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OPTIMAL ASSET CONTROL OF THE DIFFUSION MODEL UNDER CONSIDERATION OF THE TIME VALUE OF RUIN

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ABSTRACT. In this paper, we consider the optimal asset control of a financial company which can control its liquid reserves by paying dividends and by issuing new equity. We assume that the liquid surplus of the company in the absence of control is modeled by the diffusion model. It is a hot topic to maximize the expected present value of dividends payout minus equity issuance until the time of bankruptcy. We study this problem under consideration of the time value of ruin. By constructing two categories of suboptimal models, one with classical model without equity issuance, and the other which never goes bankrupt by equity issuance, the optimal problem is addressed. At the end, some numerical examples and interesting economic interpretations are presented.

Keywords: Optimal dividend control, optimal financing control, time value of ruin.

MSC(2010): Primary: 49J20; Secondary: 49J30, 60G40.

1. Introduction

Optimizing dividends payout is a classical problem starting from the early work of Borch [2,3] and Gerber [5]. Diffusion models for companies that can control their risk exposure by means of dividends have attracted significant interest recently. We refer the readers to Asmussen et al. [1], Paulsen and Gjessing [11], Højgaard and Taksar [8,9], Radner and Shepp [12] and the references therein. However, one often raised disadvantage of the optimal dividend strategies is the fact that such a strategy does

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not take the lifetime of the controlled process into account. Scholars have paid much more attentions to this problem in recent years. Thonhauser and Albrecher [14] considered the time value of ruin when the company can control its exposure only by paying dividends.

In the real financial market, equity issuance is an important approach for the company to raise capitals and reduce risk. Sethi and Taksar [13] considered the model for the company that can control its exposure by issuing new equity and by paying dividends. For more information, we refer the readers to Løkka and Zervos [10], He and Liang [6, 7], Yao and their coauthors [15, 16] and the references therein.

In this paper, we study the optimal asset control of the diffusion model. Similar to Thonhauser and Albrecher [14], a new component that penalizes early ruin of the controlled risk process is added to the objective function. In particular, this additional term can be interpreted as a continuous payment of a (discounted) constant intensity during the lifetime of the controlled process.

The rest of this paper is organized as follows. In Section 2, the diffusion model is shortly discussed, and two categories of suboptimal models are constructed. In Section 3, we solve the control problem without equity issuance. In Section 4, we solve the control problem that arises when the admissible strategies are constrained to allow for no bankruptcy. In Section 5, we solve the general control problem. In Section 6, some numerical examples are presented and some economic interpretations are discussed.

2. Mathematical model

We start with a complete probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ endowed with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ and a standard Brownian motion $B = \{B_t\}_{t \geq 0}$ adapted to that filtration. The liquid surplus of the company evolves according to the following equation

$$R_t = x + \mu t + \sigma B_t,$$

where $x \geq 0$ is the initial capital, $\mu > 0$ and $\sigma > 0$. We denote by L_t the cumulative amount of the dividends paid from time 0 up to time t , and by G_t the total amount raised by issuing equity from time 0 up to time t . We assume that both $L = \{L_t\}_{t \geq 0}$ and $G = \{G_t\}_{t \geq 0}$ are non-decreasing, (\mathcal{F}_t) -adapted and right-continuous with left limits.

A control strategy π is described by the stochastic process $\{L^\pi, G^\pi\}$. Given a control strategy π , we assume that the liquid surplus of the

company is modeled by

$$dR_t^\pi = \mu dt + \sigma dB_t - dL_t^\pi + dG_t^\pi.$$

Let Π denote the set of all admissible strategies. For each $\pi \in \Pi$, we define the time of bankruptcy by

$$\tau^\pi = \inf \left\{ t \geq 0 : R_t^\pi < 0 \right\}.$$

Since the company is allowed to issue new equity, the time of bankruptcy could be infinite.

Inspired by Sethi and Taksar [13], Løkka and Zervos [10] and He and Liang [6], we also consider the proportional transaction costs in our model. If the company pays l as the dividends, then the shareholders can get $\beta_1 l$, for some $\beta_1 \in (0, 1)$. In the meanwhile, the shareholders must pay out $\beta_2 g$, for some $\beta_2 > 1$ to meet the cost of getting the amount of g by issuing new equity.

The management of the company should maximize the performance index

$$V(x, \pi) = \mathbb{E} \left[\int_0^{\tau^\pi} e^{-rt} \beta_1 dL_t^\pi - \int_0^{\tau^\pi} e^{-rt} \beta_2 dG_t^\pi + \int_0^{\tau^\pi} e^{-rt} \Lambda dt \right],$$

where r denotes the discounted rate and $\Lambda > 0$. The additional term $e^{-rt} \Lambda$ can be interpreted as the present value of an amount which the shareholders can earn as long as the company is alive. In this way, the lifetime of the portfolio becomes a part of the value function and is weighted according to the choice of Λ .

Definition 2.1. *Given an initial capital $x \geq 0$, we define the value function $V(x)$ by*

$$(2.1) \quad V(x) = \sup_{\pi \in \Pi} V(x, \pi).$$

In this paper, we mainly aim at finding the value function $V(x)$ and an optimal strategy. In order to reach our aim, we need to consider two categories of suboptimal problems, each corresponding to the maximization of the performance index $V(x, \pi)$ over a subset of Π .

Definition 2.2. *Let $\Pi_p = \{ \pi_p = (L^{\pi_p}, G^{\pi_p}) \in \Pi : G_t^{\pi_p} = 0 \text{ for all } t \geq 0 \} \subset \Pi$. Given an initial capital $x \geq 0$, we define the value function $V_p(x)$ by*

$$(2.2) \quad V_p(x) = \sup_{\pi_p \in \Pi_p} V(x, \pi_p).$$

Definition 2.3. Let $\Pi_s = \{\pi_s = (L^{\pi_s}, G^{\pi_s}) \in \Pi : R_t^{\pi_s} \geq 0 \text{ for all } t \geq 0\} \subset \Pi$. Given an initial capital $x \geq 0$, we define the value function $V_s(x)$ by

$$(2.3) \quad V_s(x) = \sup_{\pi_s \in \Pi_s} V(x, \pi_s).$$

Remark 2.4. Since $\pi_p, \pi_s \in \Pi$, $V(x) \geq \max\{V_p(x), V_s(x)\}$ for all $x \geq 0$.

At the end of this section, we introduce a technical tool. The analysis of the control problems that we consider below involves the following ordinary differential equation

$$(2.4) \quad \frac{1}{2}\sigma^2 g''(x) + \mu g'(x) - rg(x) + \Lambda = 0.$$

The general solution to the equation (2.4) is given by

$$(2.5) \quad g(x) = c_1 e^{k_1 x} + c_2 e^{k_2 x} + \frac{\Lambda}{r},$$

where $c_1, c_2 \in \mathbb{R}$ are constants, and the real numbers k_1, k_2 are given by

$$\begin{aligned} k_1 &= -\frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} > 0, \\ k_2 &= -\frac{\mu}{\sigma^2} - \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} < 0. \end{aligned}$$

Remark 2.5. For convenience, let us define the operator \mathcal{A} by

$$\mathcal{A}[q(x)] = \frac{1}{2}\sigma^2 q''(x) + \mu q'(x) - rq(x)$$

for each function $q(x) \in \mathcal{C}^2[0, \infty)$.

3. The case without equity issuance

In this section, we address the problem that arises in the context of Definition 2.2. Our aim is to find the value function $V_p(x)$ and an optimal strategy π^* . Similar to He and Liang [6, 7] and Løkka and Zervos [10], we try to construct a twice continuously differentiable concave solution to this problem. We call such a solution as a classical solution to the optimal control problem (2.2).

It follows from the standard optimal control theory (see Fleming and Soner [4]) and Højgaard and Taksar [8] that the Hamilton-Jacobi-Bellman (HJB) equation associated with this problem is given by

$$(3.1) \quad \max \left\{ \frac{1}{2} \sigma^2 W''(x) + \mu W'(x) - rW(x) + \Lambda, \beta_1 - W'(x) \right\} = 0$$

with the boundary condition

$$(3.2) \quad W(0) = 0.$$

Since the reserve $x = 0$ corresponds to the bankruptcy, the boundary condition arises naturally.

The value function $V_p(x)$ should identify with a solution $W(x)$ to the HJB equation (3.1) satisfying

$$(3.3) \quad \frac{1}{2} \sigma^2 W''(x) + \mu W'(x) - rW(x) + \Lambda = 0, \quad 0 < x \leq b^*,$$

$$(3.4) \quad \beta_1 - W'(x) = 0, \quad x \geq b^*$$

for some constant $b^* > 0$. From (2.4) and (2.5), we would consider a solution to the equations (3.3) and (3.4) of the form

$$(3.5) \quad W(x) = \begin{cases} c_1 e^{k_1 x} + c_2 e^{k_2 x} + \frac{\Lambda}{r}, & 0 < x \leq b^*, \\ \beta_1(x - b^*) + c_1 e^{k_1 b^*} + c_2 e^{k_2 b^*} + \frac{\Lambda}{r}, & x \geq b^*. \end{cases}$$

Next, we evaluate c_1, c_2 and b^* . Our aim is to find a classical solution, so we need

$$(3.6) \quad c_1 k_1 e^{k_1 b^*} + c_2 k_2 e^{k_2 b^*} = \beta_1,$$

$$(3.7) \quad c_1 k_1^2 e^{k_1 b^*} + c_2 k_2^2 e^{k_2 b^*} = 0.$$

From (3.5), (3.6) and (3.7), we have

$$(3.8) \quad \begin{cases} c_1(b^*) = -\frac{\beta_1 k_2}{k_1(k_1 - k_2)} e^{-k_1 b^*} > 0, \\ c_2(b^*) = \frac{\beta_1 k_1}{k_2(k_1 - k_2)} e^{-k_2 b^*} < 0. \end{cases}$$

Since $W(0) = 0$,

$$(3.9) \quad \frac{\Lambda}{r} + c_1(b^*) + c_2(b^*) = 0.$$

Lemma 3.1. *There exists a unique solution $b^* > 0$ to the equation (3.9). The function $W(x)$ given by (3.5) with b^* being the unique solution to (3.9) and with c_1, c_2 being given by (3.8) is concave on $[0, \infty)$ and satisfies the HJB equation (3.1) and the boundary condition (3.2).*

Proof. For $b \geq 0$, define

$$(3.10) \quad C(b) = \frac{\Lambda}{r} + c_1(b) + c_2(b).$$

Hence,

$$\begin{aligned} C'(b) &= \frac{\beta_1 k_2}{k_1 - k_2} e^{-k_1 b} - \frac{\beta_1 k_1}{k_1 - k_2} e^{-k_2 b} < 0, \\ C(0) &= \frac{\Lambda}{r} + \frac{\beta_1 k_1}{k_2(k_1 - k_2)} - \frac{\beta_1 k_2}{k_1(k_1 - k_2)} > 0. \end{aligned}$$

Therefore, $C(b)$ is strictly decreasing on $[0, +\infty)$. Since $C(0) > 0$ and $\lim_{b \rightarrow +\infty} C(b) = -\infty$, (3.9) has a unique positive root $b^* > 0$.

Since

$$W''(b^*) = 0$$

and for $x < b^*$,

$$W'''(x) = c_1 k_1^3 e^{k_1 x} + c_2 k_2^3 e^{k_2 x} > 0,$$

we can obtain that $W''(x) < 0$ for all $x \in [0, b^*)$. Thus $W''(x) \leq 0$ for all $x \geq 0$. Therefore $W(x)$ is concave on $[0, \infty)$.

It follows from (3.9) that $W(x)$ satisfies the boundary condition (3.2).

The problem remained is to prove that $W(x)$ satisfies the HJB equation (3.1). Noting (3.5), we only need to prove the following:

$$\begin{aligned} W'(x) &\geq \beta_1, \quad x \in [0, b^*], \\ \mathcal{A}[W(x)] + \Lambda &\leq 0, \quad x \geq b^*. \end{aligned}$$

The proof is as follows. The concavity of $W(x)$ implies that $W'(x)$ is decreasing. From (3.6), we get that for any $x \in [0, b^*]$, $W'(x) \geq W'(b^*) = \beta_1$. Moreover, for $x \geq b^*$,

$$\begin{aligned} \mathcal{A}[W(x)] + \Lambda &= \mu\beta_1 - r\beta_1(x - b^*) - rW(b^*) \\ &\leq \mu\beta_1 - rW(b^*) \\ &= \lim_{x \downarrow b^*} \mathcal{A}[W(x)] + \Lambda = \lim_{x \uparrow b^*} \mathcal{A}[W(x)] + \Lambda = 0. \end{aligned}$$

So $W(x)$ satisfies the HJB equation (3.1). The proof is completed. \square

Theorem 3.2. *The value function V_p identifies with the concave solution $W(x)$ given by (3.5) to the HJB equation (3.1). Moreover, if an*

admissible strategy $\pi^* = (L^{\pi^*}, 0)$ satisfies

$$(3.11) \quad \begin{cases} R_t^{\pi^*} = x + \mu t + \sigma B_t - L_t^{\pi^*}, \\ R_t^{\pi^*} \leq b^*, \\ \int_0^\infty I(R_t^{\pi^*} < b^*) dL_t^{\pi^*} = 0, \end{cases} \quad t \geq 0,$$

where $I(\cdot)$ is an indicator function, then π^* is an optimal strategy, i.e., $V_p(x) = V(x, \pi^*)$.

The proof of Theorem 3.2 can be developed by a straightforward modification of the proof of Theorem 5.3 below, so we omit it.

Remark 3.3. We can see that without equity issuance the optimal strategy is a barrier strategy. That is, any surplus above the level b^* would be paid as dividends to the shareholders of the company.

Remark 3.4. Thonhauser and Albrecher [14] studied the same problem with a different method.

4. The case that the company never goes bankrupt

In this section we address the problem that arises in the context of Definition 2.3. In this case, the bankruptcy is prohibited. Thereby, the surplus of the company has to stay nonnegative all the time.

With reference to the optimal control theory, the associated HJB equation takes the form

$$(4.1) \quad \max \left\{ \frac{1}{2} \sigma^2 H''(x) + \mu H'(x) - rH(x) + \Lambda, \beta_1 - H'(x), H'(x) - \beta_2 \right\} = 0.$$

We now construct a classical solution $H(x)$ to the HJB equation (4.1). It is clear that, because of the time value of money, it can not be optimal to issue new equity before it is really necessary. We conclude that it is optimal to postpone the new equity issuance as long as possible. This strategy is associated with a solution to the HJB equation (4.1). It should be characterized by

$$\begin{aligned} H'(0) &= \lim_{x \downarrow 0} H(x) = \beta_2, \\ \frac{1}{2} \sigma^2 H''(x) + \mu H'(x) - rH(x) + \Lambda &= 0, \quad 0 < x < b^{**}, \\ H'(x) &= \beta_1, \quad x \geq b^{**}. \end{aligned}$$

By (2.4) and (2.5), we would conjecture

$$(4.2) \quad H(x) = \begin{cases} d_1 e^{k_1 x} + d_2 e^{k_2 x} + \frac{\Lambda}{r}, & 0 \leq x < b^{**}, \\ \beta_1(x - b^{**}) + d_1 e^{k_1 b^{**}} \\ \quad + d_2 e^{k_2 b^{**}} + \frac{\Lambda}{r}, & x \geq b^{**}. \end{cases}$$

In order to find the solution, we must determine the parameters d_1, d_2 and the free-boundary point b^{**} . Our aim is to find a \mathcal{C}^2 solution, so we need

$$(4.3) \quad d_1 k_1 e^{k_1 b^{**}} + d_2 k_2 e^{k_2 b^{**}} = \beta_1,$$

$$(4.4) \quad d_1 k_1^2 e^{k_1 b^{**}} + d_2 k_2^2 e^{k_2 b^{**}} = 0.$$

By using equations (4.3) and (4.4), we can express d_1 and d_2 as

$$(4.5) \quad \begin{cases} d_1(b^{**}) = -\frac{\beta_1 k_2}{k_1(k_1 - k_2)} e^{-k_1 b^{**}} > 0, \\ d_2(b^{**}) = \frac{\beta_1 k_1}{k_2(k_1 - k_2)} e^{-k_2 b^{**}} < 0. \end{cases}$$

Moreover,

$$(4.6) \quad d_1(b^{**})k_1 + d_2(b^{**})k_2 = \beta_2.$$

Lemma 4.1. *The equation (4.6) has a unique solution $b^{**} > 0$. The function $H(x)$ defined by (4.2) with b^{**} being the unique solution to (4.6) and with d_1, d_2 being given by (4.5) is concave on $[0, \infty)$, and satisfies the HJB equation (4.1).*

Proof. For $b \geq 0$, define

$$D(b) = d_1(b)k_1 + d_2(b)k_2.$$

We have

$$D'(b) = \frac{\beta_1 k_1 k_2}{(k_1 - k_2)} e^{-k_1 b} - \frac{\beta_1 k_2 k_1}{(k_1 - k_2)} e^{-k_2 b},$$

$$D'(0) = 0,$$

$$D''(b) = -\frac{\beta_1 k_1^2 k_2}{(k_1 - k_2)} e^{-k_1 b} + \frac{\beta_1 k_1 k_2^2}{(k_1 - k_2)} e^{-k_2 b} > 0.$$

Hence the function $D'(b)$ is strictly increasing on $[0, +\infty)$. Since $D'(0) = 0$, $D'(b) > 0$ for all $b > 0$. Then $D(b)$ is strictly increasing on $(0, +\infty)$. On the other hand, $D(0) = \beta_1 < \beta_2$ and $\lim_{b \rightarrow +\infty} D(b) = +\infty$. Since $D(b)$ is a continuous function on $[0, +\infty)$, we get that (4.6) has a unique positive solution $b^{**} > 0$.

Using the same method as in the proof of Lemma 3.1, we can show that $H(x)$ is concave and satisfies the HJB equation (4.1). \square

Theorem 4.2. *The value function $V_s(x)$ defined by (2.3) identifies with the concave solution $H(x)$ given by (4.2) to the HJB equation (4.1). Moreover, if an admissible strategy $\pi^{**} = (L^{\pi^{**}}, G^{\pi^{**}})$ satisfies*

$$(4.7) \quad \begin{cases} R_t^{\pi^{**}} = x + \mu t + \sigma B_t - L_t^{\pi^{**}} + G_t^{\pi^{**}}, \\ 0 \leq R_t^{\pi^{**}} \leq b^{**}, & t \geq 0, \\ \int_0^\infty \mathcal{I}(R_t^{\pi^{**}} \leq b^{**}) dL_t^{\pi^{**}} = 0, \\ \int_0^\infty \mathcal{I}(R_t^{\pi^{**}} \neq 0) dG_t^{\pi^{**}} = 0, \end{cases}$$

then π^{**} is an optimal strategy, i.e., $V_s(x) = V(x, \pi^{**}) = H(x)$.

Using the same method as in the proof of Theorem 5.3 below, we can easily prove Theorem 4.2. We omit the proof here.

5. The solution to the general problem

We now address the general problem of maximizing the expected discounted dividends payout minus the expected discounted equity issuance over all admissible strategies (see Definition 2.1). Our aim is to find the value function $V(x)$ and an optimal strategy. We first derive some properties of the value function $V(x)$.

The following lemma can be proved similar to Højgaard and Taksar [8], thus we omitted the proof here.

Lemma 5.1. *The function $V(x)$ given by (2.1) is a nonnegative concave function.*

Lemma 5.2. *The function $V(x)$ given by (2.1) satisfies*

$$V(x) \leq \frac{\mu + \Lambda}{r} + x$$

for every $x \in [0, \infty)$.

Proof. By the Itô formula, we have

$$e^{-r\tau^\pi} R_{\tau^\pi}^\pi = x - r \int_0^{\tau^\pi} e^{-rs} R_s^\pi ds + \int_0^{\tau^\pi} e^{-rs} dR_s^\pi.$$

Since $R_{\tau^\pi}^\pi = 0$ and $R_t^\pi \geq 0$ for all $t \leq \tau^\pi$,

$$-\mathbb{E} \left[\int_0^{\tau^\pi} e^{-rs} dR_s^\pi \right] = x - r \mathbb{E} \left[\int_0^{\tau^\pi} e^{-rs} R_s^\pi ds \right] \leq x.$$

Finally, we have

$$\mathbb{E} \left[\int_0^{\tau^\pi} e^{-rs} dR_s \right] = \mathbb{E} \left[\int_0^{\tau^\pi} e^{-rs} u ds \right] \leq \frac{u}{r}.$$

Thus, we get

$$\begin{aligned} V(x, \pi) &= \mathbb{E} \left[\int_0^{\tau^\pi} e^{-rt} \beta_1 dL_t^\pi - \int_0^{\tau^\pi} e^{-rt} \beta_2 dG_t^\pi + \int_0^{\tau^\pi} e^{-rt} \Lambda dt \right] \\ &< \mathbb{E} \left[\int_0^{\tau^\pi} e^{-rt} dL_t^\pi - \int_0^{\tau^\pi} e^{-rt} dG_t^\pi + \int_0^{\tau^\pi} e^{-rt} \Lambda dt \right] \\ &= \mathbb{E} \left[\int_0^{\tau^\pi} e^{-rs} dR_s - \int_0^{\tau^\pi} e^{-rs} dR_s^\pi + \int_0^{\tau^\pi} e^{-rt} \Lambda dt \right] \\ (5.1) \quad &\leq \frac{\mu + \Lambda}{r} + x. \end{aligned}$$

It follows from (2.1) and (5.1) that the lemma holds. \square

Theorem 5.3. *Fix any initial capital $x \geq 0$, and consider the problem of maximizing the performance index $V(x, \pi)$ over all admissible strategies.*

- (i) *If $b^* \leq b^{**}$, then $V(x) = W(x) = V_p(x)$. An optimal strategy is $\pi^* = \{L^{\pi^*}, 0\}$ which is given by (3.11).*
- (ii) *If $b^* \geq b^{**}$, then $V(x) = H(x) = V_s(x)$. An optimal strategy is $\pi^{**} = \{L^{\pi^{**}}, G^{\pi^{**}}\}$ which is given by (4.7).*

In order to prove Theorem 5.3, we need the following two lemmas.

Lemma 5.4. (i) *If $b^* \geq b^{**}$, then $H(0) \geq 0$.*
(ii) *If $b^* \leq b^{**}$, then $W'(x) \leq \beta_2$.*

Proof. We first prove the case (i). It follows from (3.8) and (4.5) that $c_1(x) = d_1(x)$ and $c_2(x) = d_2(x)$. Thus

$$\begin{aligned} H(0) &= d_1(b^{**}) + d_2(b^{**}) + \frac{\Lambda}{r} \\ &= c_1(b^{**}) + c_2(b^{**}) + \frac{\Lambda}{r} \\ &= C(b^{**}), \end{aligned}$$

where $C(b)$ is given by (3.10). Since $C(b)$ is strictly decreasing on $(0, +\infty)$ and $b^* \geq b^{**} > 0$,

$$(5.2) \quad \begin{aligned} H(0) = C(b^{**}) &= c_1(b^{**}) + c_2(b^{**}) + \frac{\Lambda}{r} \\ &\geq c_1(b^*) + c_2(b^*) + \frac{\Lambda}{r}. \end{aligned}$$

On the other hand, from (3.9) and Lemma 3.1,

$$(5.3) \quad W(0) = c_1(b^*) + c_2(b^*) + \frac{\Lambda}{r} = C(b^*) = 0.$$

From (5.2) and (5.3), we get $H(0) \geq 0$.

Next, we prove the case (ii). From Lemma 3.1, we get that $W(x)$ is concave. The concavity of $W(x)$ implies that for all $x \geq 0$,

$$(5.4) \quad W'(x) \leq \beta_2 \Leftrightarrow W'(0) = c_1(b^*)k_1 + c_2(b^*)k_2 \leq \beta_2.$$

Since $D(b)$ is non-decreasing on $[0, \infty)$ and $0 < b^* \leq b^{**}$,

$$(5.5) \quad \begin{aligned} \beta_2 = D(b^{**}) &= d_1(b^{**})k_1 + d_2(b^{**})k_2 \\ &\geq d_1(b^*)k_1 + d_2(b^*)k_2 \\ &= c_1(b^*)k_1 + c_2(b^*)k_2 \\ &= W'(0). \end{aligned}$$

It follows from (5.4) and (5.5) that $W'(x) \leq \beta_2$. □

Lemma 5.5. *If $Q(x)$ satisfies the HJB equation*

$$(5.6) \quad \begin{aligned} \max \left\{ \frac{1}{2}\sigma^2 Q''(x) + \mu Q'(x) - rQ(x) + \Lambda, \right. \\ \left. \beta_1 - Q'(x), Q'(x) - \beta_2 \right\} = 0, \end{aligned}$$

and

$$(5.7) \quad \max \left\{ -Q(0), Q'(0) - \beta_2 \right\} = 0,$$

then $Q(x) \geq V(x, \pi)$ for any admissible strategy π .

Proof. We choose any strategy π and let $\mathcal{D} = \{s : L_{s-}^\pi \neq L_s^\pi\}$ and $\mathcal{D}' = \{s : G_{s-}^\pi \neq G_s^\pi\}$. Moreover, let \hat{L}_t^π be the discontinuous part of L_t^π , and \tilde{L}_t^π be the continuous part of L_t^π . Similarly, \hat{G}_t^π and \tilde{G}_t^π stand for the discontinuous and continuous parts of G_t^π , respectively.

By the generalized Itô formula,

$$(5.8) \quad \begin{aligned} e^{-r(t \wedge \tau^\pi)} Q(R_{t \wedge \tau^\pi}^\pi) &= Q(x) + \int_0^{t \wedge \tau^\pi} e^{-rs} \mathcal{A}[Q(R_s^\pi)] ds + \\ &\sigma \int_0^{t \wedge \tau^\pi} e^{-rs} Q'(R_s^\pi) dB_s - \int_0^{t \wedge \tau^\pi} e^{-rs} Q'(R_s^\pi) d\tilde{L}_s^\pi + \\ &\int_0^{t \wedge \tau^\pi} e^{-rs} Q'(R_s^\pi) d\tilde{G}_s^\pi + \sum_{s \in \mathcal{D} \cup \mathcal{D}', s \leq t \wedge \tau^\pi} e^{-rs} [Q(R_s^\pi) - Q(R_{s-}^\pi)]. \end{aligned}$$

Since $\beta_1 \leq Q'(x) \leq \beta_2$, the third term on the right-hand side of (5.8) is a square integrable martingale. Taking expectations at both sides of (5.8) gives

$$\begin{aligned} &\mathbb{E} \left[e^{-r(t \wedge \tau^\pi)} Q(R_{t \wedge \tau^\pi}^\pi) \right] \\ &\leq Q(x) - \mathbb{E} \left[\int_0^{t \wedge \tau^\pi} e^{-rs} Q'(R_s^\pi) d\tilde{L}_s^\pi + \int_0^{t \wedge \tau^\pi} e^{-rs} Q'(R_s^\pi) d\tilde{G}_s^\pi \right] \\ &+ \mathbb{E} \left[\sum_{s \in \mathcal{D} \cup \mathcal{D}', s \leq t \wedge \tau^\pi} e^{-rs} [Q(R_s^\pi) - Q(R_{s-}^\pi)] \right. \\ &\quad \left. - \int_0^{t \wedge \tau^\pi} e^{-rs} \Lambda ds \right]. \end{aligned}$$

Since $\beta_1 \leq Q'(x) \leq \beta_2$,

$$Q(R_s^\pi) - Q(R_{s-}^\pi) \leq \beta_2(G_s^\pi - G_{s-}^\pi) - \beta_1(L_s^\pi - L_{s-}^\pi).$$

So

$$\begin{aligned} \mathbb{E} \left[e^{-r(t \wedge \tau^\pi)} Q(R_{t \wedge \tau^\pi}^\pi) \right. &+ \int_0^{t \wedge \tau^\pi} e^{-rs} \beta_1 dL_s^\pi - \int_0^{t \wedge \tau^\pi} e^{-rs} \beta_2 dG_s^\pi \\ &\left. + \int_0^{t \wedge \tau^\pi} e^{-rs} \Lambda ds \right] \leq Q(x). \end{aligned}$$

From Lemma 5.2, we have

$$\mathbb{E} \left[\int_0^{\tau^\pi} e^{-rs} \beta_1 dL_s^\pi - \int_0^{\tau^\pi} e^{-rs} \beta_2 dG_s^\pi + \int_0^{\tau^\pi} e^{-rs} \Lambda ds \right] \leq Q(x).$$

Therefore

$$V(x, \pi) \leq Q(x).$$

The proof has been done. \square

Next, we prove the main result of this paper.

Proof of Theorem 5.3: We first prove the case (i) of the theorem. Since $b^* \leq b^{**}$, we deduce from Lemmas 3.1 and 5.4 that $W(x)$ satisfies (5.6)

and (5.7). So $W(x) \geq V(x)$. On the other hand, we get from Remark 2.4 that $W(x) \leq V(x)$. Hence $W(x) = V(x)$.

Next, we show that $V(x, \pi^*) = W(x) = V(x)$, i.e., π^* is an optimal strategy. We deduce from Lemma 3.1 and Theorem 3.2 that for all $t \geq 0$,

$$\mathcal{A}\left[W(R_t^{\pi^*})\right] + \Lambda = 0.$$

By the generalized Itô formula, we have

$$\begin{aligned} e^{-r(t \wedge \tau^{\pi^*})} W(R_{t \wedge \tau^{\pi^*}}^{\pi^*}) &= W(x) + \int_0^{t \wedge \tau^{\pi^*}} e^{-rs} \mathcal{A}\left[W(R_s^{\pi^*})\right] ds - \\ &\int_0^{t \wedge \tau^{\pi^*}} e^{-rs} W'(R_s^{\pi^*}) d\tilde{L}_s^{\pi^*} + \sum_{s \in \mathcal{D}, s \leq t \wedge \tau^{\pi^*}} e^{-rs} \left[W(R_s^{\pi^*}) - W(R_{s-}^{\pi^*}) \right] \\ &+ \int_0^{t \wedge \tau^{\pi^*}} \sigma e^{-rs} W'(R_s^{\pi^*}) dB_s \\ &= W(x) - \int_0^{t \wedge \tau^{\pi^*}} \beta_1 e^{-rs} dL_s^{\pi^*} + \int_0^{t \wedge \tau^{\pi^*}} \sigma e^{-rs} W'(R_s^{\pi^*}) dB_s \\ (5.9) \quad &- \int_0^{t \wedge \tau^{\pi^*}} e^{-rs} \Lambda ds. \end{aligned}$$

It follows from (3.11) that $W(R_{t \wedge \tau^{\pi^*}}^{\pi^*})$ is bounded by $W(b^*)$. So

$$\liminf_{t \rightarrow \infty} e^{-r(t \wedge \tau^{\pi^*})} W(R_{t \wedge \tau^{\pi^*}}^{\pi^*}) = e^{-r\tau^{\pi^*}} W(0) = 0.$$

Taking expectations at both sides of (5.9) gives

$$W(x) = \mathbb{E} \left[\liminf_{t \rightarrow \infty} \int_0^{t \wedge \tau^{\pi^*}} e^{-rs} \beta_1 dL_s^{\pi^*} + \int_0^{t \wedge \tau^{\pi^*}} e^{-rs} \Lambda ds \right] = V(x, \pi^*).$$

So $W(x) = V(x) = V_p(x)$.

Next, we prove the case (ii) of the theorem. Using the same approach as in the case (i), we deduce from Lemmas 4.1 and 5.4 that $H(x)$ satisfies (5.6) and (5.7). So $H(x) \geq V(x)$. On the other hand, we get from Remark 2.4 that $H(x) \leq V(x)$. Hence $H(x) = V(x)$.

Next, we prove that π^{**} is an optimal strategy, i.e., $V(x, \pi^{**}) = H(x)$. The proof is as follows.

We deduce from Lemma 4.1 and Theorem 4.2 that for all $t \geq 0$,

$$\mathcal{A}\left[H(R_t^{\pi^{**}})\right] + \Lambda = 0.$$

By the generalized Itô formula,

$$\begin{aligned}
e^{-r(t \wedge \tau^{\pi^{**}})} H(R_{t \wedge \tau^{\pi^{**}}}^{\pi^{**}}) &= H(x) + \int_0^{t \wedge \tau^{\pi^{**}}} e^{-rs} \mathcal{A} \left[H(R_s^{\pi^{**}}) \right] ds \\
&\quad - \int_0^{t \wedge \tau^{\pi^{**}}} e^{-rs} H'(R_s^{\pi^{**}}) d\tilde{L}_s^{\pi^{**}} + \int_0^{t \wedge \tau^{\pi^{**}}} e^{-rs} H'(R_s^{\pi^{**}}) d\tilde{G}_s^{\pi^{**}} \\
&\quad + \sum_{s \in \mathcal{D}, s \leq t \wedge \tau^{\pi^{**}}} e^{-rs} \left[H(R_s^{\pi^{**}}) - H(R_{s-}^{\pi^{**}}) \right] \\
&\quad + \sum_{s \in \mathcal{D}', s \leq t \wedge \tau^{\pi^{**}}} e^{-rs} \left[H(R_s^{\pi^{**}}) - H(R_{s-}^{\pi^{**}}) \right] \\
&\quad + \int_0^{t \wedge \tau^{\pi^{**}}} \sigma e^{-rs} H'(R_s^{\pi^{**}}) dB_s \\
&= H(x) - \int_0^{t \wedge \tau^{\pi^{**}}} e^{-rs} \beta_1 dL_s^{\pi^{**}} + \int_0^{t \wedge \tau^{\pi^{**}}} e^{-rs} \beta_2 dG_s^{\pi^{**}} \\
&\quad + \int_0^{t \wedge \tau^{\pi^{**}}} \sigma e^{-rs} H'(R_s^{\pi^{**}}) dB_s \\
(5.10) \quad &\quad - \int_0^{t \wedge \tau^{\pi^{**}}} e^{-rs} \Lambda ds.
\end{aligned}$$

From (4.7), we obtain that $H(R_{t \wedge \tau^{\pi^{**}}}^{\pi^{**}})$ is bounded by $H(b^{**})$. Then,

$$\liminf_{t \rightarrow \infty} e^{-r(t \wedge \tau^{\pi^{**}})} H(R_{t \wedge \tau^{\pi^{**}}}^{\pi^{**}}) = 0.$$

Taking expectations at both sides of (5.10) yields

$$H(x) = V(x, \pi^{**}).$$

So $V(x) = H(x) = V_s(x)$. □

6. Numerical examples and economic interpretations

In this section, we present some numerical examples. Similar to Yao et al. [16], some interesting economic interpretations are also discussed.

6.1. The effect of β_1 . We first propose an economic interpretation of the parameter β_1 . In the real world, $1 - \beta_1$ can be thought as the tax rate. If the company pays l as the dividends, the shareholders can get $\beta_1 l$. The rest part $(1 - \beta_1)l$ can be regarded as the tax. When β_1 is large,

it implies that the shareholders can get the most part of the dividends, so they would like to pay dividends. However, when β_1 is small, the shareholders can only obtain a little part of the dividends. In such case, dividend payments may be delayed. In our model, we can delay dividend payments by increasing the dividend barrier.

TABLE 1. Effect of β_1

β_1	0.95	0.9	0.895	0.8	0.7
b^*	1.8673	1.9161	1.9213	2.0296	2.1698
b^{**}	1.7065	1.9022	1.9213	2.2829	2.6678
$\min\{b^*, b^{**}\}$	1.7065	1.9022	1.9213	2.0296	2.1698
π	π^{**}	π^{**}	π^{**} or π^*	π^*	π^*

In Table 1, we let $(r, \mu, \sigma, \beta_2, \Lambda) = (0.18, 0.16, 1.5, 1.2, 0.2)$ and β_1 vary. From Table 1, we can see that when $\beta_1 < 0.895$, the optimal strategy π coincides with the strategy π^* . When $\beta_1 > 0.895$, the optimal strategy π is the same as the strategy π^{**} . Moreover, both b^* and b^{**} increase, as β_1 decreases in our example.

6.2. The effect of β_2 . In Table 2, we let β_2 vary and $(r, \mu, \sigma, \beta_2, \Lambda) = (0.18, 0.16, 1.5, 0.8, 0.2)$. The parameter β_2 is a measure for the proportional costs arising from capital injections, for example, the bond issuance fee etc. From Section 3, we see that β_2 is independent of the dividend barrier b^* . Table 2 is an example of this conclusion. In Table 2, when β_2 is small, the company prefers to inject capitals whenever it is on the edge of bankruptcy. The corresponding optimal strategy π coincides with the strategy π^{**} . However, as β_2 becomes large, once it exceeds some critical level, $\beta_2 = 1.1076$ in Table 2, the shareholders will change their strategy and then choose the strategy π^* as the optimal strategy π . Such a decision is reasonable. Under the situation that β_2 is large, if the shareholders want to rescue the company by injecting capitals, then they need to inject much more capitals than the amount the company needs.

6.3. The effect of μ . In our model, μ describes the expected growth rate. The large value of μ means high profitability. So, when u is large enough, the shareholders may increase the dividend barrier to delay the bankruptcy of the company. On the other hand, the shareholders also need to consider the time value of the dividends, so they may want to pay dividends as early as possible. If so, the dividend barrier should

TABLE 2. Effect of β_2

β_2	1.3	1.2	1.1076	1.1	1.05
b^*	2.0296	2.0296	2.0296	2.0296	2.0296
b^{**}	2.5181	2.2829	2.0296	2.0068	1.8469
$\min\{b^*, b^{**}\}$	2.0296	2.0296	2.0296	2.0068	1.8469
π	π^*	π^*	π^{**} or π^*	π^{**}	π^{**}

be lowered. Two conflicting forces are at work at the same time. In Table 3, we assume that $(r, \sigma, \beta_1, \beta_2, \Lambda) = (0.18, 1.2, 0.8, 1.2, 0.2)$ and μ varies. We see that the force of u dominates in the strategy π^* . We can also get that as μ becomes large, the dividend barrier b^* increases in our example. Conversely, in the strategy π^{**} , the force of the discount rate r dominates. As μ increases, the dividend barrier b^{**} decreases.

TABLE 3. Effect of μ

μ	0.15	0.14	0.1302	0.13	0.12
b^*	1.8940	1.8594	1.8248	1.8240	1.7879
b^{**}	1.8104	1.8176	1.8248	1.8249	1.8322
$\min\{b^*, b^{**}\}$	1.8104	1.8176	1.8248	1.8240	1.7879
π	π^{**}	π^{**}	π^{**} or π^*	π^*	π^*

In Table 3, when $\mu > 0.1302$, the optimal strategy π is the same as the strategy π^{**} . When $\mu < 0.1302$, the corresponding optimal strategy π coincides with the strategy π^* .

6.4. The effect of σ . The parameter σ has mixed effect. σ can result in high profitability. If so, the shareholders would like to delay the bankruptcy. One way to reach this goal is to increase the dividend barrier. On the other hand, σ can also carry a high risk. Therefore, a lower dividend barrier may be appropriate. These two forces work at the same time.

In Table 4, we let $(r, \mu, \beta_1, \beta_2, \Lambda) = (0.18, 0.16, 0.8, 1.2, 0.2)$ and σ vary. In our example, as σ increases, the reserves of the company become more and more unstable. Hence, the manager has to increase the barrier level to delay the bankruptcy. If $\sigma > 1.3035$, the optimal strategy π is identical to the strategy π^* . If $\sigma < 1.3035$, the optimal strategy π coincides with the strategy π^{**} .

TABLE 4. Effect of σ

σ	1.5	1.4	1.3035	1.3	1.2
b^*	2.0296	2.0009	1.9686	1.9674	1.9279
b^{**}	2.2829	2.1230	1.9686	1.9631	1.8033
$\min\{b^*, b^{**}\}$	2.0296	2.0009	1.9686	1.9631	1.8033
π	π^*	π^*	π^{**} or π^*	π^{**}	π^{**}

6.5. The effect of Λ . The parameter Λ can be thought as the bonus that the shareholders can earn as long as the company runs. When Λ is large, the shareholders will try their best to prevent the bankruptcy of the company. There are two ways to delay the bankruptcy in our model, one is to increase the dividend barrier, the other is to inject capitals. The shareholders can choose one or both of them to prevent the bankruptcy of the company.

TABLE 5. Effect of Λ

Λ	0.19	0.18	0.1724	0.17	0.16
b^*	1.8836	1.8384	1.8033	1.7923	1.7454
b^{**}	1.8033	1.8033	1.8033	1.8033	1.8033
$\min\{b^*, b^{**}\}$	1.8033	1.8033	1.8033	1.7923	1.7454
π	π^{**}	π^{**}	π^{**} or π^*	π^*	π^*

In Table 5, we assume that $(r, \mu, \beta_1, \beta_2, \sigma) = (0.18, 0.16, 0.8, 1.2, 1.2)$ and Λ varies. When $\Lambda > 0.1724$, the optimal strategy π coincides with the strategy π^{**} . When $\Lambda < 0.1724$, the shareholders choose the strategy π^* as the optimal strategy π .

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