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ON THE EIGENVALUES OF NORMAL EDGE-TRANSITIVE CAYLEY GRAPHS

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Abstract. A graph Γ is said to be vertex-transitive or edge- transitive if the automorphism group of Γ acts transitively on $V(\Gamma)$ or $E(\Gamma)$, respectively. Let $\Gamma = Cay(G, S)$ be a Cayley graph on G relative to S. Then, Γ is said to be normal edge-transitive, if $N_{Aut(\Gamma)}(G)$ acts transitively on edges. In this paper, the eigenvalues of normal edge-transitive Cayley graphs of the groups D_{2n} and T_{4n} are given.
Keywords: Eigenvalues, Cayley graphs, normal graph.
MSC(2010): Primary: 05C40; Secondary: 05C90.

1. Introduction

Throughout this paper, all graphs are finite, simple, undirected and connected. For a graph Γ, we denote the vertex set, the edge set and the automorphism group of Γ by $V(\Gamma), E(\Gamma)$ and $Aut(\Gamma)$, respectively. Let G be a finite group and S a subset of G such that $1 \notin S$, $S = S^{-1}$ and $G = \langle S \rangle 5$. The Cayley graph $\Gamma = Cay(G, S)$ on G is a graph with vertex set $V(\Gamma) = G$ and two vertices $x, y \in G$ are adjacent if and only if $xy^{-1} \in S$. The Cayley graph $\Gamma = Cay(G, S)$ is normal if G is a normal subgroup of $Aut(\Gamma)$.

Recently, edge-transitive Cayley graphs of small valency are considered by mathematicians. In [7], all edge-transitive Cayley graphs of valency four and odd order are characterized.

Normal edge-transitive Cayley graphs on the groups $\mathbb{Z}_{pq}$, where p and q are distinct primes, are classified by Houlis [6]. In [1] the authors

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studied normal edge-transitive Cayley graphs on some abelian groups of valency at most 5 and in [2] edge-transitive Cayley graphs of valency 4 on non-abelian simple groups are considered.

In this paper, we compute the eigenvalues of normal edge-transitive Cayley graphs on the groups D_{2n} and T_{4n}. It should be noted that for the group T_{4n}, we will investigate all cases for which the Cayley graph $\Gamma = Cay(T_{4n}, S)$ is normal edge-transitive of valency four.

In the next section, we give necessary definitions and some preliminary results. Section 3 contains the main results, i.e., the explicit formulas for eigenvalues of normal edge-transitive Cayley graphs $Cay(T_{4n}, S)$ and $Cay(D_{2n}, S)$.

2. **Definitions and preliminaries**

Our notation is standard and mainly taken from the standard books of graph theory such as [5]. A graph Γ is said to be vertex-transitive if $Aut(\Gamma)$ acts transitively on $V(\Gamma)$, that is, for every pair of vertices $u, v \in V(\Gamma)$ there exits an automorphism $\alpha \in Aut(\Gamma)$ such that $\alpha(u) = v$. An edge-transitive graph can be defined similarly.

Given any element $g \in G$, we define the permutation ρ_g on G by $\rho_g(x) = xg$ for all $x \in G$. Then $R(G) = \{\rho_g | g \in G\}$ is a permutation group isomorphic to G, which is called the right regular representation of G. Then the subgroup $Aut(G, S)$ of $Aut(G)$ is defined as $Aut(G, S) = \{\alpha \in Aut(G), S^\alpha = S\}$. In [1] it is proved that $Aut(G, S)$ is a subgroup of $Aut(Cay(G, S))_1$, the stabilizer of the vertex 1 in $Aut(Cay(G, S))$.

Given a positive integer s an $s-$arc is a sequence $(v_0, v_1, ..., v_s)$ of $s+1$ vertices of $V(\Gamma)$ such that $(v_{i-1}, v_i) \in E(\Gamma)$ and $v_{i-1} \neq v_{i+1}$ for all i.

Definition 2.1. *A Cayley graph Γ is called normal edge-transitive or normal arc-transitive if $N_A(R(G))$ acts transitively on the set of edges or arcs of Γ, respectively. If Γ is normal edge-transitive, but not normal arc-transitive, then it is called a normal half- arc-transitive Cayley graph.*

Let Γ be a graph with vertex set $V(\Gamma) = \{v_1, v_2, \cdots, v_n\}$, the adjacency matrix $A(\Gamma)$ of Γ is the $n \times n$ symmetric matrix $[a_{ij}]$, such that $a_{ij} = 1$ if v_i and v_j are adjacent and 0, otherwise. The characteristic polynomial $\phi(\Gamma, x)$ of the graph Γ is defined [5] as:

$$\phi(\Gamma, x) = det(xI - A).$$

The roots of the characteristic polynomial are the eigenvalues of the graph G and form the spectrum of this graph.

A circulant matrix is a matrix whose rows are a cyclic permutation of the first row. Thus,

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix}$$

is a circulant matrix, denoted by $[[a_1, a_2, \cdots, a_n]]$. The eigenvectors of a circulant matrix are given by

$$v_j = (1, \omega^j, \omega^{2j}, \cdots, \omega^{(n-1)j}), j = 0, 1, \cdots, n-1,$$

where $\omega = e^{\frac{2\pi}{n}i}$ are the $n-$th roots of unity and $i = \sqrt{-1}$ is the imaginary unit. The corresponding eigenvalues are then given by

$$\lambda_j = a_1 + a_2\omega^j + a_3\omega^{2j} + \cdots + a_n\omega^{(n-1)j}.$$

A block matrix M is a matrix whose entries are again a matrix. Suppose A, B, C, and D are matrices of dimension $(n \times n), (n \times m), (m \times n)$, and $(m \times m)$, respectively. Then

$$det \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = det(A)det(D).$$

Motivated by the above results, we can prove that for the matrix

$$C = \begin{pmatrix} 0 & A \\ A^t & B \end{pmatrix}$$

the characteristic polynomial is

$$\phi(C, \lambda) = |\lambda I - C| = |\lambda I - AA^t|.$$

3. **Main results**

In this section we present the eigenvalues of some normal edge-transitive Cayley graphs. Throughout this paper, the following results are crucial and play a significant role in computing the eigenvalues of Cayley graphs.

Lemma 3.1. *[8] Let $\Gamma = Cay(G, S)$ be a connected Cayley graph on S. Then*

(1) Γ *is normal edge-transitive if and only if $Aut(G, S)$ is either transitive on S, or has two orbits in S in the form of T and T^{-1}, where T is a non-empty subset of S such that $S = T \cup T^{-1}$.*

(2) Γ *is normal arc-transitive if and only if* $Aut(G, S)$ *is transitive on* S.

Corollary 3.2. *Let* $\Gamma = Cay(G, S)$ *and* H *be the subset of all involutions of the group* G. *If* $H \neq G$ *and* Γ *with respect connected normal edge-transitive, then its valency is even.*

Let $T_{4n} = \langle a, b, a^{2n} = 1, b^2 = a^n, bab^{-1} = a^{-1}\rangle$. It is easy to prove that the elements of T_{4n} are

$$1, a, \cdots, a^{2n-1}, b, ba, \cdots, ba^{2n-1}.$$

We can also prove that for $1 \leq i \leq 2n-1$, $ba^iba^i = b^2$ and so $o(ba^i) = 4$.

Theorem 3.3. *We have*

$$|Aut(T_{4n})| = 2n\varphi(2n),$$

where, φ *is Euler function.*

Proof. Consider the map $f_{i,j} : T_{4n} \to T_{4n}$, where

$$f_{i,j} : \begin{cases} a \rightsquigarrow a^i \\ b \rightsquigarrow ba^j \end{cases}$$

and set $Y = \{f_{i,j} | (i, 2n) = 1, 0 \leq j \leq 2n-1\}$. All elements of Y are automorphism. Conversely, let α be an automorphism of T_{4n}. Since, $\langle a\rangle$ is characteristic subgroup of T_{4n}, then necessarily, under every automorphism of T_{4n}, a maps to $a^i, (i, 2n) = 1$, and b maps to an element of order 4, e.g. ba^i. This implies that $\alpha \in Y$. On the other hand, assume that $f_{i,j}, f_{r,s} \in Y$. By definition, $f_{i,j} o f_{r,s}(a) = a^{ir}$ and $f_{i,j} o f_{r,s}(b) = f_{i,j}(ba^s) = ba^{si+j}$. This means that $f_{i,j} o f_{r,s} = f_{ir,si+j} (0 \leq si + j \leq 2n-1, (ir, 2n) = 1)$ and so Y is closed respect to multiplication. One can also prove easily that all elements have an inverse and this completes the proof. $\square$

Theorem 3.4. *Let* $S = \{ba^i, b, (ba^i)^{-1}, b^{-1}\}$, *then* $\Gamma = Cay(G, S)$ *is a normal edge-transitive Cayley graph.*

Proof. Assume that $f_{i,j}(b) = ba$ so that $ba^j = ba$, then $j = 1$. On the other hand, $f_{i,j}(ba) = b$ implies that $i = 2n-1$. Hence, $f_{2n-1,1}(b) = ba$ and $f_{2n-1,1}(ba) = b$. Further, if $f_{r,s}(b) = b^{-1}$, then $ba^s = b^{-1}$ and thus

$s = n$. Since $f_{r,s}(ba) = (ba)^{-1}$, one can conclude that $r = 1$. This implies that $f_{r,s} = f_{1,n}$. By continuing this method one can see that

$$id = f_{1,0}, f_{n-1,n+1}, f_{n+1,n}, f_{n+1,0}, f_{n-1,1} \in Aut(G,S).$$

Suppose now

$$id \cong \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, f_{2n-1,1} \cong \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

$$f_{1,n} \cong \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, f_{n+1,0} \cong \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix},$$

$$f_{n-1,n+1} \cong \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, f_{n+1,n} \cong \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix},$$

$$f_{n-1,1} \cong \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, f_{2n-1,n+1} \cong \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

By using a GAP program, we can prove that $Aut(G,S) \cong D_{2n}$ and $Aut(G,S)$ acts transitively on S. □

Corollary 3.5. *Let $S = \{ba^i, ba^t, (ba^i)^{-1}, (ba^t)^{-1}\}$, where $1 \le i, t \le 2n$ and $i \ne t$, then $\Gamma = Cay(G,S)$ is a normal edge-transitive Cayley graph.*

Proof. Similar to the last theorem, we have

$$\{id, f_{2n-1,t+i}, f_{1,n}, f_{2n-1,n+i+t}\} \subseteq Aut(G,S)$$

and so $S = T \cup T^{-1}$, where $T = \{ba^i, ba^t\}$. □

Corollary 3.6. *The tetravalent Cayley graph $\Gamma = Cay(G,S)$ for $S = \{b, b^{-1}, ba, (ba)^{-1}\}$ is normal arc-transitive.*

Theorem 3.7. *For $S = \{b, ba, b^{-1}, (ba)^{-1}\}$, the spectrum of $\Gamma = Cay(G,S)$ is*

$$Spec(\Gamma) = \begin{pmatrix} -4 & \pm\alpha & 4 \\ 1 & 1 & 1 \end{pmatrix},$$

where $\omega = e^{\frac{\pi}{n}i}$, $\alpha = 1 + \omega + \omega^{nr} + \omega^{(n+1)r}$ and $r = 1, \cdots, 2n-1$.

Proof. Let $S = \{b, ba, b^{-1}, (ba)^{-1}\}$. We claim that the Cayley graph $\Gamma = Cay(G,S)$ be a circulant bipartite graph. Let

$$X = \{1, a, \cdots, a^{2n-1}\}, Y = \{b, ba, \cdots, ba^{2n-1}\}.$$

The vertices of X are not adjacent, since for all integers n, $a^n \notin S$ and similarly, all vertices of Y are not adjacent. This implies that $\Gamma =$

$Cay(G, S)$ is bipartite and so, the adjacency matrix of Γ can be written as the following form

$$A = \begin{pmatrix} O & B \\ B^t & O \end{pmatrix},$$

where B is the circulant matrix $[[1, 1, 0, 0, \cdots, 0, \overbrace{1}^{n}, \overbrace{1}^{n+1}, 0, \cdots, 0]]$. Hence, $\phi(\Gamma, \lambda) = det(\lambda^2 I - B^2) = det(\lambda I - B)det(\lambda I + B)$. Since B is a circulant matrix, its eigenvalues are

$$\lambda_r = 1 + \sum_{j=2}^{2n} \omega^{(j-1)r}, \ r = 0, 1, \cdots, 2n-1, \ \omega = e^{\frac{\pi}{n}i}.$$

If $r = 0$, then $\lambda_0 = 4$ and so ± 4 are eigenvalues of Γ, because it is bipartite. If $r \geq 1$, then $\lambda_r = 1 + \omega + \omega^{nr} + \omega^{(n+1)r}$ and the proof is completed.

□

It is well-known fact that the dihedral group D_{2n} can be presented as follows:

$$D_{2n} = \langle a, b : a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle$$

Similar to group T_{4n}, we compute the eigenvalues of $\Gamma = Cay(D_{2n}, S)$, where Γ is normal edge-transitive. First let us recall the following lemma which present conditions that Γ is normal edge-transitive:

Theorem 3.8. *[9] Let $\Gamma = Cay(D_{2n}, S)$ is a Cayley graph on the dihedral group D_{2n} of valency four. If $S = \{b, ba, ba^i, ba^{1-i}\}$ such that $(n, 2i-1) = 1, 2i(1-i) \equiv 0(mod\ n)$, then Γ is normal edge-transitive.*

We claim that the Cayley graph $\Gamma = Cay(D_{2n}, S)$ be a circulant bipartite graph. Let

$$X = \{1, a, \cdots, a^{n-1}\}, Y = \{b, ba, \cdots, ba^{n-1}\}.$$

Similar to the proof of Theorem 3.7, one can prove that the elements of X and Y are not adjacent with themselves. Hence, $\Gamma = Cay(D_{2n}, S)$ is a bipartite Cayley graph. It should be noted that 1 is adjacent with all elements of S. If ba^j be adjacent with a^i, then $ba^{j-i} \in S$ and thus, $j - i \equiv 0, 1, i$ or $1 - i(mod\ n)$. This implies the adjacency matrix of Γ is as follows,

$$A = \begin{pmatrix} O & B \\ B^t & O \end{pmatrix},$$

where B is the circulant matrix $[[1,0,\cdots,0,\overbrace{1}^{i},0,\cdots,\overbrace{1}^{n-i+1},0,\cdots,0]]$. Hence, $\phi(\Gamma,\lambda)=det(\lambda^2I-B^2)=det(\lambda I-B)det(\lambda I+B)$. Since B is a circulant matrix, its eigenvalues are

$$\lambda_r=1+\sum_{j=2}^{n}\omega^{(j-1)r},\ r=0,1,\cdots,2n-1,\ \omega=e^{\frac{\pi}{n}i}.$$

If $r=0$, then $\lambda_0=4$ and so ± 4 are eigenvalues of Γ, because it is bipartite. If $r\geq 1$, then $\lambda_r=1+\omega^r+\omega^{ir}+\omega^{(n-i+1)r}$ and we have proved the following theorem.

Theorem 3.9. *Let $S=\{b,ba,ba^i,ba^{1-i}\}$ and $\Gamma=Cay(D_{2n},S)$ be normal edge-transitive Cayley graph on D_{2n} respect with S, then the spectrum of Γ is*

$$Spec(\Gamma)=\begin{pmatrix}-4 & \pm\beta & 4\\ 1 & 1 & 1\end{pmatrix},$$

where $\omega=e^{\frac{2\pi}{n}i},\beta=1+\omega^k+\omega^{ik}+\omega^{(n-i+1)k}$ and $k=1,\cdots,n-1$.

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