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THE CONTINUITY OF LINEAR AND SUBLINEAR CORRESPONDENCES DEFINED ON CONES

M. AGHAJANI, K. NOUROUZI* AND D. O'REGAN

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ABSTRACT. In this paper, we investigate the continuity of linear and sublinear correspondences defined on cones in normed spaces. We also generalize some known results for sublinear correspondences.

Keywords: Linear correspondence, sublinear correspondence, cone.
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1. Introduction

An investigation of linear correspondences defined on cones in normed spaces was given in [3]. In particular, the existence of a unique iteration semigroup of continuous linear selections of an iteration semigroup of linear correspondences defined on a cone with a finite cone basis is shown in [3]. It is shown in [5] that a regular cosine family consisting of super-additive mappings continuous and homogeneous with respect to positive rationals with compact values has exponential growth. The continuity of a regular cosine family consisting of continuous and additive mappings with compact and convex values defined on cones with nonempty interior in Banach spaces is established in [5]. A generalization of these results in normed spaces can be found in [1].

In this paper, we reintroduce linear and sublinear correspondences on cones in real normed spaces and give some results on continuity. A general form of linear and sublinear correspondences with convex and compact values is given. We also present some results on invertibility of selections of sublinear correspondences and some results for an iteration

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semigroup of sublinear correspondences. More precisely, the outline of this paper is as follows.

In Lemma 2.2 we give a necessary and sufficient condition for upper semicontinuity of a sublinear correspondence. In Lemma 2.6 we show that the inequality given in Lemma 2 of [3] can be replaced by equality. Corollaries 2.7 and 2.8 show the validity of Lemma 2 and Lemma 3 in [3] for sublinear correspondences, respectively. Theorem 3.4 is a restatement of Theorem 1 in [3] for sublinear correspondences.

We begin with some basic concepts which are needed in this paper.

A subset C of a real normed space X is a cone if $tC \subseteq C$ for every $t > 0$. A linearly independent set E is said to be a basis of cone C if

$$C = \{x \in X : x = \sum_{i=1}^n \lambda_i e_i, n \in \mathbb{N}, e_i \in E, \lambda_i \geq 0, i = 1, \dots, n\}.$$

Throughout this paper we assume that X and Y are two real normed spaces and C is a convex cone of X .

Let $c(X)$ denote the set of all nonempty and compact subsets of X and $cc(X)$ be the family of all convex sets of $c(X)$.

We recall that a correspondence φ on any subset E of X is a relation which assigns a nonempty set of Y to each element of E . We use the notations $\varphi : C \rightarrow c(Y)$ and $\varphi : C \rightarrow cc(Y)$ for correspondences with compact values and convex and compact values, respectively.

Definition 1.1. [3] A correspondence $\varphi : C \rightarrow Y$ is called:

- (1) linear if $\varphi(x+y) = \varphi(x) + \varphi(y)$ (additivity) and $\varphi(\lambda x) = \lambda\varphi(x)$, for every $x, y \in C$ and $\lambda > 0$;
- (2) sublinear if $\varphi(x+y) \subseteq \varphi(x) + \varphi(y)$ and $\varphi(\lambda x) = \lambda\varphi(x)$, for every $x, y \in C$ and $\lambda > 0$.

It is clear that every linear correspondence is sublinear but the converse is not true.

Definition 1.2. [5] A correspondence $\varphi : C \rightarrow Y$ is said to be bounded if for every bounded subset E of C the subset $\varphi(E)$ is bounded in Y .

We recall that a neighborhood of a set A is any set B for which there is an open set V satisfying $A \subseteq V \subseteq B$.

Definition 1.3. [2] A correspondence $\varphi : C \rightarrow Y$ is said to be:

- (1) upper semicontinuous at the point x if for every neighborhood U of $\varphi(x)$, there is a neighborhood V of x such that $z \in V$ implies $\varphi(z) \subseteq U$. Also φ is upper semicontinuous on C , if it is upper semicontinuous at every point of C .

- (2) lower semicontinuous at the point x if for every open set U that $\varphi(x) \cap U \neq \emptyset$ there is a neighborhood V of x such that $z \in V$ implies $\varphi(z) \cap U \neq \emptyset$. φ is lower semicontinuous on C , if it is lower semicontinuous at every point of C .
- (3) continuous at x if it is both upper and lower semicontinuous at x . It is continuous if it is continuous at each point of C .

For each pair of nonempty and compact subsets A and B of X , the Hausdorff metric \mathfrak{h} is defined as

$$\mathfrak{h}(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\},$$

where $d(a, B) = \inf_{b \in B} \|a - b\|$.

Every correspondence with compact values $\varphi : X \rightrightarrows Y$ is continuous if and only if $\varphi : X \rightarrow (c(Y), \mathfrak{h})$ is continuous in the sense of a single-valued function (see Theorem 17.15 in [2]).

2. Continuity of linear and sublinear correspondences

In this section we study the continuity of linear and sublinear correspondences defined on cones with a finite basis in real normed spaces. We start with the following.

Lemma 2.1. [5] *A sublinear correspondence $\varphi : C \rightrightarrows Y$ is bounded if and only if there exists a positive constant M such that*

$$(2.1) \quad \|\varphi(x)\| := \sup\{\|y\| : y \in \varphi(x)\} \leq M\|x\|, \quad (x \in C).$$

Lemma 1 in [3] gives a necessary condition for upper semicontinuity of a linear correspondence.

Lemma 2.2. *Let $0 \in C \subseteq X$. If $\varphi : C \rightrightarrows Y$ is a bounded-valued sublinear correspondence, then φ is upper semicontinuous at zero if and only if φ is bounded.*

Proof. If φ is upper semicontinuous at zero, then by an argument similar to that in the proof of ([3], Lemma 1) and Lemma 2.1 we get the boundedness of φ . Conversely, suppose that φ satisfies (2.1) and U is a neighborhood of $\varphi(0) = \{0\}$. Then, there exists $\varepsilon > 0$ such that $N_\varepsilon(0) \subseteq U$. Now for every $z \in N_{\frac{\varepsilon}{M}}(0)$ we have $\varphi(z) \subseteq U$, i.e., φ is upper semicontinuous at zero. \square

We define the norm of a bounded sublinear correspondence $\varphi : C \rightrightarrows Y$ by

$$\|\varphi\| = \inf\{M > 0 : \|\varphi(x)\| \leq M\|x\|, x \in C\}.$$

Theorem 2.3. *Let $E = \{e_1, e_2, \dots, e_n\}$ be a basis of C . If $\varphi : C \rightarrow c(Y)$ is linear, then φ is continuous.*

Proof. Let \sim denote the Rådström's equivalence relation between pairs of members of $cc(Y)$ defined by

$$(A, B) \sim (C, D) \Leftrightarrow A + D = B + C, \quad (A, B \in cc(X))$$

and $[A, B]$ denote the equivalence class of (A, B) (see [4]). The set of all equivalence classes Δ with the operations

$$[A, B] + [C, D] = [A + C, B + D],$$

$$\lambda[A, B] = [\lambda A, \lambda B] \quad (\lambda \geq 0),$$

$$\lambda[A, B] = [-\lambda B, -\lambda A] \quad (\lambda < 0),$$

and the norm

$$\|[A, B]\| := \mathfrak{h}(A, B),$$

constitute a real linear normed space (see [4]). The function $f : C \rightarrow \Delta$ defined by

$$f(x) = [\varphi(x), \{0\}],$$

is linear and can be extended to a linear operator $\hat{f} : C - C \rightarrow \Delta$ by

$$\hat{f}(x - y) = f(x) - f(y), \quad (x, y \in C).$$

Since $C - C$ is of finite dimension, \hat{f} and consequently f are continuous. Let $x_0 \in C$ and (x_n) be a sequence of C converging to x_0 . Then

$$\lim_{n \rightarrow \infty} \mathfrak{h}(\varphi(x_n), \varphi(x_0)) = \lim_{n \rightarrow \infty} \|f(x_n) - f(x_0)\| = 0,$$

that is, φ is continuous. \square

Each set of the form $M := M_1 \times \dots \times M_n$, where $M_i \subseteq \mathbb{R}^n$ ($i = 1, \dots, n$) will be called a multimatrix. If $E = \{e_1, e_2, \dots, e_n\}$ is a basis of C and $\varphi : C \rightarrow C$ is linear, then there exists an isomorphism $l : C - C \rightarrow \mathbb{R}^n$ defined by

$$(2.2) \quad l\left(\sum_{j=1}^n \lambda_j e_j\right) = (\lambda_1, \dots, \lambda_n)^T,$$

that maps C onto $[0, +\infty)^n$ and $M_\varphi := l(\varphi(e_1)) \times \dots \times l(\varphi(e_n))$ is a nonempty convex multimatrix [3].

Corollary 2.4. *Let $E = \{e_1, e_2, \dots, e_n\}$ be a basis of C . If $\varphi : C \rightarrow C$ is a linear correspondence, then*

$$(2.3) \quad \varphi(x) = \{l^{-1}Al(x)\}_{A \in M_\varphi}, \quad (0 \neq x \in C)$$

and φ is lower semicontinuous at every point.

Proof. If $x = \sum_{j=1}^n \lambda_j e_j \in C$, we have $l^{-1}Al(x) = \sum_{i=1}^n \sum_{j=1}^n (\lambda_j a_{ij}) e_i$, for every $A = [a_{ij}] \in M_\varphi$ and therefore

$$\left\{ \sum_{i=1}^n \sum_{j=1}^n (\lambda_j a_{ij}) e_i : A = [a_{ij}] \in M_\varphi \right\} = \{l^{-1}Al(x)\}_{A \in M_\varphi}.$$

Thus, it suffices to show that

$$\varphi(x) = \left\{ \sum_{i=1}^n \sum_{j=1}^n (\lambda_j a_{ij}) e_i : A = [a_{ij}] \in M_\varphi \right\} \quad (0 \neq x = \sum_{j=1}^n \lambda_j e_j \in C).$$

Let $\varphi_M(x)$ be the quantity of the right hand and $z \in \varphi(x)$. We may find $u_j \in \varphi(e_j)$ and then $A = [a_{ij}] \in M_\varphi$ such that

$$z = \sum_{j=1}^n \lambda_j u_j = \sum_{j=1}^n \lambda_j \sum_{i=1}^n (a_{ij} e_i) = \sum_{j=1}^n \sum_{i=1}^n (\lambda_j a_{ij}) e_i.$$

Therefore $z \in \varphi_M(x)$. It is easy to see that $\varphi_M(x) \subseteq \varphi(x)$ and therefore $\varphi(x) = \varphi_M(x)$ for every $x \in C \setminus \{0\}$. To see the lower semicontinuity of φ , let $x \in C \setminus \{0\}$ and U be an open set with $U \cap \varphi(x) \neq \emptyset$. From (2.3) and the continuity of $l^{-1}Al$ for each $A \in M_\varphi$ there exists an open neighborhood V of x such that $\varphi(z) \cap U \neq \emptyset$, for each $z \in V$. Now to see the lower semicontinuity of φ at zero, let $(x_n)_n$ be a sequence convergent to zero and $y \in \varphi(0)$. We can assume that all x_n 's are nonzero. Fix an $A \in M_\varphi$ so $\lim_{n \rightarrow \infty} l^{-1}Al(x_n) = 0$ and the sequence (z_n) with

$$z_n = y + l^{-1}Al(x_n) \in \varphi(0) + \varphi(x_n) = \varphi(x_n),$$

tends to y . Thus, by Theorem 17.21 in [2], the proof is complete. \square

In Corollary 2.4 if $\varphi : C \rightarrow c(C)$, then φ is continuous by Theorem 2.3.

Let C be a cone with a finite basis $\{e_1, e_2, \dots, e_n\}$. For every sub-linear correspondence $\varphi : C \rightarrow c(Y)$ there exists a linear continuous correspondence $\widehat{\varphi} : C \rightarrow cc(Y_0)$ containing φ , defined by

$$(2.4) \quad \widehat{\varphi}(x) = \sum_{j=1}^n \lambda_j \overline{c\mathcal{O}}(\varphi(e_j)),$$

for every $x = \sum_{j=1}^n \lambda_j e_j$, where Y_0 and $\overline{c\mathcal{O}}(\varphi(e_j))$ denote the completion of Y and the closed convex hull of the set $\varphi(e_j)$ in Y_0 , respectively.

Corollary 2.5. *Let $E = \{e_1, e_2, \dots, e_n\}$ be a basis of C . If $\varphi : C \rightarrow c(Y)$ is a sublinear correspondence, then*

i) φ is upper semicontinuous at every point;

ii) moreover, if $\varphi : C \twoheadrightarrow C$, then for every $x \in C \setminus \{0\}$ we have

$$\varphi(x) \subseteq (l^{-1}Al(x))_{A \in M_{c\mathcal{O}}(\varphi)},$$

where l is the isomorphism given in (2.2).

Proof. i) Let Y_0 be the completion of Y . Consider $\widehat{\varphi}$ as given in (2.4). Obviously, $\widehat{\varphi} : C \twoheadrightarrow Y_0$ is a linear correspondence with convex and compact values. From Theorem 2.3, since $\widehat{\varphi}$ is upper semicontinuous at zero and $\|\varphi\| \leq \|\widehat{\varphi}\|$, Lemma 2.2 implies that φ is upper semicontinuous at zero. Now let x_0 be a nonzero element in C . For any neighborhood U of $\varphi(x_0)$, there exists an open ball U_0 centered at zero such that,

$$\varphi(x_0) - U_0 + U_0 \subseteq U.$$

Since φ is upper semicontinuous at zero then there is an open neighborhood V_0 of zero in C such that $\varphi(z) \subseteq U_0$ for every $z \in V_0$. We may suppose that $\alpha x_0 \in V_0$, for some $\alpha \in (0, 1)$. Now there is an open ball W_0 of αx_0 such that $W_0 \subseteq V_0$ with $x_0 \notin \overline{W_0}$. Putting $W_{x_0} = x_0 - \alpha x_0 + W_0$ we see that W_{x_0} is open. Since $x_0 = \sum_{i=1}^n \lambda_i e_i \notin \overline{W_0}$ there exist $r > 0$ and open balls $N_r^C(x_0)$ and $N_r^C(\alpha x_0)$ in C such that $N_r^C(\alpha x_0) \subseteq W_0$, $N_r^C(x_0) \cap W_0 = \emptyset$ and $r < \alpha \lambda_i$ for $i = 1, \dots, n$ with $\lambda_i \neq 0$. We now show that

$$N_r^C(x_0) \subseteq N_r^C(\alpha x_0) + (1 - \alpha)x_0.$$

Without loss of generality we assume that $\|e_1\| = \|e_2\| = \dots = \|e_n\| = 1$. Let $z = \sum_{i=1}^n \mu_i e_i \in N_r^C(x_0)$, so $-r < \mu_i - \lambda_i < r$ for each $1 \leq i \leq n$ with $\lambda_i \neq 0$. Therefore,

$$0 < -r + \alpha \lambda_i < \mu_i - \lambda_i + \alpha \lambda_i < r + \alpha \lambda_i \quad (i = 1, \dots, n), \quad \lambda_i \neq 0,$$

and

$$0 < \mu_i < r \quad (i = 1, \dots, n), \quad \lambda_i = 0.$$

Thus $\sum_{i=1}^n (\mu_i - (1 - \alpha)\lambda_i)e_i \in C$. Since

$$\|z - (1 - \alpha)x_0 - \alpha x_0\| = \|z - x_0\| < r,$$

so

$$N_r^C(x_0) - (1 - \alpha)x_0 \subseteq N_r^C(\alpha x_0).$$

Now for every $z \in W_{x_0}$, there is a point $z_0 \in W_0$ such that $z = x_0 - \alpha x_0 + z_0$ and from above (note $\alpha x_0, z_0 \in V_0$)

$$\begin{aligned} \varphi(z) = \varphi(x_0 - \alpha x_0 + z_0) &\subseteq (1 - \alpha)\varphi(x_0) + \varphi(z_0) \\ &\subseteq \varphi(x_0) - \alpha\varphi(x_0) + \varphi(z_0) \\ &\subseteq \varphi(x_0) - U_0 + U_0 \\ &\subseteq U. \end{aligned}$$

Thus, φ is upper semicontinuous at x_0 .

ii) Consider the linear correspondence $\widehat{\varphi}$ as (2.4). By Corollary 2.4, $\widehat{\varphi}$ is of the form (2.3). Since $M_{co(\varphi)} = M_{\widehat{\varphi}}$ and $\varphi(x) \subseteq \widehat{\varphi}(x)$ for each nonzero x , we obtain the desired inclusion. \square

The following example shows that a sublinear correspondence need not be lower semicontinuous at every point.

Example 1. Define $\varphi : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty) \times [0, +\infty)$ by

$$\varphi(x, y) = \begin{cases} \{(0, 0)\} & x \geq 0, y > 0; \\ \{(t, 0) : 0 \leq t \leq x\} & x \geq 0, y = 0. \end{cases}$$

It is easy to see that the sublinear correspondence φ is not lower semicontinuous at every point $(x, 0)$ where $x > 0$.

For the rest of this section we consider, inspired by [3], the relations between Hausdorff distance of the unit matrix and multimatrix of a linear correspondence and invertibility of its selections.

Every cone C with a finite basis $E = \{e_1, \dots, e_n\}$ induces a norm on the vector space of all $n \times n$ matrices $\mathbb{M}_n(\mathbb{R})$ by

$$(2.5) \quad \|A\| = \sup\left\{ \left\| \sum_{i=1}^n \left(\sum_{j=1}^n \lambda_j a_{ij} \right) e_i \right\| : \sum_{j=1}^n \lambda_j e_j \in C, \left\| \sum_{j=1}^n \lambda_j e_j \right\| = 1 \right\},$$

for every $A = [a_{ij}]$ (see [3]).

In the following, \mathfrak{h}_1 and \mathbb{I} will denote the Hausdorff metric derived from the norm given in (2.5) and the unit matrix, respectively.

Lemma 2.6. *Suppose that C has a finite cone basis. If $\varphi : C \rightarrow c(C)$ is a linear correspondence, then*

$$\mathfrak{h}_1(M_\varphi, \{\mathbb{I}\}) = \sup\{\mathfrak{h}(\varphi(x), \{x\}) : x \in C, \|x\| = 1\}.$$

Proof. From Lemma 2 in [3], we have

$$\mathfrak{h}_1(M_\varphi, \{\mathbb{I}\}) \leq \sup\{\mathfrak{h}(\varphi(x), \{x\}) : x \in C, \|x\| = 1\},$$

and from Corollary 2.4, $\varphi(x) = \{l^{-1}Al(x)\}_{A \in M_\varphi}$ for each $x \in C \setminus \{0\}$. If $x \in C$ with $\|x\| = 1$, then by Lemma 3.76 in [2] there exists $A_x \in M_\varphi$ such that

$$\mathfrak{h}(\varphi(x), \{x\}) = \|l^{-1}A_x l(x) - x\| \leq \|A_x - \mathbb{I}\|.$$

Therefore

$$\begin{aligned} \sup\{\mathfrak{h}(\varphi(x), \{x\}) : x \in C, \|x\| = 1\} &\leq \sup\{\|A_x - \mathbb{I}\| : x \in C, \|x\| = 1\} \\ &\leq \sup\{\|A - \mathbb{I}\| : A \in M_\varphi\} \\ &= \mathfrak{h}_1(M_\varphi, \{\mathbb{I}\}). \end{aligned}$$

□

Corollary 2.7. *Suppose that C has a finite cone basis. If $\varphi : C \rightarrow c(C)$ is a sublinear correspondence, then*

$$\mathfrak{h}_1(M_{\widehat{\varphi}}, \{\mathbb{I}\}) \geq \sup\{\mathfrak{h}(\varphi(x), \{x\}) : x \in C, \|x\| = 1\},$$

where $\widehat{\varphi}$ is given in (2.4).

Proof. The proof is an easy application of Lemma 2.6 and $\varphi(x) \subseteq \widehat{\varphi}(x)$ for each $x \in C$. □

In [3], it is shown that for a cone C with a finite basis there exists an $\eta > 0$ such that for every linear correspondence $\varphi : C \rightarrow c(C)$ satisfying

$$\mathfrak{h}_1(M_\varphi, \{\mathbb{I}\}) < \eta,$$

every $A \in M_\varphi$ is invertible. If $co\varphi : C \rightarrow cc(C)$ denotes the correspondence $x \rightarrow co\varphi(x)$, then we have the following result.

Corollary 2.8. *Let $\{e_1, e_2, \dots, e_n\}$ be a finite basis of C . Then, there exists an $\eta > 0$ such that for every sublinear correspondence $\varphi : C \rightarrow c(C)$ satisfying $\mathfrak{h}_1(M_{co(\varphi)}, \{\mathbb{I}\}) < \eta$, each $A \in M_\varphi$ is invertible.*

Proof. Consider $\widehat{\varphi}$ as given in (2.4). Since $\widehat{\varphi}$ is linear with convex and compact values, by Lemma 3 in [3], there exists $\eta > 0$ such that for every linear correspondence $\widehat{\varphi}$ with $\mathfrak{h}_1(M_{\widehat{\varphi}}, \{\mathbb{I}\}) < \eta$, then $A \in M_{\widehat{\varphi}}$ is invertible. Since $M_{co(\varphi)} = M_{\widehat{\varphi}}$ and $M_{\varphi} \subseteq M_{co(\varphi)}$, every $A \in M_{\varphi}$ is invertible. \square

3. Iteration semigroups of sublinear correspondences

In this section we investigate the continuity of an iteration semigroup of sublinear correspondences. Theorem 3.4 is, in fact, a generalization of Theorem 1 in [3].

Recall that the composition of two correspondences $\varphi : X \rightarrow Y$ and $\psi : Y \rightarrow Z$ is defined by

$$\psi \circ \varphi(x) = \cup_{y \in \varphi(x)} \psi(y), \quad (x \in X).$$

Definition 3.1. [3] A family $\{\varphi^t : t \geq 0\}$ of correspondences $\varphi^t : C \rightarrow C$ is called an iteration semigroup if $\varphi^t \circ \varphi^s = \varphi^{t+s}$ for all $t, s \geq 0$. An iteration semigroup $\{\varphi^t : t \geq 0\}$ of correspondences $\varphi^t : C \rightarrow cc(C)$ is said to be continuous if for every $x \in C$ the correspondence $t \rightarrow \varphi^t(x)$ is continuous.

Lemma 3.2. [5] *Let C be convex with nonempty interior. Then there exists $M > 0$ such that for every linear continuous correspondence $\varphi : C \rightarrow c(Y)$ the inequality*

$$\mathfrak{h}(\varphi(x), \varphi(y)) \leq M \|\varphi\| \|x - y\|, \quad (x, y \in C)$$

holds.

As a direct result of Lemma 3.2, we get the following result. If $E = \{e_1, e_2, \dots, e_n\}$ is a finite basis of C , then there is $M > 0$ such that for every compact-valued sublinear correspondence $\varphi, \psi : C \rightarrow c(C)$,

$$\mathfrak{h}(\widehat{\varphi} \circ \widehat{\psi}(x), \widehat{\varphi}(x)) \leq M \|\widehat{\varphi}\| \mathfrak{h}(\widehat{\psi}(x), \{x\}), \quad (x \in C)$$

where $\widehat{\varphi}$ is of the form (2.4).

Lemma 3.3. *Let C be a cone with finite basis $\{e_1, e_2, \dots, e_n\}$. If $\{\varphi^t : C \rightarrow cc(C)\}_{t \geq 0}$ is an iteration semigroup of sublinear correspondences φ^t with $\varphi^0(x) = \{x\}$, then there exists $M > 1$ such that*

$$(3.1) \quad \mathfrak{h}(\widehat{\varphi^{w+s}}(x), \widehat{\varphi^w}(x)) \leq M \|\widehat{\varphi^w}\| \|\widehat{\varphi^s} - \varphi^0\| \|x\|,$$

for each $w, s \geq 0$ and $x \in C$.

Proof. For every $w \geq 0, s \geq 0$ and $z \in \widehat{\varphi^{s+w}}(x) \subseteq \widehat{\varphi^w} \circ \widehat{\varphi^s}(x)$, we get

$$d(z, \widehat{\varphi^w}(x)) \leq \mathfrak{h}(\widehat{\varphi^w} \circ \widehat{\varphi^s}(x), \widehat{\varphi^w}(x)).$$

Therefore by Lemma 3.2,

$$d(z, \widehat{\varphi^w}(x)) \leq M \|\widehat{\varphi^w}\| \mathfrak{h}(\widehat{\varphi^s}(x), \{x\}),$$

and so

$$(3.2) \quad \sup_{z \in \widehat{\varphi^{w+s}}(x)} d(z, \widehat{\varphi^w}(x)) \leq M \|\widehat{\varphi^w}\| \|\widehat{\varphi^s} - \varphi^0\| \|x\|.$$

Without loss of generality we can assume that $M > 1$. On other hand for $z \in \widehat{\varphi^w}(x)$, there exist $z_i \in \varphi^w(e_i)$, $i = 1, \dots, n$ such that $z = \sum_{i=1}^n \lambda_i e_i$. Thus

$$\begin{aligned} \widehat{\varphi^{w+s}}(x) &= \sum_{i=1}^n \lambda_i \varphi^s \varphi^w(e_i) \\ &\supseteq \sum_{i=1}^n \lambda_i \varphi^s(z_i) \\ &\supseteq \varphi^s(\sum_{i=1}^n \lambda_i z_i), \end{aligned}$$

and consequently

$$\begin{aligned} d(z, \widehat{\varphi^{w+s}}(x)) &\leq d(z, \varphi^s(z)) \\ &\leq \sup_{y \in \widehat{\varphi^s}(z)} \|z - y\| \\ &= \|(\widehat{\varphi^s} - \varphi^0)(z)\| \\ &\leq \|z\| \|\widehat{\varphi^s} - \varphi^0\| \\ &\leq \|\widehat{\varphi^w}\| \|\widehat{\varphi^s} - \varphi^0\| \|x\|. \end{aligned}$$

Hence

$$(3.3) \quad \sup_{z \in \widehat{\varphi^w}(x)} d(z, \widehat{\varphi^{w+s}}(x)) \leq M \|\widehat{\varphi^w}\| \|\widehat{\varphi^s} - \varphi^0\| \|x\|.$$

Now, (3.2) and (3.3) imply (3.1). \square

Theorem 3.4. *Let C be a cone with finite basis $\{e_1, \dots, e_n\}$ and let B be a bounded subset of C . If $\{\varphi^t : C \rightarrow cc(C)\}_{t \geq 0}$ is an iteration semigroup of sublinear correspondences satisfying the conditions:*

- i) $\varphi^0(x) = \{x\}$, for all $x \in C$;*
- ii) $\lim_{t \rightarrow 0} \|\varphi^t - \varphi^0\| = 0$;*

then, there exists $\beta_0 > 0$ and $\gamma > 0$ such that $\|\varphi^t\| \leq \beta_0 e^{\gamma t}$, for each $t \geq 0$ and

(3.4)

$$\forall w \geq 0 \forall \varepsilon > 0 \exists \delta > 0 \forall x \in B(|s - w| < \delta \Rightarrow \mathfrak{h}(\widehat{\varphi^w}(x), \widehat{\varphi^s}(x)) < \varepsilon).$$

In particular, $\{\widehat{\varphi^t} : t \geq 0\}$ is continuous, where $\widehat{\varphi^t}$ is of the form (2.4).

Proof. We assume that $\|e_1\| = \dots = \|e_n\| = 1$. Let $\{\varphi^t : t \geq 0\}$ be an iteration semigroup satisfying i) and ii). Consider $\widehat{\varphi^t}$ as in (2.4) and $\|\cdot\|_0$ the norm induced by the basis $\{e_1, \dots, e_n\}$ of $C - C$ with $\|x\|_0 = \sum_{i=1}^n |\lambda_i|$, where $x = \sum_{i=1}^n \lambda_i e_i$. Since

$$\begin{aligned} \|\widehat{\varphi^t} - \varphi^0\| &= \sup\{\|\widehat{\varphi^t}(x) - x\| : \|x\| = 1, x \in C\} \\ &= \sup\{\|\sum_{i=1}^n \lambda_i \varphi^t(e_i) - \sum_{i=1}^n \lambda_i e_i\| : \|x\| = 1, x = \sum_{i=1}^n \lambda_i e_i\} \\ &\leq \sup\{\sum_{i=1}^n \lambda_i \|\varphi^t(e_i) - e_i\| : \|x\| = 1, x = \sum_{i=1}^n \lambda_i e_i\} \\ &\leq k \|\varphi^t - \varphi^0\|, \end{aligned}$$

for some $k > 0$ where $\|x\|_0 \leq k\|x\|$, for all $x \in C - C$. From (ii), we have $\lim_{t \rightarrow 0} \|\widehat{\varphi^t} - \varphi^0\| = 0$. Therefore there exist $\alpha > 0$ and $\beta > 1$ such that $\|\widehat{\varphi^t}\| \leq \beta$, for all $t \in [0, \alpha]$. According to the equality $\varphi^{t+s} = \varphi^t \circ \varphi^s$ and Corollary 1 in [5] we have

$$\|\varphi^{t+s}\| \leq \|\varphi^t\| \|\varphi^s\|.$$

Putting $t = r \cdot \alpha + \delta$, where $0 \leq \delta < \alpha$ and r is a nonnegative integer we obtain

$$\begin{aligned} \|\widehat{\varphi^t}\| &= \sup\{\|\sum_{i=1}^n \lambda_i \varphi^{r\alpha+\delta}(e_i)\| : x = \sum_{i=1}^n \lambda_i e_i, \|x\| = 1\} \\ &\leq \sup\{\sum_{i=1}^n \lambda_i \|\varphi^{r\alpha+\delta}(e_i)\| : x = \sum_{i=1}^n \lambda_i e_i, \|x\| = 1\} \\ &\leq k \|\varphi^{r\alpha+\delta}\|. \end{aligned}$$

Therefore for $t \geq 0$, we have (note $t = r\alpha + \delta$ and so $r < \frac{t}{\alpha}$)

$$\begin{aligned} \|\widehat{\varphi^t}\| &\leq k \|\varphi^\alpha\|^r \|\varphi^\delta\| \\ &\leq k \beta^{r+1} \\ &= k \beta \beta^r \\ &\leq \beta_0 e^{\gamma t}, \end{aligned}$$

where $\gamma := \frac{1}{\alpha} \ln \beta$ and $\beta_0 = k\beta$.

Now we will show that (3.4) can be established by an argument similar to that in the proof of Theorem 1 in [3]. Let $w > 0$ and B be a bounded set. By Lemma 3.3, there exists $\rho > 1$ such that for each $x \in B$ and $s \geq 0$

$$\begin{aligned} \mathfrak{h}(\widehat{\varphi^{s+w}}(x), \widehat{\varphi^w}(x)) &\leq \rho \|\widehat{\varphi^w}\| \|\widehat{\varphi^s} - \varphi^0\| \|x\| \\ &\leq \rho \beta_0 e^{\gamma w} \|\widehat{\varphi^s} - \varphi^0\| \|x\| \\ &\leq \rho \beta_0 e^{\gamma w} \|\widehat{\varphi^s} - \varphi^0\| \|B\|. \end{aligned}$$

On the other hand for every $x \in B$ and $w \geq s \geq 0$,

$$\begin{aligned} \mathfrak{h}(\widehat{\varphi^w}(x), \widehat{\varphi^{w-s}}(x)) &\leq \rho \|\widehat{\varphi^{w-s}}\| \|\widehat{\varphi^s} - \varphi^0\| \|x\| \\ &\leq \rho \beta_0 e^{\gamma(w-s)} \|\widehat{\varphi^s} - \varphi^0\| \|x\| \\ &\leq \rho \beta_0 e^{\gamma w} \|\widehat{\varphi^s} - \varphi^0\| \|B\|. \end{aligned}$$

Now, by (ii), statement (3.4) holds. Finally for every $x \in C$, putting $B = \{x\}$, we get $\lim_{s \rightarrow w} \mathfrak{h}(\widehat{\varphi^s}(x), \widehat{\varphi^w}(x)) = 0$ and $\{\widehat{\varphi^t} : t \geq 0\}$ is continuous. \square

Example 2. Let C be a cone with a finite basis. Then for the iteration semigroup $\{\varphi^t : t \geq 0\}$ of sublinear correspondences $\varphi^t : C \rightarrow cc(C)$ given by

$$\varphi^t(x) = [e^{\frac{t}{2}}, e^t]x \quad (x \in C),$$

we have

$$\|\varphi^t\| \leq e^t,$$

and

$$\lim_{s \rightarrow w} \mathfrak{h}(\varphi^s(x), \varphi^w(x)) = 0,$$

that is the given family and therefore the family of their linear extensions are continuous.

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