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THE CATEGORY OF MONOID ACTIONS IN CPO

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ABSTRACT. In this paper, some categorical properties of the category $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ of all cpo S-acts, cpo's equipped with actions of a monoid S on them, and strict continuous action-preserving maps between them is considered. In particular, we describe products and co-products in this category, and consider monomorphisms and epimorphisms. Also, we show that the forgetful functor from $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ to the category of cpo's has both a left and right adjoint.
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1. Introduction

The categories **Dcpo** (and **Cpo**) of directed complete partially ordered sets (with a bottom element) and (strict) continuous maps between them play an important role in domain theory and in theoretical computer science. It has been shown (see, for example [1, 9, 11]) that these categories are complete and co-complete.

The action of a monoid on a set is an important algebraic structure in mathematics as well as in computer science. For example, computer scientists use the notion of a projection algebra (sets with an action of the monoid $(\mathbb{N}^{\infty}, \min)$) as a convenient means of algebraic specification of process algebras (see [8]). Combining the notions of a poset and an act, many algebraic and categorical properties of the category of actions of a monoid on a poset have been studied (see for example [2, 3, 7]).

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In this paper, considering actions of a monoid S on a set, namely S-acts, as unary algebraic structures, are investigated as algebras in the category **Cpo**. We describe products, and co-products in this category and show that they are both complete and co-complete. We find free and co-free objects over cpo's, and show that this category is not cartesian closed. Also, monomorphisms and epimorphisms in this category are considered.

2. **Preliminaries**

In this section, we briefly recall the preliminary notions about the actions of a monoid on a set and on a poset. For more information, see [3, 12].

2.1. **The category Act-S.**

In this subsection, we briefly recall the preliminary notions about the action of a monoid on a set. For more information, see [6, 12].

Definition 2.1. Let S be a monoid with 1 as its identity. A (right) *S-act* (also called *S-set, S-polygon, S-system, S-transition system, S-automata*) is a set A equipped with an *action* $\lambda : A \times S \to A$, ($\lambda(a, s)$ is denoted by as) such that $a1 = a$ and $a(st) = (as)t$, for all $a \in A$ and $s, t \in S$.

Remark 2.2. (a) Let A be an S-act with the action $A \times S \to A$. Then, we have the following right and left translations:
(*Right translation*) For each $s \in S$, $R_s : A \to A$, $R_s(a) = as$.
(*Left translation*) For each $a \in A$, $L_a : S \to A$, $L_a(t) = at$.

(b) Let S be a monoid, A a set, and $A \times S \to A$ a function. Then, A is an S-act if and only if $(A; (R_s)_{s\in S})$ is a *unary algebra* with $R_s \circ R_t = R_{ts}$ and $R_1 = \mathrm{id}_A$.

(c) Each monoid S can clearly be considered as an S-act with the action given by its binary operation $S \times S \to S$. Note that the unary algebra related to this S-act is $(S; (R_s)_{s\in S})$, where $R_s : S \to S$ is defined by $R_s(t) = ts$.

Definition 2.3. An *S-map* $f : A \to B$ between S-acts is an action-preserving map, that is $f(as) = f(a)s$ for each $a \in A, s \in S$. The category of all S-acts and S-maps between them is denoted by **Act**-S.

Definition 2.4. An element a of an S-act A is said to be a *zero* (or a *fixed*) element of A if $as = a$, for all $s \in S$.

Note that if S is a monoid with a zero element z, then for each S-act A and $a \in A$, az is a zero element of A.

In the following, we recall some categorical ingredients of **Act**-S needed in the sequel (for more information, see [6]).

Theorem 2.5. (1) *The product in the category of S-acts is the cartesian product with the component-wise action. In particular, the terminal S-act is the singleton S-act.*

(2) *The equalizer of a pair $f, g : A \to B$ of S-maps is given by $E = \{a \in A : f(a) = g(a)\}$ with the action inherited from A.*

(3) *The pullback of S-maps $f : A \to C$ and $g : B \to C$ is the sub S-act $P = \{(a, b) : f(a) = g(b)\}$ of $A \times B$.*

(4) *The co-product in* **Act**-S *is the disjoint union with the natural action. In particular, the initial S-act is the empty S-act.*

2.2. **The category Pos-S.** In this subsection we recall the definition and give some categorical ingredients of **Pos**-S needed in the sequel. For more information, see [3].

For a monoid S we have three functions: the binary operation $S \times S \to S$ and the right and the left translations $R_s, L_s : S \to S$, for each $s \in S$. Using these functions we have the following ordered monoids (borrowing some terms from topological semigroups).

Definition 2.6. Let S be a monoid with a partial order $\leq$. Then:

(a) $(S, \leq)$ is a *pomonoid* if the partial order $\leq$ is compatible with the monoid operation; that is, for $s, t, s', t' \in S$, $s \leq t$, $s' \leq t'$ imply $ss' \leq tt'$. In other words,

$$(s, s') \leq (t, t') \;\Rightarrow\; ss' \leq tt'.$$

(b) $(S, \leq)$ is a *semi pomonoid* (or *separately pomonoid*) if for each $s \in S$, both the right and the left translations $R_s, L_s : S \to S$ are order-preserving.

(c) $(S, \leq)$ is a *right* (*left*) *semi pomonoid* if for each $s \in S$, the right (left) translation R_s (L_s) is order-preserving.

Remark 2.7. Note that $(S, \leq)$ is a pomonoid if and only if it is a *semi pomonoid*. This remark also shows that a right or a left semi pomonoid is not necessarily a pomonoid.

Similar to Definition 2.6, one has the following.

Definition 2.8. Let S be a pomonoid, A an S-act, and $\leq$ a partial order on A. Then:

(a) $(A, \leq)$ is an *S-poset* if the action on A is order-preserving; that is,

$$(a, s) \leq (b, t) \quad \Rightarrow \quad as \leq bt.$$

(b) $(A, \leq)$ is a *semi S-poset* (or *separately S-poset*) if for each $s \in S$ and $a \in A$, both the right and the left translations $R_s : A \to A$, $L_a : S \to A$ are order-preserving.

(c) $(A, \leq)$ is a *weak semi S-poset* if for each $s \in S$, the right translation $R_s : A \to A$ is order-preserving.

Remark 2.9. (a) Let S be a pomonoid, A an S-act, and $\leq$ a partial order on A. Then A is an S-poset if and only if it is a semi S-poset.

(b) Let S be a pomonoid, A an S-act, and $\leq$ a partial order on A. Then, A is a semi S-poset if and only if $(A; (R_s)_{s \in S})$ is a *unary algebra* in the category **Pos** of posets, satisfying $R_s \circ R_t = R_{ts}$ and $R_1 = \mathrm{id}_A$.

(c) Notice that if $(S, \leq)$ is a left or right semi pomonoid or $(A, \leq)$ has some extra properties, then the above remarks may change accordingly. For example, as in this paper, $\leq$ may be the equality on S and each S-poset A may have the smallest element $\perp_A$, in which case the action may preserve the smallest element $(\perp_A, \perp_S)$ but the right translations may not preserve $\perp_S$, and vice versa (see Remark 3.2).

Definition 2.10. An *S-poset map* $f : P \to Q$ between S-posets is an action-preserving monotone map. The category of all S-posets with action-preserving monotone maps between them is denoted by **Pos**-S.

Recall from [3] that products, terminal object, equalizers, pullbacks, and co-products of S-posets are as in **Act**-S with the obvious order.

3. Acts in Cpo

In the following, we first recall the category **Cpo** of cpo's (see [11]) and then introduce the category of acts in **Cpo**.

Recall that a nonempty subset D of a partially ordered set P is called *directed*, denoted by $D \subseteq^d P$, if for every $a, b \in D$ there exists $c \in D$ such that $a, b \leq c$, and P is called a *directed complete poset*, or briefly a *dcpo*, if for every $D \subseteq^d P$, the directed join $\bigsqcup D$ exists in P. A dcpo P which has a bottom element $\perp_P$ is said to be a *cpo*.

A *continuous map* $f : P \to Q$ between dcpo's is a map with the property that for every directed subset D of P, the subset $f(D)$ of Q is directed and $f(\bigsqcup D) = \bigsqcup f(D)$. By a *cpo map* between cpo's, we mean a continuous map which is *strict* ; that is, it preserves the bottom element. We denote the category of all cpo's with cpo maps between them by **Cpo**.

Recall from [1] that the product of a family of cpo's is their cartesian product, with component-wise order and ordinary projection maps. In particular, the terminal object of **Cpo** is the one element poset. Also, the co-product of a family of cpo's is their coalesced sum. Recall that *the coalesced sum* of the family $\{A_i : i \in I\}$ of cpo's is defined to be

$$\biguplus_{i\in I} A_i = \bot \oplus \dot{\bigcup}_{i\in I}(A_i \setminus \{\bot_{A_i}\})$$

In particular, the initial object of **Cpo** is the singleton poset $\{\theta\}$.

Now, by Remark 2.2, an S-act can be considered as a unary algebra in the category **Set** of sets. In the following, we consider S-acts in the category **Cpo**, the category which will be studied in this paper.

Definition 3.1. Let S be a monoid with identity 1. By a *cpo S-act* we mean an S-act in the category **Cpo**. In other words, a pair $(A; (R_s)_{s\in S})$ is called a *cpo S-act* if A is a cpo and each $R_s : A \to A, R_s(a) = as$, is a cpo map, called *an action*, such that for all $s, t \in S$, $R_s \circ R_t = R_{ts}, R_1 = id_A$. That is, $a(st) = (as)t, a1 = a$.

By a *cpo S-map* between cpo S-acts, we mean a cpo map which is also an S-map. The category of all cpo S-acts with cpo S-maps between them is denoted by $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$.

Remark 3.2. (1) First, note that in the definition of a cpo S-act, S need not be a pomonoid. Thus, S itself need not be a cpo S-act.

(2) Notice that a cpo A which is also an S-act whose action $\lambda : A \times S \to A$ is strict continuous, need not be a cpo S-act. For example, take $S = \{1, a\}$ with the identity 1, the order $1 < a$, and binary operation $aa = a$. Then, S is a cpo which is also an S-act whose action is its binary operation which is clearly strictly continuous. However, S is not a cpo S-act, because $a1 = a$ and so $R_a : S \to S$ is not strict.

(3) The converse of (2) need not be true as well. That is, the action of a cpo S-act is not necessarily continuous. For example, consider the monoid $S = (\mathbb{N} \cup \{\infty\}, \min)$ with the usual order of natural numbers, and $n < \infty$ for all $n \in \mathbb{N}$. Then, S is a cpo, and a monoid whose binary operation is strict continuous. In fact, for every subset X of S, we have $\bigsqcup X = \max X$ if X is a finite set and $\bigsqcup X = \infty$ if X is an infinite set. Now, take A to be the chain $\mathbf{2} = \{0, a\}$ with the right translations defined by $0n = 0$ for all $n \in \mathbb{N} \cup \{\infty\}$ and $a\infty = a$ and $an = 0$ for all $n \in \mathbb{N}$.

Then, A is a cpo S-act but the action $\lambda : A \times S \to A$ is not continuous. Notice that the right translations $R_s : A \to A$ are strict (by definition), and are continuous since for $n \in \mathbb{N}$, $(\bigsqcup\{0, a\})n = an = 0 = \bigsqcup\{0n, an\}$, and $(\bigvee\{0, a\})\infty = a\infty = a = \bigsqcup\{0\infty, a\infty\}$. But the action $A \times S \to A$ is not continuous, since $a(\bigsqcup \mathbb{N}) = a\infty = a$ and $\bigsqcup_{n\in\mathbb{N}} an = \bigsqcup\{0\} = 0$.

(4) If we define the notion of a dcpo S-act by replacing cpo by dcpo in Definition 3.1, then the above Remarks (2) and (3) do not hold. This is because a map $g : A \times B \to C$ is continuous if and only if it is separately continuous in each of its component (that is the induced maps $g_b : A \to C$, and $g_a : B \to C$ are continuous), and this gives that an action is continuous if and only if the right and the left translations are continuous.

Notice that both the *terminal and the initial* objects in the category $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ are the one element cpo S-act. In the following, we consider products and co-products in the category $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$. To describe the products of cpo S-acts, we need the following lemma.

Lemma 3.3. *[5, 11] Let $\{A_i : i \in I\}$ be a family of dcpo's. Then, the directed join of a directed subset $D \subseteq^d \prod_{i\in I} A_i$ is calculated as $\bigsqcup D = (\bigsqcup D_i)_{i\in I}$ where*

$$D_i = \{a \in A_i : \exists d = (d_k)_{k\in I} \in D, a = d_i\}$$

for all $i \in I$.

Theorem 3.4. *The product of a family of cpo S-acts is their cartesian product with componentwise actions and order.*

Proof. Let $\{A_i : i \in I\}$ be a family of cpo S-acts, and $A =: \prod_{i\in I} A_i$. Then, we show that A with the right translations defined by $(a_i)_{i\in I}s = (a_is)_{i\in I}$ is a cpo S-act. Recall from preliminaries that A is a cpo and an S-act. Now, to see that A is a cpo S-act, it is enough to show that the right translations on A are continuous. Let $D \subseteq^d A$ and $s \in S$. Then we show that $(\bigsqcup D)s = \bigsqcup_{x\in D} xs$. By Lemma 3.3, $\bigsqcup D = (\bigsqcup D_i)_{i\in I}$, where $D_i = \{a \in A_i : \exists (d_k)_{k\in I} \in D,\ d_i = a\}$ is a directed subset of A_i, for all $i \in I$. Thus we have $(\bigsqcup D)s = (\bigsqcup D_i)_{i\in I}s = ((\bigsqcup D_i)s)_{i\in I} = (\bigsqcup D_is)_{i\in I}$, where the latter equality is true, because the action on each A_i is continuous. Now, we see that $(\bigsqcup D_is)_{i\in I} = \bigsqcup_{x\in D} xs$. Firstly, notice that $(\bigsqcup D_is)_{i\in I}$ is an upper bound of the set $\{xs : x \in D\}$, since for $x = (d_i)_{i\in I} \in D$, we have $d_i \in D_i$, for all $i \in I$, and so $xs = (d_is)_{i\in I} \leq ((\bigsqcup D_i)s)_{i\in I} = (\bigsqcup D_is)_{i\in I}$. Secondly, if $c = (c_i)_{i\in I}$ is any upper bound of the set $\{xs : x \in D\}$, then for $i \in I$ and $a \in D_i$, taking $x = (d_i)_{i\in I}$

with $d_i = a$, we have $as = d_i s \leq c_i$. Thus $(\bigsqcup D_i s)_{i\in I} \leq c$, as required. Also, since A with the ordinary projection maps $p_i : A \to A_i$ is the product in the category of S-acts and in the category of cpo's, we have that they are cpo S-act maps. The universal property of products in the category $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$, follows from the universal property of A as the product in the categories **Cpo** and **Act**-S. $\square$

Since the underlying set of the co-product of S-acts is different from that of cpo's, the description of the co-product in $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ is not as straight as that of products. To describe it, first we see the following lemmas.

Lemma 3.5. *Let A be a dcpo. Then $D \subseteq A_\perp = \perp \oplus A$ is directed if and only if $D = \{\perp\}$ or $D = \{\perp\} \cup D'$ or $D = D'$ where $D' \subseteq^d A$.*

Proof. It is clear that all of $D = \{\perp\}$, $D = D'$ and $D = \{\perp\} \cup D'$ where $D' \subseteq^d A$ are directed subsets of $A_\perp$. Conversely, let $D \subseteq^d A_\perp$. Then in the case that $\perp \notin D$, we have $D = D' \subseteq^d A$, and in the case where $\perp \in D$, we have $D = \{\perp\} \cup D'$ with $D' \subseteq^d A$, because for $x, y \in D'$, there exists $z \in D$ such that $x, y \leq z$, and this gives $z \neq \perp$. $\square$

Lemma 3.6. *The coalesced sum of a family of cpo S-acts is a cpo S-act.*

Proof. Let $\{A_i : i \in I\}$ be a family of cpo S-acts. It is known that the coalesced sum $A =: \biguplus_{i\in I} A_i$ is a cpo (see Proposition 3.2.8 of [1]). Define the action on A as

$$a.s = \begin{cases} as & \text{if} \quad as \neq \perp_{A_i} \\ \perp_A & \text{if} \quad as = \perp_{A_i} \end{cases}$$

for $a \in A_i$, $i \in I$, $s \in S$, and $\perp_A.s = \perp_A$. In particular, $\perp_A.1 = \perp_A$. Also, for $a \neq \perp_A$, we have $a.1 = a$. This is because, for some $i \in I$, $a \in A_i$, and so $a.1 = a1 = a$. Also, $a.(st) = (a.s).t$, for $a \in A$, $s, t \in S$. This is because, $\perp_A.(st) = (\perp_A.s).t$, by the definition. Also, for $a \neq \perp_A$, $a \in A_i$ for some $i \in I$. First, if $a(st) \neq \perp_{A_i}$, then $as \neq \perp_{A_i}$ (otherwise, since $\perp_{A_i}$ is a zero element, $a(st) = (as)t = \perp_{A_i} t = \perp_{A_i}$), so $(as).t = (as)t$. Also, $(as)t = a(st) \neq \perp_{A_i}$. Therefore, $(a.s).t = (as).t = (as)t = a(st) = a.(st)$. Secondly, if $a(st) = \perp_{A_i}$, then $a.(st) = \perp_A$. Now, if $as = \perp_{A_i}$ then $a.s = \perp_A$ and so $(a.s).t = \perp_A.t = \perp_A$. Also, if $as \neq \perp_{A_i}$ then $a.s = as$, and since $(as)t = a(st) = \perp_{A_i}$, $(a.s).t = \perp_A$. Thus $(a.s).t = (a.s).t = \perp_A$, as required.

Now, we show that the right translations are continuous. Let $D \subseteq^d A$ and $s \in S$. Then, by Lemma 3.5, $D \subseteq A$ is directed if and only if

$D \subseteq^d A_i$, for some $i \in I$, or $D = D' \cup \{\perp_A\}$, where $D' = \emptyset$ or D' is a directed subset of A_i, for some $i \in I$. Now, two cases may occur:

Case (i): $D \subseteq^d A_i$, for some $i \in I$.

Subcase (i1): If $(\bigsqcup D)s \neq \perp_{A_i}$, then we have $(\bigsqcup D).s = (\bigsqcup D)s = \bigsqcup_{x\in D} xs$, where the last equality is true because A_i is a cpo S-act. Now we claim that

$$\bigsqcup_{x\in D} xs = \bigsqcup_{x\in D} x.s \qquad (*)$$

Let $K = \{x \in D : xs \neq \perp_{A_i}\}$. Then we have

(1) $K \neq \emptyset$, because otherwise $(\bigsqcup D)s = \bigsqcup_{x\in D} xs = \perp_{A_i}$, which is a contradiction.

(2) For all $x \in K$, $x.s = xs$, by the definition of right translations on A.

(3) For all $x \in K$ and $x' \notin K$, there exists $x'' \in K$ with $x \leq x''$ and $x' \leq x''$. This is because D is directed. Then $xs \leq x''s$, and hence $x'' \in K$, since $x \in K$.

Now to prove $(*)$, first we see that $\bigsqcup_{x\in D} xs$ is an upper bound of the set $\{x.s : x \in D\}$. Also, for all $x \in K$, $x.s = xs \leq \bigsqcup_{x\in D} xs$. For $x \notin K$, $x.s = \perp_A \leq \bigsqcup_{x\in D} xs$, as required. Secondly, if c is an upper bound of the set $\{x.s : x \in D\}$, then for all $x \in K$ we have $x.s = xs \leq c$. For $x \notin K$ and $x' \in K$ (which exists, since $K \neq \emptyset$), there exists, by (3), $x'' \in K$ such that $x \leq x''$ and $x' \leq x''$. This gives $xs \leq x''s = x''.s \leq c$. Then, for all $x \in D$, we have $xs \leq c$, and so $\bigsqcup_{x\in D} xs \leq c$, as required.

Subcase (i2): If $(\bigsqcup D)s = \perp_{A_i}$, then again we have $(\bigsqcup D)s = \bigsqcup_{x\in D} xs$. This is because the action on A_i is continuous. Also, $(\bigsqcup D)s = \perp_{A_i}$ gives $xs = \perp_{A_i}$, for all $x \in D$. This is because $\perp_{A_i} = (\bigsqcup D)s = \bigsqcup_{x\in D} xs$. Hence, by the definition of the action on A, $(\bigsqcup D).s = \bigsqcup_{x\in D} x.s = \perp_A$.

Case (ii): $D = D' \cup \{\perp_A\}$, where $D' \subseteq^d A_i$, for some $i \in I$.

By case (i), we have $(\bigsqcup D').s = \bigsqcup_{x'\in D'} x'.s$. Also, we have $(\bigsqcup D).s = (\bigsqcup D').s = \bigsqcup_{x'\in D'} x'.s = \bigsqcup_{x'\in D'} x'.s \vee \perp_A = \bigsqcup_{x'\in D'} x'.s \vee \perp_A.s = \bigsqcup_{x\in D} x.s$, as required. Therefore, the right translations on A are strict continuous, and so $A = \biguplus_{i\in I} A_i$ is a cpo S-act. $\square$

Theorem 3.7. *The co-product of a family of cpo S-acts is their coalesced sum.*

Proof. Let $A = \biguplus_{i \in I} A_i$. By Proposition 3.6, A is a cpo S-act. It is known that the injections $u_i : A_i \to A$, $i \in I$, defined by

$$u_i(x) = \begin{cases} x & \text{if} \quad x \neq \perp_{A_i} \\ \perp_A & \text{if} \quad x = \perp_{A_i} \end{cases}$$

are cpo maps (see [1]). In addition, we show that $u_i : A_i \to A$, $i \in I$, are action-preserving. First, notice that $u_i(\perp_{A_i} s) = u_i(\perp_{A_i}) = \perp_A = \perp_A.s = u_i(\perp_{A_i}).s$. Now, let $\perp_{A_i} \neq x \in A_i$ and $s \in S$. If $xs = \perp_{A_i}$, then, by the definition of the action on A, $x.s = \perp_A$, and so $u_i(xs) = \perp_A = x.s = u_i(x).s$. If $xs \neq \perp_{A_i}$, then $x.s = xs$, and so $u_i(xs) = xs = x.s = u_i(x).s$. Moreover, for every cpo S-act B with cpo S-maps $f_i : A_i \to B$, $i \in I$, the unique cpo map $f : A \to B$ given by

$$f(a) = \begin{cases} f_i(a) & \text{if} \quad a \in A_i \\ \perp_B & \text{if} \quad a = \perp_A \end{cases}$$

which exists by the universal property of co-products in **Cpo**, and satisfies $f \circ u_i = f_i$ for all $i \in I$, is action-preserving. First notice that $f(\perp_A.s) = f(\perp_A) = \perp_B = \perp_B s = f(\perp_A)s$, for all $s \in S$. Now, let $a \neq \perp_A$. Then $a \in A_i$, for some $i \in I$. If $as = \perp_{A_i}$ then $a.s = \perp_A$, and so $f(a.s) = f(\perp_A) = \perp_B = f_i(\perp_{A_i}) = f_i(as) = f_i(a)s = f(a)s$. If $as \neq \perp_{A_i}$, then $a.s = as$, and so $f(a.s) = f(as) = f_i(as) = f_i(a)s = f(a)s$. $\square$

4. Adjoint relations for $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$

In this section, we show that the forgetful functor $U : \mathbf{Cpo}_{\mathbf{Act}\text{-}S} \to \mathbf{Cpo}$ has both a left and a right adjoint. In other words, we find the free and co-free cpo S-acts over a cpo.

Theorem 4.1. *Free cpo S-act over a cpo exists.*

Proof. For a given cpo P, we show that the free cpo S-act over P is $F = F(P) = (P \setminus \{\perp_P\} \times S)_\perp$ with the order defined by $(x, s) \leq (y, t) \Leftrightarrow x \leq y$ and $s = t$ and the right translations given by $(x, t).s = (x, ts)$ and $\perp.s = \perp$, for all $x \in P \setminus \{\perp_P\}$ and $s, t \in S$. Also, we see that the unit of the adjunction, namely the free map, is $\tau : P \to F$ given by $x \mapsto (x, 1)$ and $\perp_P \mapsto \perp$.

First, we see that $(P \setminus \{\perp_P\}) \times S$ being the product of dcpo's is a dcpo and so F is a cpo. Also, the right translations on it work since $(x, s).(tt') = (x, stt') = (x, st).t = ((x, s).t).t'$ and $(x, s).1 = (x, s1) = (x, s)$ for all $s, t \in S$ and $x \in P \setminus \{\perp_P\}$. Furthermore, the right translations are strict continuous. To see the continuity, let

$D \subseteq^d (P \setminus \{\bot_P\} \times S)_\bot$. Then, by Lemma 3.5, $D = D' \times \{t\}$ or $D = \{\bot\} \cup (D' \times \{t\})$ where $D' \subseteq^d P \setminus \{\bot_P\}$ and $t \in S$. Now $\bigsqcup D = (\bigsqcup D', t)$, and

$$\begin{aligned}(\bigsqcup D).s &= (\bigsqcup D', t).s = (\bigsqcup D', ts) = \bigsqcup_{x \in D'} (x, ts) \\ &= \bigsqcup_{x \in D'} ((x,t).s) = \bigsqcup D.s\end{aligned}$$

Therefore, F is a cpo S-act.

Secondly, we see that the defined map $\tau : P \to F$ is a universal strict continuous map. It is strict by its definition. For continuity, let $D \subseteq^d P$. Then

$$\tau(\bigsqcup D) = (\bigsqcup D, 1) = \bigsqcup_{x \in D} (x, 1) = \bigsqcup \tau(D).$$

To prove the universal property, let $f : P \to B$ be a strict continuous map to a cpo S-act B. Then the map $\overline{f} : (P \setminus \{\bot_P\} \times S)_\bot \to B$ defined by $\overline{f}(x,s) = f(x)s$ and $\overline{f}(\bot) = \bot_B$ is the unique cpo S-map with $\overline{f} \circ \tau = f$. It is strict by its definition. To prove continuity, let $D \subseteq^d (P \setminus \{\bot_P\} \times S)_\bot$. If $D = D' \times \{t\}$ where $D' \subseteq^d P \setminus \{\bot_P\}$, then

$$\begin{aligned}\overline{f}(\bigsqcup D) &= \overline{f}((\bigsqcup D', t)) = f(\bigsqcup D')t = (\bigsqcup_{x \in D'} f(x))t \\ &= \bigsqcup_{x \in D'} (f(x)t) = \bigsqcup_{x \in D'} \overline{f}((x,t)) = \bigsqcup \overline{f}(D)\end{aligned}$$

where the third equality is true because f is continuous and the fourth equality follows from the fact that B is a cpo S-act. If $D = \bot \oplus (D' \times \{t\})$ where $D' \subseteq^d P \setminus \{\bot_P\}$, then

$$\overline{f}(\bigsqcup D) = \overline{f}((\bigsqcup D', t)) = (f(\bigsqcup D'))t = (\bigsqcup f(D'))t = \bigsqcup_{x \in D'} (f(x)t)$$

$$= \bigsqcup_{x \in D'} \overline{f}(x,t) = \bigsqcup_{x \in D'} \overline{f}(x,t) \vee \bot_B = \bigsqcup_{x \in D'} \overline{f}(x,t) \vee \overline{f}(\bot) = \bigsqcup \overline{f}(D).$$

The proof of the fact that $\overline{f}$ is action-preserving, and is unique, is straightforward. □

Now, the assignment $P \mapsto F(P)$, which maps a cpo P to the free cpo S-act, defines a left adjoint to the forgetful functor, and the free map is the unit of the adjunction.

Corollary 4.2. *The forgetful functor* $U : \mathbf{Cpo}_{\mathbf{Act}\text{-}S} \to \mathbf{Cpo}$ *has a left adjoint.*

Theorem 4.3. *Co-free cpo* S*-act over a cpo exists.*

Proof. For a given cpo P, we show that the set $K = P^S$, of all maps from S to P with the pointwise order and the right translations given by $(fs)(t) = f(st)$, for $s, t \in S$ and $f \in P^S$ is the co-free cpo S-act over P. Also, we show that $\sigma : P^S \to P$, given by $\sigma(f) = f(1)$, is the co-unit of the adjunction, or in other words the co-free map.

Firstly, we show that P^S is a cpo S-act. It is clear that P^S is a cpo in which the supremum of each directed subset $D \subseteq P^S$ is given by $(\bigsqcup D)(s) = \bigsqcup_{f\in D} f(s)$ for all $s \in S$ and the bottom element is the constant mapping $f_\perp : S \to P$, $s \mapsto \perp_P$ for all $s \in S$. To see that the right translations defined above are strict continuous maps, let $D \subseteq^d P^S$ and $s \in S$. Then

$$((\bigsqcup D)s)(t) = (\bigsqcup D)(st) = \bigsqcup_{f\in D} f(st) = \bigsqcup_{f\in D} (fs)(t) = (\bigsqcup_{f\in D} fs)(t)$$

where the third and the last equalities hold because the supremum in P^S is calculated pointwise. Also $(f_\perp s)(t) = f_\perp(st) = \perp_P = f_\perp(t)$, for all $s, t \in S$, so the action is strict.

Secondly, we see that the co-free map $\sigma : P^S \to P$ is a universal strict continuous map. It is strict by its definition. To prove continuity, let $D \subseteq^d P^S$. Then

$$\sigma(\bigsqcup_{f\in D} f) = (\bigsqcup_{f\in D} f)(1) = \bigsqcup_{f\in D} f(1) = \bigsqcup_{f\in D} \sigma(f).$$

Finally, given a strict continuous map $g : A \to P$ from a cpo S-act A, it is routine to see that the map $\overline{g} : A \to P^S$, given by $\overline{g}(a)(s) = g(as)$, is the unique cpo S-map which satisfies $\sigma \circ \overline{g} = g$. □

Corollary 4.4. *The forgetful functor* $U : \mathbf{Cpo}_{\mathbf{Act}\text{-}S} \to \mathbf{Cpo}$ *has a right adjoint.*

Remark 4.5. (1) Notice that the forgetful functor $U : \mathbf{Cpo}_{\mathbf{Act}\text{-}S} \to \mathbf{Set}$ does not have a right adjoint. This is because it does not preserve the initial object. Recall that the initial object in $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ is the singleton cpo whereas in $\mathbf{Set}$ is the empty set.

(2) Notice that the free cpo S-act on a dcpo S-act A is proved directly to be $A_\perp$ with the right translations defined by:

$$a.s = \begin{cases} as & \text{if} \quad a \in A \\ \perp & \text{if} \quad a = \perp \end{cases}$$

for all $a \in A_\perp$ and $s \in S$.

Regarding the adjoint functors, we show that the product endofunctor $A \times -$ of cpo S-acts does not necessarily have a right adjoint; that is:

Theorem 4.6. *The category* $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ *is not cartesian closed.*

Proof. For a non trivial cpo S-act A, since $A \times \{\perp\} \ncong \{\perp\}$, the functor $A \times - : \mathbf{Cpo}_{\mathbf{Act}\text{-}S} \to \mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ does not preserve the initial object, and so it does not have a right adjoint. □

5. Monomorphisms and epimorphisms in $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$

In this section, we show that monomorphisms in $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ are exactly one-one morphisms while its counterpart is not true for epimorphisms. It is shown for morphisms whose images are Scott-closed, being epic is equivalent to being onto.

Remark 5.1. Recall from [4] that monomorphisms in the category **Dcpo** are exactly one-one morphisms. The same is true for the category **Cpo**. In fact, since the inclusion functor from **Cpo** to **Dcpo** has a left adjoint (see [1]), it preserves monomorphisms, and consequently monomorphisms in **Cpo** are monomorphisms in **Dcpo**, and hence they are one-one.

Theorem 5.2. *Monomorphisms in* $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ *are exactly one-one cpo* S*-maps.*

Proof. Having the adjunction given in 4.2, and the fact that right adjoints preserve limits, and in particular monomorphisms, we get that monomorphisms in $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ are exactly cpo S-maps which are monomorphisms in **Cpo**. Then the result follows by the above remark. □

In the following remark, we consider epimorphisms in $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$.

Remark 5.3. Epimorphisms in $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ are not necessarily onto. For example, take S to be an arbitrary monoid, $A = \perp \oplus \mathbb{N}$ in which the order on $\mathbb{N}$ is discrete, and $B = \perp \oplus \mathbb{N} \oplus \top$ in which the order on $\mathbb{N}$ is the usual order. Then, A and B with the identity right translations are cpo S-acts. Define the map $h : A \to B$ by $h(\perp) = \perp$ and $h(n) = n$ for all

$n \in \mathbb{N}$. We show that h is a cpo S-act epimorphism which is not onto. It clearly preserves the action. To see that h is (strict) continuous, let $D \subseteq^d A$. Then $D = \{\perp\}$, or there exists $n \in \mathbb{N}$ such that $D = \{\perp, n\}$ or $D = \{n\}$. For the case that $D = \{\perp, n\}$, $n \in \mathbb{N}$, we have

$$h(\bigsqcup D) = h(n) = n = \bigsqcup\{\perp, n\} = \bigsqcup\{h(\perp), h(n)\} = \bigsqcup h(D).$$

The other cases are clear. To show that h is an epimorphism, let $f_1, f_2 : B \to P$ be two cpo S-maps with $f_1 \circ h = f_2 \circ h$. Then, we have

$$f_1(\perp) = f_1(h(\perp)) = f_2(h(\perp)) = f_2(\perp),$$
$$f_1(n) = f_1(h(n)) = f_2(h(n)) = f_2(n)$$

for all $n \in \mathbb{N}$, and

$$f_1(\top) = f_1(\bigsqcup \mathbb{N}) = \bigsqcup_{n\in\mathbb{N}} f_1(n) = \bigsqcup_{n\in\mathbb{N}} f_2(n) = f(\bigsqcup \mathbb{N}) = f_2(\top).$$

Therefore, $f_1 = f_2$.

In the following, we show that epimorphisms $f : A \to B$ in $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ are onto if $f(A)$ is a Scott-closed subset of B. Recall that (see [1]), $C \subseteq B$ is Scott-closed if and only if $B \setminus C$ is an upper closed subset of B, and for $D \subseteq^d B$ if $\bigsqcup D \in (B \setminus C)$ then $D \cap (B \setminus C) \neq \emptyset$.

Lemma 5.4. *Let B be a cpo S-act, and I be a nonempty Scott-closed subact of B. Then*

(i) *$B^* = (B \setminus I) \cup \{\perp\}$ is a cpo S-act.*

(ii) *The mapping $g : B \to B^*$ defined by $g(x) = \perp$ for $x \in I$, $g(x) = x$ for $x \notin I$ is a cpo S-map.*

Proof. (i) First we show that B^* with the order defined by $b \leq b'$ if and only if $b \leq b'$ in B or $b = \perp$, is a cpo. We show that the supremum of every directed subset of B^* exists. For this, let $D \subseteq^d B^*$. Then, two cases may occur:

Case (1): $\perp \notin D$. In this case, $D \subseteq^d B$ and $\bigsqcup D$ exists in B. Also, $\bigsqcup D \notin I$ and so $\bigsqcup D \in B^*$. This is because, if $\bigsqcup D \in I$ then $D \subseteq I$ which is a contradiction.

Case (2): $\perp \in D$. In this case, $D' = D \setminus \{\perp\}$ is a directed subset of B, and so $\bigsqcup D = \bigsqcup D'$ exists in B^*. This is because, if $\bigsqcup D' \in I$ then $D' \subseteq I$ which is a contradiction.

Now, we define the right translations on B^* as

$$a.s = \begin{cases} as & \text{if} \quad as \notin I \\ \perp & \text{if} \quad as \in I \text{ or } a = \perp \end{cases}$$

for $a \in B^*$ and $s \in S$. We show that for $s, t \in S$ and $a \in B^*$, we have $a.(st) = (a.s).t$. If $a(st) \in I$, then $a.(st) = \perp$. Now, if $as \in I$, then $(a.s).t = \perp.t = \perp$; and if $as \notin I$, then $(a.s).t = (as).t = \perp$, since $(as)t = a(st) \in I$. Also, if $a(st) \notin I$, then $as \notin I$, since I is a subact, and so $a.(st) = a(st) = (as)t = (a.s).t$. The right translations are also continuous, since for $D \subseteq^d B^*$ and $s \in S$, $(\bigsqcup D).s = \bigsqcup_{y \in D} y.s$. This is because, if $\bigsqcup D = c$ and $cs \in I$, then $ys \in I$ for all $y \in D$. Thus, $c.s = \perp = \bigsqcup_{y \in D} y.s$, as required. Also, if $cs \notin I$, then $c.s = cs$ and cs is an upper bound of $y.s$ for all $y \in D$ (this is because $y.s = ys$ or $y.s = \perp$). Further, if $b \in B^*$ is an upper bound of the set $\{y.s : y \in D\}$, then taking $K = \{y \in D : ys \notin I\}$, we have $K \neq \emptyset$ (since otherwise $ys \in I$ for all $y \in D$, and so $\bigsqcup_{y \in D} ys \in I$ which gives the contradiction that $cs = (\bigsqcup D)s = \bigsqcup_{y \in D} ys \in I$). Therefore, there exists $y' \in K$. Now, since D is directed, for $y \notin K$ there exists $y_0 \in D$ such that $y, y' \leq y_0$. We show that $y_0 \in K$. To get this, we show $y_0 s \notin I$. On the contrary, let $y_0 s \in I$, then $ys \in I$ (since I is a down-set) which is a contradiction. $z.s = zs \leq b$ for all $z \in K$, and for all $y \notin K$ there exists $y_0 \in K$ such that $y \leq y_0$ and so $ys \leq y_0 s \leq b$. So b is an upper bound for the set of $\{ys : y \in D, y \neq \perp\}$, and $c.s = cs = \bigsqcup_{(y \in D \setminus \{\perp\})} ys \leq b$, as required. Hence B^* with the action and order defined above is a cpo S-act.

(ii) We see that g is strict by its definition. To show that g is continuous, let $D \subseteq^d B$. We consider two cases:

Case (1): $\bigsqcup D \in I$. In this case, $g(\bigsqcup D) = \perp = \bigsqcup g(D)$, where the last equality holds because $D \subseteq I$.

Case (2): $\bigsqcup D \notin I$. In this case, $g(\bigsqcup D) = \bigsqcup D$. Now, using the fact that for $y \in D$, $g(y) = y$ or $g(y) = \perp$, we have $\bigsqcup_{y \in D} g(y) = \bigsqcup D$, as required.

Finally, to prove that g is action-preserving, let $b \in B$ and $s \in S$. Then two cases may occur:

Case (a): $bs \in I$. In this case, $g(bs) = \perp$. If $b \notin I$, then $g(b).s = b.s = \perp$. Also if, $b \in I$, then $g(b).s = \perp.s = \perp$. Therefore $g(bs) = g(b).s$.

Case (b): $bs \notin I$. In this case, $b \notin I$ and $g(bs) = bs$. Also $g(b).s = b.s = bs$. Therefore $g(bs) = g(b).s$. □

Theorem 5.5. *Let $f : A \to B$ be an epimorphism in $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$. Then, f is onto if and only if its image is a Scott-closed subset of B.*

Proof. Let $f(A)$ be a Scott-closed subset of B. Consider the cpo S-act B^* with $I = f(A)$, and the cpo S-map $h : B \to B^*$ defined in Lemma 5.4. Also, consider the constant cpo S-map $k : B \to B^*$, $b \mapsto \perp$. Then

$h \circ f = k \circ f$. Since f is an epimorphism, $h = k$ and so $f(A) = B$, as required. The converse is clear. $\square$

6. Completeness and co-completeness of $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$

In this final section, applying the results of the above sections, we show that the category of cpo S-acts is both complete and co-complete.

Proposition 6.1. *The category* $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ *is complete.*

Proof. Applying Theorem 3.4, it is enough to show that equalizers exist in the category $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$. Let $f, g : A \to B$ be cpo S-maps. Then, from the fact that

$$E = \{x \in A : f(x) = g(x)\}$$

is their equalizer in both categories **Cpo** and **Act**-S, we get that it is the equalizer of f and g in the category of cpo S-acts. $\square$

In the following, we show that the category $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ is co-complete. To this end, applying Theorem 23.14 of [10], it is enough to show that $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ has a coseparator and is well-powered.

Recall that an object C of a category $\mathcal{C}$ is called a *coseparator* (or *cogenerator*) if the functor $\hom(-, C) : \mathcal{C}^{op} \to \mathbf{Set}$ is faithful; in other words, for each distinct arrows $f, g : A \to B$ there exists an arrow $h : B \to C$ such that $h \circ f \neq h \circ g$.

Lemma 6.2. *The category* $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ *has a coseparator.*

Proof. We show that for each cpo P with $|P| \geq 2$, the co-free object P^S described in Proposition 4.3 is a co-separator. Let P be a cpo with $|P| \geq 2$, and $f, g : A \to B$ be cpo S-maps with $f \neq g$. First, we define a cpo map $k : B \to P$ such that $k \circ f \neq k \circ g$. Since $f \neq g$, there exists $a \in A$ with $f(a) \neq g(a)$. Three cases may occur:

$$(1)\ \ f(a) < g(a) \qquad (2)\ \ g(a) < f(a) \qquad (3)\ \ f(a) \parallel g(a)$$

Let $f(a) < g(a)$. Then, taking $B' = \{b \in B \mid \ b \leq f(a)\}$, define $k : B \to P$ by

$$k(b) = \begin{cases} \perp_P & \text{if } b \in B' \\ y & \text{otherwise} \end{cases}$$

where $y \in P$ is chosen with $y \neq \perp_P$ (such y exists since $|P| \geq 2$). First, we show that k is order-preserving, and hence it take directed subsets to directed ones. Let $b_1, b_2 \in B$ with $b_1 \leq b_2$. If $b_1 \in B'$, then for the case where $b_2 \in B'$, $\perp_P = k(b_1) = k(b_2)$; and for the case where $b_2 \notin B'$, $\perp_P = k(b_1) < y = k(b_2)$. Also, if $b_1 \notin B'$, then $b_2 \notin B'$ and so

$k(b_1) = k(b_2) = y$. To prove the continuity of k, let $D \subseteq^d B$. It is clear, by the definition of B', that $\bigsqcup D \in B' \Leftrightarrow D \subseteq B'$. Now, if $\bigsqcup D \in B'$, then $D \subseteq B'$ and so $k(\bigsqcup D) = \bot_P = \bigsqcup_{z \in D} k(z)$. Also, if $\bigsqcup D \notin B'$, then $k(\bigsqcup D) = y$ and $D \nsubseteq B'$. Thus $D \setminus B' \neq \emptyset$, and

$$\bigsqcup_{z \in D} k(z) = \bigsqcup_{z \in (D \setminus B') \cup (B' \cap D)} k(z) = y \vee \bot_P = y$$

as required. Finally, since P^S is the co-free cpo S-act on P, there exists a unique cpo S-map $h : B \to P^S$ such that $\sigma \circ h = k$, where σ is the co-free map defined in Proposition 4.3. This gives that $h \circ f \neq h \circ g$, and so P^S is a co-separator.

The case (2) is proved similarly. And about case (3), take $B' = \{b \in B : b \leq f(a)\}$ or $B' = \{b \in B : b \leq g(a)\}$ in the proof of case (1). □

Recall for instance from [10] that a category is well-powered if for each of its objects, the class of its subobjects (considered up to isomorphisms) forms a set. Also, for the categories in which monomorphisms coincide with one-one morphisms, the class of subobjects of an object B would be a subset of the power set of B, and therefore is a set. This also holds for the category of cpo S-acts, by Lemma 5.2. Therefore,

Lemma 6.3. *The category* $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ *is well-powered.*

Proposition 6.4. *The category* $\mathbf{Cpo}_{\mathbf{Act}\text{-}S}$ *is co-complete.*

Proof. By Theorem 23.14 [10] and Lemmas 6.2, 6.3 and 6.1, the category is co-complete. □

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