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STRICTLY KÄHLER-BERWALD MANIFOLDS WITH CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

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Abstract. In this paper, the authors prove that a strictly Kähler-Berwald manifold with nonzero constant holomorphic sectional curvature must be a Kähler manifold.
Keywords: Complex Finsler manifold, holomorphic sectional curvature, Kähler-Berwald manifold.
MSC(2010): Primary: 53C56; Secondary: 32Q99.

1. **Introduction and preliminaries**

Recently, more and more people have been attracted to the study of Finsler geometry. The study of Finsler spaces has many applications in physics and biology. In complex Finsler geometry, people think the notion of Kähler-Finlser metrics is the extension of the Kähler metrics. Actually, the Kähler-Berwald metrics may be the closest non-Hermitian complex Finsler metrics to the Kähler metrics. Therefore, to explore the properties of the Kähler-Finsler metrics and the Kähler-Berwald metrics is one of the most important tasks in complex Finsler geometry.

In real Finsler geometry, it has been known that a Berwald manifold with constant flag curvature c must be a Riemann space ($c \neq 0$) or a locally Minkowski space ($c = 0$). In complex cases, the authors [4] prove that a strictly Kähler-Berwald manifold is a complex locally Minkowski space if and only if it has vanishing holomorphic sectional curvature.

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In this paper, we will prove that a strictly Kähler-Berwald manifolds with nonzero constant holomorphic sectional curvature must be a Kähler manifold.

Definition 1.1. *A strongly pseudoconvex complex Finsler metric (we shall simply call it complex Finsler metric below) on a complex manifold* M *is a continuous function* $F : T^{1,0}M \to R^+$ *satisfying:*

(i) $G = F^2$ *is smooth on* $\tilde{M}(= T^{1,0}M - \{0\})$*;*

(ii) $F(v) > 0$ *for all* $v \in \tilde{M}$*;*

(iii) $F(\zeta v) = |\zeta|F(v)$ *for all* $v \in T^{1,0}M$ *and* $\zeta \in C$*;*

(iv) for any $p \in M$*, the* F*-indicatrix* $I_F(p) = \{v \in T_p^{1,0}M | F(v) < 1\}$ *is strongly pseudoconvex.*

A complex manifold M endowed with a complex Finsler metric will be called a complex Finsler manifold.

In the study of complex Finsler geometry, there are several important classes of special metrics with additional properties, in which we are more interested.

Let (M, F) be a complex manifold M of complex dimension n with a complex Finsler metric F. Let $\{z^1, \cdots, z^n\}$ be a set of local complex coordinates, with $\{y^1, \cdots, y^n\}$ the induced holomorphic tangent space coordinates. We shall denote by indexes after G the derivatives with respect to the y-coordinates and the derivatives with respect to the z-coordinates after a semicolon. For instance,

$$G_{\alpha\bar{\beta}} = \frac{\partial^2 G}{\partial y^\alpha \partial \bar{y}^\beta} \quad or \quad G_{;\mu\bar{\nu}} = \frac{\partial^2 G}{\partial z^\mu \partial \bar{z}^\nu}.$$

Definition 1.2. *A complex Finsler manifold* (M, F) *is said to be complex locally Minkowskian if, at every point* $z \in M$*, there is a local coordinate system* (z^α)*, with induced holomorphic tangent space coordinates* (y^α)*, such that* F *has no dependence on the* z^α*. Equivalently speaking,* $G_{\alpha\bar{\beta}}$ *has no dependence on the* z^α*.*

Definition 1.3. *A complex Finsler metric* F *is said to be a complex Berwald metric if the Christoffel symbols* $\Gamma^\alpha_{\beta;\gamma}$ *of Chern-Finsler connection induced by* F *have no* y *dependence in natural coordinates, where*

$$\Gamma^\alpha_{\beta;\gamma} = G^{\bar{\tau}\alpha} \frac{\delta G_{\beta\bar{\tau}}}{\delta z^\gamma};$$

$(G^{\bar{\tau}\alpha})$ *is the inverse matrix of* $(G_{\alpha\bar{\tau}})$, *and* $\frac{\delta}{\delta z^\mu} = \frac{\partial}{\partial z^\mu} - \Gamma^\alpha_{;\mu}\frac{\partial}{\partial y^\alpha}$ *are vectors on* $T^{1,0}M$. *Here* $\Gamma^\alpha_{;\mu} = G^{\bar{\tau}\alpha}G_{\bar{\tau};\mu}$. *Clearly,* $\Gamma^\alpha_{;\mu} = y^\gamma\Gamma^\alpha_{\gamma;\mu}$ *and* $\Gamma^\alpha_{\gamma;\mu} = \frac{\partial}{\partial y^\gamma}\Gamma^\alpha_{;\mu}$.

Definition 1.4. *In local coordinates, a complex Finsler metric is called strongly-Kähler if and only if* $\Gamma^\alpha_{\mu;\nu} = \Gamma^\alpha_{\nu;\mu}$; *it is called Kähler if and only if* $\Gamma^\alpha_{\mu;\nu}y^\mu = \Gamma^\alpha_{\nu;\mu}y^\mu$; *it is called weakly-Kähler if and only if* $G_\alpha[\Gamma^\alpha_{\mu;\nu} - \Gamma^\alpha_{\nu;\mu}]y^\mu = 0$.

Recently, it has been shown in [3] that a Kähler-Finsler metric must be a strongly Kähler-Finsler one.

Definition 1.5. *A Kähler-Finsler metric is called a strictly Kähler-Finsler metric if it satisfies* $\langle\bar{\partial}_H\theta(H,\chi,\bar{K}),\chi\rangle = 0$, *for all* $H, K \in \mathcal{H}$, *where* χ *is the complex radial horizontal vector field,* $\mathcal{H}$ *is the complex horizontal bundle,* θ *is the (2,0)-torsion of the Chern-Finsler connection and* $\bar{\partial}_H$ *is the horizontal part of* $\bar{\partial}$. *We refer to [1] for all notations.*

Abate and Patrizio [1] have done much research on the strictly Kähler-Finsler manifolds, although they haven't given an explicit definition for them.

Let (M, F) be a complex Finsler manifold and $\sigma : [a, b] \to M$ a regular curve on M. We define $\dot{\sigma} : [a, b] \to \tilde{M}$ by setting

$$\dot{\sigma}(t) = \frac{d\sigma^i}{dt}(t)\frac{\partial}{\partial z^i}|_{\sigma(t)}.$$

Then the length of σ is given by

$$L(\sigma) = \int_a^b F(\dot{\sigma}(t))dt.$$

Let $\Sigma : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a regular variation of a given regular curve $\sigma : [a, b] \to M$. We set $l_\Sigma(s) = L(\sigma_s)$, where $\sigma_0 = \sigma$.

Definition 1.6. *We shall say that a regular curve* σ *is a geodesic for* F *if* $\frac{dl_\Sigma}{ds}(0) = 0$ *for all fixed regular variations* Σ *of* σ. *The vector field* $U(t) = U^i(t)\frac{\partial}{\partial z^i}|_{\sigma(t)} := \frac{\partial l_\Sigma}{\partial u}(0, t)$ *is called the variation field of* l_Σ.

Observe that

$$\begin{aligned}\frac{dl_\Sigma}{ds}(0) &= \frac{d}{ds}\int_a^b F(\dot{\sigma}_s(t))dt|_{s=0}\\ &= \int_a^b \frac{1}{2F}\{[F^2]_{z^k}U^k + [F^2]_{\bar{z}^k}\bar{U}^k + [F^2]_{y^k}\frac{dU^k}{dt} + [F^2]_{\bar{y}^k}\frac{d\bar{U}^k}{dt}\}dt\\ &= Re(\int_a^b \frac{1}{2F}\{[F^2]_{z^k}U^k + [F^2]_{y^k}\frac{dU^k}{dt}\}dt)\\ &= Re(\int_a^b\{\frac{1}{2F}[F^2]_{z^k} - \frac{d}{dt}(\frac{1}{2F}[F^2]_{y^k})\}U^k dt + \frac{1}{2F}[F^2]_{y^k}U^k|_a^b)\\ &= Re(\int_a^b \frac{1}{2F}\{[F^2]_{z^k} - \frac{d}{dt}[F^2]_{y^k}\}U^k dt).\end{aligned}$$

Thus σ is a geodesic if and only if the following equality holds on σ:

$$G_{;k} - \frac{d}{dt}G_k = 0,$$

where $G_{;k} = \frac{\partial G}{\partial z^k}, G_k = \frac{\partial G}{\partial y^k}$.

If we further assume M is weak kählerian, we have (see [1]):

$$\ddot{\sigma}^i + N^i_j\dot{\sigma}^j = 0. \quad (1.1)$$

Then for any given $p \in M$ and $v \in \tilde{M}_p$, there exists a unique geodesic $\sigma : (-\varepsilon, \varepsilon) \to M$ such that $\sigma(0) = p$ and $\dot{\sigma}(0) = v$.

2. Holomorphic sectional curvatures for Kähler-Finsler manifolds

Suppose that Y is a non-zero geodesic field on an open subset $\mathcal{U} \subset M$, then G_Y (or g_Y) is a naturally induced smooth Hermitian metric on $\mathcal{U}$, where $G_Y(Z, W) = G_{\alpha\bar{\beta}}(Y)Z^\alpha \bar{W}^\beta$. G_Y is called the Y-Hermitian metric on M.

Let $\bar{D}$ denote the (1,0)-compatible connection of $\bar{G} := G_Y$. It is well known that

$$\bar{D}_Y W = \{y^j\frac{\partial w^i}{\partial z^j} + w^j\tilde{\Gamma}^i_{j;k}y^k\}\frac{\partial}{\partial z^i},$$

where under the local coordinates,

$$Y = y^i\frac{\partial}{\partial z^i}, W = w^i\frac{\partial}{\partial z^i}, and \quad \tilde{\Gamma}^i_{j;k} = (G_{j\bar{h};k} + G_{j\bar{h}r}\frac{\partial y^r}{\partial z^k})G^{\bar{h}i}.$$

Since g_Y is an Hermitian metric on $\mathcal{U}$, we can define the curvature under this metric. Hence for $p \in \mathcal{U}$,we have a well-known quadrilinear function

$$R_Y : T_p\mathcal{U} \times T_p\mathcal{U} \times T_p\mathcal{U} \times T_p\mathcal{U} \to C.$$

Let $p \in M, 0 \neq X \in T_pM$. We write $X = X^{1,0} + X^{0,1}$,where $X^{1,0} \in T^{1,0}M, X^{0,1} \in T^{0,1}M$, and $X^{0,1} = \overline{X^{1,0}}$.We extend $X^{1,0}$ to be a geodesic field on $\mathcal{U}$, and denote it by Y, then $X = Y(p) + \bar{Y}(p)$.

For $X \neq 0$, let

$$\begin{aligned} K(p,X) &= -\frac{R_Y(X,JX,X,JX)}{|X\wedge JX|_Y^2} \\ &= \frac{R_Y(Y,\bar{Y},Y,\bar{Y})}{g_Y^2(Y,Y)}|_p \\ &= \frac{1}{G^2(Y)}R_Y(Y,\bar{Y},Y,\bar{Y})|_p \end{aligned}$$

where J is complex structure, and if $X = 0$, let $K(p,X) = 0$.

Definition 2.1. *Let (M,F) be a Kähler-Finsler manifold. $\forall p \in M, X \in T_pM$,the above $K(p,X)$ is called the holomorphic sectional curvature of M towards the tangent vector X at p.*

We want to show the above definition is rational. For this we need the following conclusion:

Proposition 2.2. *$K(p,X)$ depends only on $Y(p)$(or X) for Kähler-Finsler manifold (M,F).*

Proof. Firstly we seek a formula in local coordinates for $R(p,X)$. To begin with, if $X = \xi^i\frac{\partial}{\partial z^i} + \bar{\xi}^i\frac{\partial}{\partial \bar{z}^i}, JX = i\xi^i\frac{\partial}{\partial z^i} - \bar{\xi}^i\frac{\partial}{\partial \bar{z}^i}.Y = y^i\frac{\partial}{\partial z^i}$, where $y^i(p) = \xi^i$.

$$K(p,X) = \frac{1}{G^2(Y)}R_Y(Y,\bar{Y},Y,\bar{Y})|_p = \frac{R_{i\bar{j}k\bar{l}}y^i\bar{y}^jy^k\bar{y}^l}{g_{i\bar{j}}g_{k\bar{l}}y^i\bar{y}^jy^k\bar{y}^l}|_p$$

where $R_{i\bar{j}k\bar{l}}$ is the curvature tensor under the Hermitian metric g_Y. Now we need to show $R_{i\bar{j}k\bar{l}}y^i\bar{y}^jy^k\bar{y}^l(p)$ depends only on $Y(p)$.

It is well known that

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2[g_{i\bar{j}}(z,Y(z))]}{\partial z^k\partial\bar{z}^l} - \frac{\partial[g_{i\bar{s}}(z,Y(z))]}{\partial z^k}\frac{\partial[g_{t\bar{j}}(z,Y(z))]}{\partial\bar{z}^l}g^{\bar{s}t} \tag{2.1}$$

Lemma 2.3. $\frac{\partial^2[g_{i\bar{j}}(z,Y(z))]}{\partial z^k\partial\bar{z}^l}y^i\bar{y}^l = g_{i\bar{j};\bar{l}}N_k^i\bar{y}^l + g_{i\bar{j}}\frac{\partial N_k^i}{\partial\bar{z}^l}\bar{y}^l - g_{i\bar{j}}\frac{\partial N_k^i}{\partial\bar{y}^s}\bar{N}_l^s\bar{y}^l - g_{i\bar{j}\bar{s}}N_k^i\bar{N}_l^s\bar{y}^l$

Proof. We have known that

$$N_k^s = y^i(g_{i\bar{j};k} + g_{i\bar{j}r}\frac{\partial y^r}{\partial z^k})g^{\bar{j}s},$$

so

$$g_{s\bar{k}}N_k^s = y^i(g_{i\bar{j};k} + g_{i\bar{j}r}\frac{\partial y^r}{\partial z^k}),$$

and

$$y^i \frac{\partial}{\partial \bar{z}^l}(g_{i\bar{j};k} + g_{i\bar{j}r}\frac{\partial y^r}{\partial z^k}) = \frac{\partial}{\partial \bar{z}^l}(g_{s\bar{j}}N_k^s)$$
$$= g_{s\bar{j};\bar{l}}N_k^s + g_{s\bar{j}}\frac{\partial N_k^s}{\partial \bar{z}^l} + (g_{s\bar{j}\bar{t}}N_k^s + g_{s\bar{j}}\frac{\partial N_k^s}{\partial \bar{y}^t})\frac{\partial \bar{y}^t}{\partial \bar{z}^l}.$$

Since Y is a geodesic field, we have

$$\bar{y}^l y^i \frac{\partial}{\partial \bar{z}^l}(g_{i\bar{j};k} + g_{i\bar{j}r}\frac{\partial y^r}{\partial z^k}) = (g_{s\bar{j};\bar{l}}N_k^s + g_{s\bar{j}}\frac{\partial N_k^s}{\partial \bar{z}^l})\bar{y}^l - (g_{s\bar{j}\bar{t}}N_k^s + g_{s\bar{j}}\frac{\partial N_k^s}{\partial \bar{y}^t})\bar{N}_l^t \bar{y}^l,$$

where we have used (1.1). So the equality in lemma holds. □

Now we return to our proof of Proposition 2.1. Dealing with each term of (2.1) similarly as Lemma 2.3, we know $R_{i\bar{j}k\bar{l}}y^i\bar{y}^j y^k\bar{y}^l$ depends only on $Y(p)$. In fact, by direct computation,

$$K(p,X) = \frac{g_{i\bar{j}}(\frac{\partial N_k^i}{\partial \bar{z}^l} - \frac{\partial N_k^i}{\partial \bar{y}^r}\bar{N}_l^r)\bar{y}^j y^k \bar{y}^l}{g_{i\bar{j}}g_{k\bar{l}}y^i\bar{y}^j y^k\bar{y}^l}|_p = \frac{g_{i\bar{j}}\frac{\delta}{\delta \bar{z}^l}N_k^i \bar{y}^j y^k \bar{y}^l}{g_{i\bar{j}}g_{k\bar{l}}y^i\bar{y}^j y^k\bar{y}^l}|_p.$$

□

Definition 2.4. *A Kähler-Finsler metric F is said to be of scalar curvature if $K(p,X) = K(p)$ is independent of the tangent vector X. In particular, if $K(p,X)$ is a constant in spite of any p and X, it is said to be of constant curvature.*

Remark 2.5. *Before the present work, there have been other notions of curvature for complex Finsler manifold (see [1, 7]). However, all of them are unanimous. The definition here seems more natural when it is viewed as an extension from an Hermitian manifold.*

Example 2.6. *Let $(M_1,\alpha),(M_2,\beta)$ be Hermitian manifolds. F_ε is the complex Szabó metric on the product manifold $M_1 \times M_2$ defined by*

$$F_\varepsilon := \sqrt{\alpha(y_1)^2 + \beta(y_2)^2 + \varepsilon(\alpha(y_1)^{2k} + \beta(y_2)^{2k})^{\frac{1}{k}}},$$

where $y = y_1 \oplus y_2 = (v^1,\cdots,v^m,v^{m+1},\cdots,v^{m+n}) \in T_z^{1,0}(M_1\times M_2)$, $z = (z_1,z_2) \in M_1\times M_2$.$y_1 = (v^1,\cdots,v^m) \in T_{z_1}^{1,0}M_1$, $y_2 = (v^{m+1},\cdots,v^{m+n}) \in T_{z_2}^{1,0}M_2$, and $k > 1$ is a positive integer.

We have known in [5] that F_ε is a strongly pseudoconvex complex Finsler metric. Furthermore, F_ε is strongly Kähler-Finslerian if α and

β are both Kähler metrics. In fact, the coefficients of Chern-Finsler connection can be written as follows:

$$N^i_j(y) = \begin{cases} \sum_{l=1}^m a^{\bar{l}i} v_{\bar{l};j} & 1 \le i,j \le m \\ \sum_{l=m+1}^{m+n} b^{\bar{l}i} v_{\bar{l};j} & m+1 \le i,j \le m+n \\ 0 & \text{otherwise} \end{cases}$$

For $X = (X_1, X_2) = y + \bar{y} \in T_z(M_1 \times M_2)$, a direct computation shows that

$$\begin{aligned} K(z,X) &= \tfrac{1}{G^2}(A a_{\alpha\bar{\delta}} \Gamma^{\alpha}_{\gamma;\mu\bar{\nu}} v^{\gamma} \bar{v}^{\delta} v^{\mu} \bar{v}^{\nu} \\ &\quad + B b_{\alpha+m\overline{\delta+m}} \Gamma^{\alpha+m}_{\gamma+m;\mu+m\overline{\nu+m}} v^{\gamma+m} \bar{v}^{\delta+m}) v^{\mu+m} \bar{v}^{\nu+m} \\ &= \tfrac{1}{G^2}(A K_{\alpha}(z_1, X_1) + B K_{\beta}(z_2, X_2)), \end{aligned}$$

where $G = F_{\varepsilon}^2, A = 1 + \varepsilon(\alpha^{2k} + \beta^{2k}))^{\frac{1}{k}-1} \alpha^{2(k-1)}, B = 1 + \varepsilon(\alpha^{2k} + \beta^{2k})^{\frac{1}{k}-1} \beta^{2(k-1)}$.

Now we can easily have:

Theorem 2.7. *Let $(M_1, \alpha), (M_2, \beta)$ be Kähler manifolds and F_{ε} the complex Szabó metric on the product manifold $M_1 \times M_2$. Then the holomorphic sectional curvature of $(M_1 \times M_2, F_{\varepsilon})$ vanishes if both (M_1, α) and (M_2, β) have vanishing holomorphic sectional curvatures.*

3. Strictly Kähler-Berwald manifolds with nonzero constant holomorphic sectional curvature

Since the Chern-Finsler connection is defined on the holomorphic tangent-tangent bundle $T^{1,0}(T^{1,0})M$ of M, we now give another connection directly defined on $T^{1,0}M$.

Let (M, F) be a complex Finsler manifold. For any $z \in M$ and $0 \neq y \in T_z^{1,0}M$, we define $\nabla^y : T_z^{1,0}M \otimes C^{\infty}(T^{1,0}M) \to T_z^{1,0}M$ by $\nabla^y_u V = \{u(V^i(z)) + V^j(z)\Gamma^i_{j;k}(z,y)u^k\} \otimes \frac{\partial}{\partial z^i}|_z$ where

$$u = u^i \frac{\partial}{\partial z^i}|_z, V = V^i \frac{\partial}{\partial z^i}.$$

The complex Berwald connection $\nabla : T_z^{1,0}M \otimes C^{\infty}(T_M^{1,0}) \to T_z^{1,0}M$ is defined by $\nabla_y V = \nabla^y_y V$ for $y \in \tilde{M}_z$ and $\nabla_y = 0$ when $y = 0$. In general, this connection isn't a linear one. However, it is linear if and only if (M, F) is Berwaldian.

Theorem 3.1. *Let (M, F) be a strictly Kähler-Berwald manifold with nonzero constant holomorphic sectional curvature c. Then (M, F) is a Kähler manifold.*

Proof. Let ∇ be the complex Berwald connection on (M,F), which is a linear connection since M is Berwaldian. The curvature forms of ∇ are

$$\Phi_{\alpha}^{\beta}=d\omega_{\alpha}^{\beta}-\omega_{\alpha}^{\gamma}\wedge\omega_{\gamma}^{\beta},$$

where $\omega_{\alpha}^{\beta}=\Gamma_{\gamma;\alpha}^{\beta}dz^{\gamma}$.

Under local coordinate system, we can write

$$\Phi_{\alpha}^{\beta}=\frac{1}{2}K_{\alpha\gamma\bar{\delta}}^{\beta}dz^{\gamma}\wedge d\bar{z}^{\delta},$$

where $K_{\alpha\gamma\bar{\delta}}^{\beta}=-2\frac{\partial\Gamma_{\alpha;\gamma}^{\beta}}{\partial\bar{z}^{\delta}}$, since $[\frac{\delta}{\delta z^{\mu}},\frac{\delta}{\delta z^{\nu}}]=0$.

We can rewrite the holomorphic sectional curvature of (M,F) as

$$K(X)=\frac{G_{\alpha\bar{\beta}}K_{\sigma\gamma\bar{\delta}}^{\alpha}y^{\sigma}\bar{y}^{\beta}y^{\gamma}\bar{y}^{\delta}}{G^{2}(y)},$$

where $X\in T_pM, p\in M$, and $X=y+\bar{y}, y\in T_p^{1,0}M, y=y^{\alpha}\frac{\partial}{\partial z^{\alpha}}$.

Now let (M,F) be a Kähler-Berwald manifold with nonzero constant holomorphic sectional curvature c; then $\Gamma_{\beta;\mu}^{\alpha}$, and $K_{\alpha\gamma\bar{\delta}}^{\beta}$ are independent on y.

Let D be the Chern-Finlser connection associated to F. In local coordinates, the curvature operator of D is given by

$$\Omega_{\beta}^{\alpha}=R_{\beta;\mu\bar{\nu}}^{\alpha}dz^{\mu}\wedge d\bar{z}^{\nu}+R_{\beta\delta;\bar{\nu}}^{\alpha}\psi^{\delta}\wedge d\bar{z}^{\nu}+R_{\beta\bar{\gamma};\mu}^{\alpha}dz^{\mu}\wedge\bar{\psi}^{\gamma}+R_{\beta\delta\bar{\gamma}}^{\alpha}\psi^{\delta}\wedge\bar{\psi}^{\gamma},$$

where

$$\begin{aligned}
R_{\beta;\mu\bar{\nu}}^{\alpha}&=-\delta_{\bar{\nu}}(\Gamma_{\beta;\mu}^{\alpha})-\Gamma_{\beta\sigma}^{\alpha}\delta_{\bar{\nu}}(\Gamma_{;\mu}^{\sigma}),\\
R_{\beta\delta;\bar{\nu}}^{\alpha}&=-\delta_{\bar{\nu}}(\Gamma_{\beta\delta}^{\alpha}),\\
R_{\beta;\bar{\gamma}\mu}^{\alpha}&=-\dot{\partial}_{\gamma}(\Gamma_{\beta;\mu}^{\alpha})-\Gamma_{\beta\sigma}^{\alpha}\Gamma_{\bar{\gamma};\mu}^{\sigma},\\
R_{\beta\delta\bar{\gamma}}^{\alpha}&=-\dot{\partial}_{\gamma}(\Gamma_{\beta\delta}^{\alpha}).
\end{aligned}$$

We refer to [1] for the notations here.

Since (M,F) is with constant holomorphic sectional curvature c, then

$$cG^{2}=-2G_{\alpha}\delta_{\bar{\nu}}(\Gamma_{;\mu}^{\alpha})y^{\mu}\bar{y}^{\nu}.$$

It is equivalent to

$$\frac{c}{2}(G_{\beta\bar{\gamma}}G_{\mu\bar{\nu}}+G_{\beta\bar{\nu}}G_{\mu\bar{\gamma}})y^{\beta}\bar{y}^{\gamma}y^{\mu}\bar{y}^{\nu}=-2G_{\alpha\bar{\gamma}}\delta_{\bar{\nu}}(\Gamma_{\beta;\mu}^{\alpha})y^{\beta}\bar{y}^{\gamma}y^{\mu}\bar{y}^{\nu}. \tag{3.1}$$

Denote $K_{\beta\bar{\sigma}\mu\bar{\nu}}=G_{\alpha\bar{\sigma}}K_{\beta\mu\bar{\nu}}^{\alpha}, R_{\beta\bar{\sigma};\mu\bar{\nu}}=G_{\alpha\bar{\sigma}}R_{\beta;\mu\bar{\nu}}^{\alpha}$, then

$$R_{\beta\bar{\sigma};\mu\bar{\nu}}=\frac{1}{2}K_{\beta\bar{\sigma}\mu\bar{\nu}}-G_{\alpha\bar{\sigma}}\Gamma_{\beta\delta}^{\alpha}\delta_{\bar{\nu}}(\Gamma_{;\mu}^{\delta}),$$

and

$$R_{\beta\bar{\sigma};\mu\bar{\nu}}y^{\beta} = \frac{1}{2}K_{\beta\bar{\sigma}\mu\bar{\nu}}y^{\beta}.$$

For a Kähler-Berwald metric, the condition $\langle\bar{\partial}_H\theta(H,\chi,\bar{K}),\chi\rangle = 0$ is equivalent to $\langle\Omega(H,\bar{K})\chi,\chi\rangle = \langle\Omega(\chi,\bar{K})H,\chi\rangle$,for all $H, K \in \mathcal{H}$. So we have

$$R_{\beta\bar{\sigma};\mu\bar{\nu}}y^{\beta}\bar{y}^{\sigma} = R_{\mu\bar{\sigma};\beta\bar{\nu}}y^{\beta}\bar{y}^{\sigma}.$$

Furthermore, by (1.4) in [2], $\overline{R_{\beta\bar{\sigma};\mu\bar{\nu}}} = R_{\sigma\bar{\beta};\nu\bar{\mu}}$. And we have $R_{\beta\bar{\sigma};\mu\bar{\nu}}y^{\beta}\bar{y}^{\sigma} = R_{\beta\bar{\nu};\mu\bar{\sigma}}y^{\beta}\bar{y}^{\sigma}$. Notice that $K_{\beta\bar{\sigma}\mu\bar{\nu}} = K_{\mu\bar{\sigma}\beta\bar{\nu}}$, we get

$$K_{\beta\bar{\sigma}\mu\bar{\nu}}y^{\beta}\bar{y}^{\sigma}y^{\mu} = K_{\beta\bar{\nu}\mu\bar{\sigma}}y^{\beta}\bar{y}^{\sigma}y^{\mu}. \tag{3.2}$$

Differentiating on $\bar{y}$ for both sides of (3.1), it turns into

$$c(G_{\beta\bar{\gamma}}G_{\mu\bar{\nu}} + G_{\beta\bar{\nu}}G_{\mu\bar{\gamma}})y^{\beta}y^{\mu}\bar{y}^{\nu} = 4G_{\alpha\bar{\gamma}}K^{\alpha}_{\beta\mu\bar{\nu}}y^{\beta}y^{\mu}\bar{y}^{\nu},$$

where we use (3.2). Hence,

$$cG_{\mu\bar{\nu}}y^{\alpha}y^{\mu}\bar{y}^{\nu} = 2K^{\alpha}_{\beta\mu\bar{\nu}}y^{\beta}y^{\mu}\bar{y}^{\nu}.$$

Differentiating again on $\bar{y}$, we have

$$cG_{\mu\bar{\nu}}y^{\alpha}y^{\mu} = 2K^{\alpha}_{\beta\mu\bar{\nu}}y^{\beta}y^{\mu}.$$

Differentiating on y^{α} and add up by α, we can get

$$c(n+1)G_{\mu\bar{\nu}}y^{\mu} = 4K^{\alpha}_{\alpha\mu\bar{\nu}}y^{\mu}.$$

Differentiating once more on y, we have

$$G_{\mu\bar{\nu}} = \frac{4}{c(n+1)}K^{\alpha}_{\alpha\mu\bar{\nu}},$$

which means $G_{\mu\bar{\nu}}$ is independent on y, and (M, F) is a Kähler manifold. □

Let's look at the example in Section 2. It is obvious that F_{ε} is a complex Berwald metric. Furthermore, F_{ε} is strictly Kähler-Finslerian if α and β are both Kähler metrics. Since F_{ε} is non-Hermitian, then it is impossible for $(M_1 \times M_2, F_{\varepsilon})$ to have nonzero constant holomorphic sectional curvature.

4. **A note on S-curvature in complex Finsler geometry**

S-curvature plays an important role in Riemann-Finsler geometry, which describes the rate of change of the distortion along geodesics (see [6, 8, 9]). Now let's take a look in the complex setting.

Let V be an n-dimensional complex vector space and let $F = F(y)$ be a Minkowski norm on V. Fix a basis $\{b_i\}$ for V and let

$$\sigma_F := \frac{vol(B^n)}{vol\{(y^i) \in C^n | F(y^i b_i) < 1\}},$$

and

$$g_{i\bar{j}} = \frac{\partial^2 F^2}{\partial y^i \partial \bar{y}^j},$$

where $y = y^i b_i$. Define

$$\tau := \ln \frac{\det(g_{i\bar{j}})}{\sigma_F}.$$

It is easily verified that τ is well-defined and real-valued. We call it the distortion of F.

Observe that

$$\tau_{y^k} = \frac{\partial}{\partial y^k}[\ln \det(g_{i\bar{j}})] = g^{\bar{j}i} \frac{\partial g_{i\bar{j}}}{\partial y^k} = I_k,$$

where $I_k = g^{\bar{j}i} \frac{\partial g_{i\bar{j}}}{\partial y^k}$ is the mean Cartan tensor given in [10]. By Deicke's Theorem on complex Minkowski space (see [10]), one concludes that F is Euclidean if and only if $\tau = constant$, in which case, $\tau = 0$.

Now we consider complex Finsler metrics. Let F be a complex Finsler metric on a complex manifold M. Since the distortion is defined for the complex Minkowski norm F_z on every holomorphic tangent space $T_z^{1,0}M$, we obtain a scalar function $\tau = \tau(z, y)$ on $T^{1,0}M \backslash \{0\}$. We call it the distortion of F. By Deicke's Theorem in [10], F is Hermitian if and only if $\tau = 0$. Thus the distortion characterizes Hermitian metrics among complex Finsler metrics.

Now we try to define the S-curvature in a complex Finsler manifold. For a vector $y \in T_z^{1,0}M \backslash \{0\}$, let $\sigma = \sigma(t)$ be a geodesic with $\sigma(0) = z$ and $\dot{\sigma}(0) = y$. Set

$$S^1(z, y) := \frac{d}{dt}[\tau(\sigma(t), \dot{\sigma}(t))]|_{t=0},$$

then S^1 is real-valued. However, S^1 is also complex y-homogeneous of degree one,

$$S^1(z, \lambda y) = \lambda S^1(z, y), \lambda \in C - \{0\}.$$

Hence, it has no choice but $S^1 \equiv 0$, which means τ is constant along geodesics. Recall that in real Finsler manifold, the S-curvature vanishes for any Berwald metric. However, in general, this is not the case.

Therefore, we retry to use complex geodesics instead of geodesic curves. For a vector $y \in T_z^{1,0}M \backslash \{0\}$, let $\varphi : \Delta_r \to M$ be a segment of c-geodesic complex curve with $\varphi(0) = z, \varphi'(0) = y$ in the sense of M.Abate and G.Patrizio [1]. We can set

$$S(z, y) = \frac{d}{dz}\tau(\varphi(z), \varphi'(z))|_{z=0}.$$

This S is also complex y-homogeneous of degree one.

However, according to the existence theorem in [1], the Cauchy problem

$$\begin{cases} D_c(\varphi) = \varphi" + A_c(\varphi') + \Gamma^{\alpha}_{;\mu}(\varphi')(\varphi') = 0 \\ \varphi(0) = z, \ \varphi'(0) = y \end{cases}$$

has a holomorphic solution for all (z, y) where $F(y) = 1$ if and only if the holomorphic sectional curvature of (M, F) is constant $2c$ and $\langle \bar{\partial}_H \theta(H, \chi, \bar{\chi}), \chi \rangle = 0$ for all $H \in \mathcal{H}$. This means complex geodesics can exist only under such strict conditions.

Furthermore, we also assert that $S(z, y)$ vanishes for any Kähler-Berwald manifolds. In fact, let $\varphi = \varphi(z)$ be any segment of c-geodesic complex curve on a Kähler-Berwald manifold M. Since the line segment $z = z(t) = \lambda t$ is the geodesic from 0 to λ on Δ_r, where $|\lambda| < r, 0 \leq t \leq 1$, $\gamma(t) = \varphi(z(t))$ is the geodesic on M. Let $\{b_i(0)\}$ be a basis of $T^{1,0}_{\varphi(0)}M$, and we extend it to be a holomorphic frame $\{b_i(z)\}$ on $\varphi(z)$ by parallel translation along the geodesic γ. Let $g_{i\bar{j}}(z) = g_{\dot{\varphi}(z)}(b_i(z), b_j(z))$; then, $g_{i\bar{j}}(z)$ is constant and $\dot{\gamma}(t) = \lambda\dot{\varphi}(z(t))$. Similarly, $F(\varphi(z), y^i b_i(z)) = constant$ for any $(y^i) \in C^n$, so $\sigma_F(\varphi(z)) = constant$ and $S = 0$.

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