^{s-r+4,tq+s-r-2,*} = 0$ for $r \geq 2$ and $n \neq 3$. The only nontrivial case is

$$E_1^{s-r+4,tq+s-r-2,*} = \mathbb{Z}/p\{a_3^{s-4}h_{3,0}h_{2,1}h_{1,2}h_{4,0}h_{2,0}h_{1,0}\}$$

for $r = 2$ and $n = 3$. Since $d_4(h_{4,0}) = h_{1,0}h_{3,1} + h_{2,0}h_{2,2} + h_{3,0}h_{1,2} \neq 0$, then $h_{4,0}$ becomes zero in $E_{k}^{*,*,*}$ for $k \geq 2$. According to the above statement $a_3^{s-4}h_{3,0}h_{2,1}$ and $h_{2,0}h_{1,0}$ are permanent cycles in the May spectral sequence, thus they are always nontrivial in $E_{k}^{*,*,*}$ for any $k \geq 1$. It follows that $a_3^{s-4}h_{3,0}h_{2,1}h_{1,2}h_{4,0}h_{2,0}h_{1,0}$ becomes trivial in $E_{k}^{*,*,*}$ for $k \geq 2$.

Since

$$E_{k}^{s-r+4,tq+s-r-2,*} \implies \Ext_{\mathcal{A}}^{s-r+4,tq+s-r-2}(\mathbb{Z}/p, \mathbb{Z}/p)$$

for $k \geq 2$, it follows that $\Ext_{\mathcal{A}}^{s-r+4,tq+s-r-2}(\mathbb{Z}/p, \mathbb{Z}/p) = 0$ for $r \geq 2$. Thus we have proven part (a).

(b) In what follows we need to show that $a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{2,0}h_{1,0}b_{1,n-1}$ can not be hit by the May differential $d_r$: $E_r^{s+3,tq+s-3,7s+p-8} \implies E_r^{s+4,tq+s-3,7s+p-8}$ for any $r \geq 1$.

According to the above analysis we only need to consider the generators of $E_1^{s+3,tq+s-3,M}$ with $M > 7s+p-8$. Hence by Theorem 3.1 we do not need to consider the generators with May filtrations $M_3^3$, $M_6^3$, $M_9^3$ as their May differentials will not touch $a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{2,0}h_{1,0}b_{1,n-1}$.

For other generators we list them up with their first May differentials in the following table:
**Table 1:**

<table>
<thead>
<tr>
<th>Term</th>
<th>Expression</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>( a_3^{-3}a_1 h_{3,0} h_{2,0} h_{1,2} b_{2,0} )</td>
<td>May differential</td>
</tr>
<tr>
<td>2nd</td>
<td>( a_3^{-3} h_{3,0} h_{2,0} h_{1,2} b_{2,0} )</td>
<td>May differential</td>
</tr>
<tr>
<td>3rd</td>
<td>( a_3^{-3} b_1 h_{3,0} h_{3,0} h_{1,2} b_{2,0} )</td>
<td>May differential</td>
</tr>
<tr>
<td>4th</td>
<td>( a_3^{-3} h_{3,0} h_{2,0} h_{1,2} b_{2,0} )</td>
<td>May differential</td>
</tr>
<tr>
<td>5th</td>
<td>( a_3^{-3} h_{3,0} h_{2,0} h_{1,2} b_{2,0} )</td>
<td>May differential</td>
</tr>
<tr>
<td>6th</td>
<td>( a_3^{-3} h_{3,0} h_{2,0} h_{1,2} b_{2,0} )</td>
<td>May differential</td>
</tr>
<tr>
<td>7th</td>
<td>( a_3^{-3} h_{3,0} h_{2,0} h_{1,2} b_{2,0} )</td>
<td>May differential</td>
</tr>
</tbody>
</table>

In the above diagram the first May differential of the seventh generator is zero since the first May differentials of \( a_3^{-3} h_{3,0} h_{2,0} h_{1,2} \) and \( h_{3,0} \) are all zero. For the first six generators with May filtration \( M_1^3 \), we see that the first May differential of each generator contains at least a term which is not in the first May differential of the other generators. It follows that the first May differentials of the generators are linearly independent and thus the cycle of \( E_r^{s+3, t_q + s - 3, M_1^3} \) must be zero. This implies that \( E_r^{s+3, t_q + s - 3, M_1^3} = 0 \) for \( r \geq 2 \) and hence

\[
a_3^{-3} h_{3,0} h_{1,2} h_{1,0} h_{2,0} b_{1,n-1} \notin d_r(E_r^{s+3, t_q + s - 3, M_1^3}) \text{ for } r \geq 1.
\]
By the same method we can similarly show that for the generators counting from 7 to 11, there is
\[ a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{1,0}h_{2,0}b_{1,n-1} \notin d_r(E_r^{s+3,tq+s-3,M^n_i}) \text{ for } r \geq 1 \]
with corresponding \( i \) and \( n \) for each mentioned generator.

We deal with the last six families of generators differently. By the table (10) in the last part of the proof of Theorem 3.1, we see that \( h_i \) \((5 \leq m \leq 10)\) all come from the first May differential of \( h_i \) \((5 \leq m \leq 10)\). Thus \( d_1(h_{5i}) \) \((5 \leq m \leq 10)\) will possibly be linearly dependent in this case. In order to avoid this, we first just consider \( h_{5i} \) \((5 \leq m \leq 10)\). Since \( d_1(h_{5i}) \neq 0 \), it follows that \( h_{5i} \) \((5 \leq m \leq 10)\) vanish in \( E_2^{s+3,tq+s-3,M^n_{11}} \). Now for the remaining five families of generators, according to the above table we see that the first May differential of each generator contains at least a term which is not in the first May differential of the other generators. It follows that the first May differentials of these generators are linearly independent. Thus \( h_{5i} \) \((6 \leq m \leq 10)\) also vanish in \( E_2^{s+3,tq+s-3,M^n_{11}} \). Hence \( E_k^{s+3,tq+s-3,M^n_{11}} = 0 \) for \( k \geq 2 \) and then
\[ a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{1,0}h_{2,0}b_{1,n-1} \notin d_r(E_r^{s+3,tq+s-3,M^n_{11}}) \text{ for } r \geq 1. \]

According to the above discussion, we see that
\[ a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{1,0}h_{2,0}b_{1,n-1} \]
cannot be hit by any May differential for \( n \geq 2 \). Thus it is a permanent cycle in the MSS and converges nontrivially to \( \tilde{\gamma}_n b_{n-1}g_0 \in \text{Ext}_A^{s+3,tq+s-3}((\mathbb{Z}/p, \mathbb{Z}/p)) \). Thus we have proven the part (b).

We need also the following lemma:

**Lemma 4.2.** [3] Let \( p \geq 5 \) and \( n \geq 2 \). Then
\[ (i_1i_0)_*(b_{n-1}g_0) \in \text{Ext}_A^{4(2+p+p^n_q)}(H^*(V(1)), \mathbb{Z}/p) \]
converges to a nontrivial homotopy element \( \zeta_n \in \pi_{2+p+p^n_q}(V(1)). \)

In what follows we give our proof of Theorem 1.1.

**Proof of Theorem 1.1.** By Lemma 4.2 we see that
\[ (i_1i_0)_*(b_{n-1}g_0) \in \text{Ext}_A^{4(2+p+p^n_q)}(H^*(V(1)), \mathbb{Z}/p) \]
converges to a nontrivial homotopy element $\zeta_n \in \pi_{(2+p+p^n)q-4} V(1)$. Consider the following composite of maps:

$$
\tilde{f}: \Sigma^{(2+p+p^n)q-4} S^{\zeta_n} \to V(1) \xrightarrow{i_2} V(2) \xrightarrow{\gamma^s} \Sigma^{-s(1+p+p^n)} q V(2)
$$

where $\gamma^s$ denotes the $n$-times of composites of $\gamma$. Thus $\tilde{f}$ is represented by

$$
\gamma_s \in \text{Ext}_A^{s,(s-3+(s-1)p+sp^2)q-3} (\mathbb{Z}/p, \mathbb{Z}/p)
$$

due to [12], by the knowledge of Yoneda products we know that the composite

$$
\text{Ext}_A^{0,0} (\mathbb{Z}/p, \mathbb{Z}/p) \xrightarrow{(i_2i_1i_0)} \text{Ext}_A^{0,0} (H^* V(2), \mathbb{Z}/p) \xrightarrow{(j_0j_1j_2)(\gamma^s)^n} \text{Ext}_A^{s,(s-3+(s-1)p+sp^2)q-3} (\mathbb{Z}/p, \mathbb{Z}/p)
$$

is multiplication (up to a nonzero scalar) by

$$
\gamma_s \in \text{Ext}_A^{s,(s-3+(s-1)p+sp^2)q-3} (\mathbb{Z}/p, \mathbb{Z}/p).
$$

Hence $\tilde{f}$ is represented up to a nonzero scalar by a non-zero element

$$
\gamma_s b_{n-1} g_0 \in \text{Ext}_A^{s+4,(s+sp+sp^2+p^n)q+s-3} (\mathbb{Z}/p, \mathbb{Z}/p)
$$

in the ASS (see Lemma 4.1(b)). Moreover, by Lemma 4.1(a), $\gamma_s b_{n-1} g_0$ cannot be hit by the Adams differential

$$
d_r: E_r^{s-r+4,(s+sp+sp^2+p^n)q+s-r-2} \to E_r^{s+4,(s+sp+sp^2+p^n)q+s-3},
$$

for $r \geq 2$, hence the corresponding homotopy element $\tilde{f} \in \pi_\ast S$ is nontrivial. Thus we have finished the proof of Theorem 1.1. □

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