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Author(s):

E. Anjidani

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ALMOST MULTIPLICATIVE LINEAR FUNCTIONALS AND APPROXIMATE SPECTRUM

E. ANJIDANI

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Abstract. We define a new type of spectrum, called δ-approximate spectrum, of an element a in a complex unital Banach algebra A and show that the δ-approximate spectrum $\sigma_\delta(a)$ of a is compact. The relation between the δ-approximate spectrum and the usual spectrum is investigated. Also, an analogue of the classical Gleason-Kahane-Żelazko theorem is established: For each $\epsilon > 0$, there is $\delta > 0$ such that if ϕ is a linear functional with $\phi(a) \in \sigma_\delta(a)$ for all $a \in A$, then ϕ is ϵ-almost multiplicative. Finally, we use these ideas to provide a sufficient condition for a δ-almost multiplicative functional to be multiplicative.
Keywords: Almost multiplicative linear functional, Ransford spectrum, pseudospectrum, condition spectrum, Gleason-Kahane-Żelazko theorem.
MSC(2010): Primary: 46H05; Secondary: 47A10, 46J05.

1. Introduction and preliminaries

Let A be a complex unital Banach algebra with unit 1 and $\delta > 0$. A linear functional ϕ on A is said to be δ-almost multiplicative if

$$|\phi(ab) - \phi(a)\phi(b)| \leq \delta\|a\|\|b\|,$$

for all $a, b \in A$. The notion of almost multiplicative linear functional has been studied intensively by a number of authors, see e.g. [4, 5, 7, 8]. The almost multiplicative linear functionals have interesting properties and applications. Jarosz [5, Proposition 5.5] proved that if ϕ is a δ-almost multiplicative linear functional on A, then ϕ is continuous and $\|\phi\| \leq 1 + \delta$. Also, it is easy to see that if $\delta < \frac{1}{4}$, then either $|\phi(1) - 1| < 2\delta$, in

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which case $\|\phi\| > 1 - 2\delta$ or $\|\phi\| < 2\delta$, in which case $|\phi(1)| < 2\delta$ [7]. In this paper, these properties are the motivation for definition of a specific type of Ransford spectrum, called the approximate spectrum.
The Ransford spectrum is an extension of the concept of spectrum (see [12]). A subset Ω of A is called a Ransford set if $1 \in \Omega$, $0 \notin \Omega$ and $\lambda\Omega \subseteq \Omega$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. The Ransford spectrum of an element a in A with respect to the Ransford set Ω is defined by

$$sp^{\Omega}(a) := \{\lambda \in \mathbb{C} : \lambda - a \notin \Omega\}.$$

A particular case of the Ransford spectrum is called the ϵ-condition spectrum and defined as follows:

$$sp_{\epsilon}(a) := \left\{\lambda \in \mathbb{C} : \|\lambda - a\|\|(\lambda - a)^{-1}\| \geq \frac{1}{\epsilon}\right\},$$

for $a \in A$ and $0 < \epsilon < 1$ (see [11]).

Let $a \in A$ and $0 < \delta < \frac{1}{4}$. We say that a is δ-invertible if $\phi(a) \neq 0$ for all $\phi \in \Phi_{\delta}(A)$, where

$$\Phi_{\delta}(A) := \{\phi |\ \phi \text{ is } \delta - \text{almost multiplicative on } A \text{ and } |\phi(1) - 1| < 2\delta\}.$$

The set $\text{Inv}_{\delta}A$ of all δ-invertible elements of A is a Ransford set. The Ransford spectrum of a with respect to the Ransford set $\text{Inv}_{\delta}A$ is called δ-approximate spectrum of element a and defined as follows:

$$\sigma_{\delta}(a) := \{\lambda \in \mathbb{C} : \lambda - a \notin \text{Inv}_{\delta}A\}.$$

In section 2, we prove that $\text{Inv}_{\delta}A$ is open and hence $\sigma_{\delta}(a)$ is compact. Moreover, the relation between the δ-approximate spectrum $\sigma_{\delta}(a)$ and the usual spectrum $\sigma(a)$ of element a is investigated. In particular, we show that $\sigma_{\delta}(a) = \sigma(a)$ if and only if $a = \lambda$ for some $\lambda \in \mathbb{C}$.

The Gleason-Kahane-Żelazko theorem states that a linear functional ϕ on complex Banach algebra A is multiplicative if

$$\phi(a) \in \sigma(a) \quad (a \in A, \|a\| = 1). \tag{1.1}$$

Some important generalizations of this result have been given by many mathematicians (see for example [3,6,13]). In addition, there are several possible approximate versions of the Gleason-Kahane-Żelazko theorem which are concerned with identifying the almost multiplicative linear functionals among all linear functionals on Banach algebra A in terms of spectra. The first result was given by Johnson [7]. He proved that linear functional ϕ is almost multiplicative if condition (1.1) is replaced by

$$d(\phi(a), \sigma(a)) < \epsilon \quad (a \in A, \|a\| = 1).$$

In [9, Theorem 5] the spectrum in condition (1.1) is replaced by the ϵ-condition spectrum while in [1, Theorem 4.2] is replaced by the so-called pseudospectrum, defined by

$$\Lambda_\epsilon(a) := \{\lambda \in \mathbb{C} : \|(\lambda - a)^{-1}\| \geq \frac{1}{\epsilon}\}$$

for $a \in A$ and $\epsilon > 0$.
In section 3, we prove an approximate version of the Gleason-Kahane-Żelazko theorem in which the spectra in condition (1.1) are replaced by the δ-approximate spectra.

Finally, in section 4, we use these ideas to provide a sufficient condition for an almost multiplicative linear functional to be multiplicative.

2. **Approximate Spectrum**

We define a specific type of Ransford spectrum, called the approximate spectrum.

Definition 2.1. *Let $a \in A$ and $0 < \delta < \frac{1}{4}$. We say that a is δ-invertible if for every $\phi \in \Phi_\delta(A)$, $\phi(a) \neq 0$, where*

$\Phi_\delta(A) := \{\phi | \ \phi$ is $\delta -$ almost multiplicative on A and $|\phi(1) - 1| < 2\delta\}$.

The set of all δ-invertible elements of A is denoted by $\mathrm{Inv}_\delta A$.

It is clear that $1 \in \mathrm{Inv}_\delta A$, $0 \notin \mathrm{Inv}_\delta A$ and $\lambda x \in \mathrm{Inv}_\delta A$ for all $\lambda \in \mathbb{C}\backslash\{0\}$ and $x \in \mathrm{Inv}_\delta A$. Hence, $\mathrm{Inv}_\delta A$ is a Ransford set.

Definition 2.2. *The Ransford spectrum of a with respect to the Ransford set* $\mathrm{Inv}_\delta A$ *is called δ-approximate spectrum of a and defined by*

$$\sigma_\delta(a) := \{\lambda \in \mathbb{C} : \lambda - a \notin \mathrm{Inv}_\delta A\}.$$

It is known that the set $\Phi(A)$ of all nonzero multiplicative linear functionals on A is a weak* compact subset of the dual space A^*. Now, we prove that so is $\Phi_\delta(A)$.
Recall that a basis for weak* topology on the dual space X^* of a Banach space X is given by sets of the form $V(\phi; x_1, \dots, x_n; \epsilon)$, where

$$V(\phi; x_1, \dots, x_n; \epsilon) = \{\psi \in X^* : |\psi(x_i) - \phi(x_i)| < \epsilon \text{ for } 1 \leq i \leq n\}$$

for arbitrary $\epsilon > 0$ and elements $x_1, \dots, x_n \in X$.

Theorem 2.3. *Let $0 < \delta < \frac{1}{4}$. The set $\Phi_\delta(A)$ is a weak* compact subset of A^*.*

Proof. At first we show that the set

$$\mathfrak{M}_\delta(A) := \{\phi : \phi \text{ is } \delta - \text{almost multiplicative on } A\}$$

is weak* compact in A^*. Since $\mathfrak{M}_\delta(A)$ is bounded, it is sufficient to prove that this set is weak* closed. For each $x, y \in A$, since the maps $\phi \mapsto \phi(x)$, $\phi \mapsto \phi(y)$ and $\phi \mapsto \phi(xy)$ are weak* continuous, the set

$$\{\phi \in A^* : |\phi(xy) - \phi(x)\phi(y)| \leq \delta\|x\|\|y\|\} \tag{2.1}$$

is weak* closed. Hence, $\mathfrak{M}_\delta(A)$ which is the intersection of sets of the form (2.1) over all choices of x, y, is weak* closed.
By [7, Proposition 2.3], we have

$$\mathfrak{M}_\delta(A) = \Phi_\delta(A) \bigcup \{\phi : \phi \text{ is } \delta - \text{almost multiplicative on } A,\ \|\phi\| < 2\delta\}.$$

Also, it is clear that

$$\{\phi : \phi \text{ is } \delta - \text{almost multiplicative on } A,\ \|\phi\| < 2\delta\} \subseteq V(0; 1; 2\delta)$$

and $V(0; 1; 2\delta) \cap \Phi_\delta(A) = \varnothing$. Therefore, $\Phi_\delta(A)$ is weak* compact. □

Theorem 2.4. *The set* $\mathrm{Inv}_\delta \mathrm{A}$ *is an open subset of* A.

Proof. Let $x \in \mathrm{Inv}_\delta \mathrm{A}$. Then $\phi(x) \neq 0$ for every $\phi \in \Phi_\delta(A)$. Since $\Phi_\delta(A)$ is weak* compact, the set $\{\phi(x) : \phi \in \Phi_\delta(A)\}$ is compact in $\mathbb{C}$. Therefore, there is $\epsilon > 0$ such that the open disc with center at the origin and radius ϵ does not intersect the set $\{\phi(x) : \phi \in \Phi_\delta(A)\}$. Choose $y \in A$ with $\|x - y\| < \frac{\epsilon}{1+\delta}$. Then $|\phi(x) - \phi(y)| \leq \|\phi\|\|x - y\| < \epsilon$ for all $\phi \in \Phi_\delta(A)$. Therefore $y \in \mathrm{Inv}_\delta \mathrm{A}$ and so $\mathrm{Inv}_\delta \mathrm{A}$ is an open subset of A. □

In the following theorem, we investigate the relation between the approximate spectrum and the usual spectrum in commutative complex Banach algebras.

Theorem 2.5. *Let A be a complex commutative unital Banach algebra with unit 1 and $a \in A$. Then,*

1) $\sigma(a) \subseteq \sigma_\delta(a)$,
2) $\sigma_\delta(a)$ *is a nonempty compact subset of* $\mathbb{C}$,
3) $\sigma(a) = \bigcap \sigma_\delta(a)$, *where the intersection is taken over* $0 < \delta < \frac{1}{4}$,
4) $\sigma_\delta(a) = \sigma(a)$ *if and only if $a = z$ for some $z \in \mathbb{C}$.*

Proof. Since A is commutative, $\lambda \in \sigma(a)$ if and only if there is $\phi \in \Phi(A)$ such that $\phi(a) = \lambda$. This implies that $\sigma(a) \subseteq \sigma_\delta(a)$ and so $\sigma_\delta(a)$ is nonempty.
Since the mapping $\lambda \mapsto \lambda - a$ is continuous and $\mathrm{Inv}_\delta(A)$ is open, $\sigma_\delta(a)$

is a closed subset of $\mathbb{C}$. Also, for every $\lambda \in \sigma_\delta(a)$ there exists $\phi \in \Phi_\delta(A)$ such that $\lambda = \frac{\phi(a)}{\phi(1)}$ and so $|\lambda| = \frac{|\phi(a)|}{|\phi(1)|} \leq \frac{1+\delta}{1-2\delta}\|a\|$. Therefore, $\sigma_\delta(a)$ is compact.

Now, it is clear that

$$\sigma(a) \subseteq \bigcap_{0<\delta<\frac{1}{4}} \sigma_\delta(a).$$

Let $\lambda \in \bigcap_{0<\delta<\frac{1}{4}} \sigma_\delta(a)$. Then, for every $n \geq 5$ there exists $\phi_n \in \Phi_{\frac{1}{n}}(A)$ such that $\lambda = \frac{\phi_n(a)}{\phi_n(1)}$. Suppose that the set $B = \{\phi_n : n \geq 5\}$ is infinite. Since $\Phi_{\frac{1}{5}}(A)$ is weak* compact and $B \subseteq \Phi_{\frac{1}{5}}(A)$, B has a limit point ϕ. We prove that $\phi \in \Phi(A)$ and $\phi(a) = \lambda$. Let $\epsilon > 0$ and $x, y \in A$. The neighborhood $V(\phi; x, y, xy, a, 1; \epsilon)$ contains infinitely many points of B. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$ and $\phi_N \in V(\phi; x, y, xy, a, 1; \epsilon)$. Then we have

$$\begin{aligned}(2.2)\ |\phi(xy) - \phi(x)\phi(y)| &\leq |\phi(xy) - \phi_N(xy)| \\ &\quad + |\phi_N(xy) - \phi_N(x)\phi_N(y)| \\ &\quad + |\phi_N(x)\phi_N(y) - \phi_N(x)\phi(y)| \\ &\quad + |\phi_N(x)\phi(y) - \phi(x)\phi(y)| \\ &< \epsilon(1 + \|x\|\|y\| + (1+\epsilon)\|x\| + \|\phi\|\|y\|).\end{aligned}$$

Also we get

$$|\phi(1) - 1| \leq |\phi(1) - \phi_N(1)| + |\phi_N(1) - 1| < 3\epsilon, \tag{2.3}$$

and

$$\begin{aligned}|\phi(a) - \lambda| &\leq |\phi(a) - \phi_N(a)| + |\phi_N(a) - \lambda| \\ &< \epsilon + |\lambda\phi_N(1) - \lambda| \\ &< \epsilon(1 + 2|\lambda|).\end{aligned} \tag{2.4}$$

Now, since $\epsilon > 0$ is arbitrary, it follows from (2.2), (2.3) and (2.4) that ϕ is multiplicative, $\phi(1) = 1$ and $\phi(a) = \lambda$. Therefore $\lambda \in \sigma(a)$.
Now consider the case where B is finite. Then there exists $\phi \in B$ such that $\phi_n = \phi$ for infinitely many $n \in \mathbb{N}$. Similar to the pervious case, it follows that $\phi \in \Phi(A)$ and $\phi(a) = \lambda$ and so $\lambda \in \sigma(a)$.
Therefore, $\sigma(a) = \bigcap \sigma_\delta(a)$.

Finally, we prove assertion (4). It is clear that $\sigma_\delta(z) = \sigma(z) = \{z\}$ for every $z \in \mathbb{C}$.
Now, suppose that $a \neq z$ for every $z \in \mathbb{C}$. At first we prove that every

point of $\sigma(a)$ is an interior point of $\sigma_\delta(a)$. Choose $\epsilon > 0$ with $\epsilon^2 + 3\epsilon < \delta$. Since a does not belong to the closed subspace $\{z.1 : z \in \mathbb{C}\}$ of A, by the Hahn-Banach theorem there exists $\Lambda \in A^*$ such that $\Lambda(a) = 1$ and $\Lambda(1) = 0$.
Now let $\lambda_0 \in \sigma(a)$ and choose $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < \frac{\epsilon}{\|\Lambda\|}$. Then we have $f(a) = \lambda_0$ for some $f \in \Phi(A)$.
Let $\phi = (\lambda - \lambda_0)\Lambda$. It is clear that $\|\phi\| = |\lambda - \lambda_0|\|\Lambda\| < \epsilon$, and so $f + \phi$ is an $\epsilon^2 + 3\epsilon$-multiplicative linear functional and $(f + \phi)(1) = 1$. Hence $f + \phi \in \Phi_\delta(A)$ and since $(f + \phi)(a) = f(a) + \phi(a) = \lambda$, it follows that $\lambda \in \sigma_\delta(a)$. Therefore $\sigma(a) \subseteq (\sigma_\delta(a))^\circ$.
Now if $\sigma_\delta(a) = \sigma(a)$, then $\sigma(a)$ is an open subset of $\mathbb{C}$. Since $\sigma(a)$ is also compact and nonempty, it follows that $\mathbb{C}$ is disconnected, which is a contradiction. Therefore, if $\sigma_\delta(a) = \sigma(a)$, then $a = z$ for some $z \in \mathbb{C}$. □

3. Gleason-Kahane-Żelazko theorem

The approximate versions of the Gleason-Kahane-Żelazko theorem that are concerned with the investigation of the relation between almost multiplicative linear functionals and spectra (see [9, Theorem 5] and [1, Theorem 4.2]) are proved by a similar method to the proof of the following theorem of Johnson (see [7]). In Theorem 3.2, we prove an approximate version of Gleason-Kahane-Żelazko theorem which connects the approximate spectrum and almost multiplicative linear functionals.

Theorem 3.1. *[7, Theorem 8.7] Let A be a complex commutative Banach algebra. There is a monotonic function ϵ from $(0, 0.1)$ into $(0, \infty)$ with $\epsilon(\delta) \to 0$ as $\delta \to 0$ such that if ϕ is a continuous linear functional on A with*

$$d(\phi(a), \sigma(a)) \leq \delta\|a\| \quad (a \in A),$$

then ϕ is $\epsilon(\delta)$-multiplicative.

Theorem 3.2. *Let A be a complex unital Banach algebra. Then the following assertions hold.*

1) *For every $\epsilon > 0$ there is $\delta > 0$ such that if ϕ is a linear functional on A with*

$$\phi(a) \in \sigma_\delta(a) \quad (a \in A, \|a\| = 1),$$

then ϕ is ϵ-almost multiplicative.

2) *Suppose that A is commutative. For every $0<\delta<\frac{1}{6}$, if ϕ is a linear functional on A with*

$$\phi(a)\in\sigma_\delta(a)\quad (a\in A,\|a\|=1),$$

then ϕ is ϵ-almost multiplicative, where

$$\epsilon=4\left(\ln\frac{1-2\delta}{\delta}\right)^{-1}\left(\frac{6}{\ln 2}+2\left(\ln\frac{1-2\delta}{\delta}\right)^{-1}\right).$$

Proof. Let $0<\delta<\frac{1}{6}$. Suppose that ϕ is a linear functional on A with

$$\phi(a)\in\sigma_\delta(a)\quad (a\in A,\|a\|=1).$$

Then $\phi(1)=1$ and $|\phi(a)|\le\frac{1+\delta}{1-2\delta}$ for all $a\in A$ with $\|a\|=1$, and so ϕ is continuous and $\|\phi\|\le\frac{1+\delta}{1-2\delta}$.
Let $a\in A$ with $\|a\|=1$. Since ϕ is continuous and linear, we have

$$\phi(\exp za)=\sum_{n=0}^{\infty}\frac{\phi(a^n)}{n!}z^n$$

for all $z\in\mathbb{C}$. Hence the function $f:\mathbb{C}\to\mathbb{C}$ defined by $f(z)=\phi(\exp za)$ is an entire function such that for all $z\in\mathbb{C}$ we have

$$|f(z)|=|\phi(\exp za)|\le\|\phi\|\|\exp za\|\le\frac{1+\delta}{1-2\delta}e^{|z|}. \tag{3.1}$$

Therefore, f has growth order less than or equal to one. Suppose that $\alpha_1,\alpha_2,\dots$ are the zeros of f indexed with

$$|\alpha_1|\le|\alpha_2|\le\cdots.$$

Using the Hadamard's factorization theorem [14] and by the same method as in the proof of [9, Theorem 5], we get

$$\phi(a^2)-\phi(a)^2=-\sum_j\frac{1}{\alpha_j^2}.$$

Now let α_j be a zero of f. Using Jensen's formula [14] and the inequality (3.1), by the same reasoning as in the proof of [9, Theorem 5] we obtain

$$j\ln 2\le\ln\frac{1+\delta}{1-2\delta}+2|\alpha_j|.$$

On the other hand, since

$$\phi\left(\frac{\exp\alpha_j a}{\|\exp\alpha_j a\|}\right)\in\sigma_\delta\left(\frac{\exp\alpha_j a}{\|\exp\alpha_j a\|}\right),$$

we have $0 \in \sigma_\delta\left(\|\exp\alpha_j a\|^{-1}\exp\alpha_j a\right)$ from which it follows that $\exp\alpha_j a \notin \mathrm{Inv}_\delta A$ and thus there exists $\psi \in \Phi_\delta(A)$ such that $\psi(\exp\alpha_j a) = 0$. Hence, we have

$$1-2\delta \le |\psi(1)| \le \delta\|\exp\alpha_j a\|\|\exp(-\alpha_j a)\| \le \delta e^{2|\alpha_j|},$$

and so $|\alpha_j| \ge \frac{1}{2}\ln\frac{1-2\delta}{\delta}$. Therefore we have

$$j\ln 2 \le \ln\frac{1+\delta}{1-2\delta} + 2|\alpha_j| \le \frac{1}{2}\ln\frac{1-2\delta}{\delta} + 2|\alpha_j| \le 3|\alpha_j|.$$

Put $M := \frac{1}{2}\ln\frac{1-2\delta}{\delta}$ and let k be the greatest integer less than or equal to $\frac{3}{\ln 2}M$. Now we find a bound for $\left|\sum_j \frac{1}{\alpha_j^2}\right|$, by using $|\alpha_j| \ge M$ for $1 \le j \le k$ and $|\alpha_j| \ge \frac{\ln 2}{3}j$ for $j > k$. Then, a simple computation shows that

$$\left|\sum_j \frac{1}{\alpha_j^2}\right| \le \sum_{j=1}^{k}\frac{1}{|\alpha_j|^2} + \sum_{j=k+1}^{\infty}\frac{1}{|\alpha_j|^2} \le \frac{1}{M}\left(\frac{6}{\ln 2} + \frac{1}{M}\right).$$

Therefore, we get

$$|\phi(a^2) - \phi(a)^2| \le \tag{3.2}$$

$$2\left(\ln\frac{1-2\delta}{\delta}\right)^{-1}\left(\frac{6}{\ln 2} + 2\left(\ln\frac{1-2\delta}{\delta}\right)^{-1}\right) := g(\delta),$$

for all $a \in A$ with $\|a\| = 1$.
Let $\epsilon > 0$. By [1, Corollary 3.6], there is $\delta_1 > 0$ such that if φ is a linear functional on A with

$$|\varphi(a^2) - \varphi(a)^2| \le \delta_1 \quad (a \in A, \|a\| = 1),$$

then φ is ϵ-almost multiplicative.
Now there is $\delta > 0$ such that $g(\delta) < \delta_1$. By (3.2), the linear functional ϕ is ϵ-almost multiplicative if $\phi(a) \in \sigma_\delta(a)$ for all $a \in A$ with $\|a\| = 1$. This proves the first assertion.

Now suppose that A is a commutative Banach algebra. Let $0 < \delta < \frac{1}{6}$ and $\phi : A \to \mathbb{C}$ be a linear functional with

$$\phi(a) \in \sigma_\delta(a) \quad (a \in A, \|a\| = 1).$$

By (3.2), the second assertion follows from [9, Lemma 4]. □

4. a sufficient condition for an almost multiplicative functional to be multiplicative

The following extension of the Gleason-Kahane-Żelazko theorem was given in [2].

Theorem 4.1. *Suppose that Ω is a Ransford subset of A and the Ransford spectrum $sp^{\Omega}(a)$ of a is bounded for all $a \in A$. Moreover, assume that there is a strictly increasing sequence $\{n_k\}$ in $\mathbb{N}$ such that if $x \in \Omega$, then $x^{n_k} \in \Omega$ for all $k \in \mathbb{N}$. Then, if ϕ is a unital (i.e. $\phi(1) = 1$) linear functional on A with $\phi(a) \in sp^{\Omega}(a)$ for all $a \in A$, then ϕ is multiplicative.*

A proof of this theorem can be found in [10]. We apply this result to get a sufficient condition for an almost multiplicative linear functional to be multiplicative.

Theorem 4.2. *Let $0 < \delta < \frac{1}{4}$ and $f \in \Phi_\delta(A)$. Assume that there exists an integer number $r \geq 2$ such that*

$$f(x) \neq 0 \Rightarrow f(x^r) \neq 0$$

holds true for all $x \in A$. Then,

$$f(xy) = \frac{f(x)f(y)}{f(1)} \quad (x, y \in A).$$

Proof. Let Ψ_r be the set of all δ-almost multiplicative functionals ψ in $\Phi_\delta(A)$ satisfying

$$\psi(x) \neq 0 \Rightarrow \psi(x^r) \neq 0 \quad (x \in A).$$

Moreover, put

$$\Omega := \{x \in A : \psi(x) \neq 0 \text{ for all } \psi \in \Psi_{\rm r}\}.$$

It is clear that $0 \notin \Omega$, $1 \in \Omega$ and $z\Omega \subseteq \Omega$ for all $z \in \mathbb{C} \setminus \{0\}$. Also, there is the strictly increasing sequence $\{r^n\}$ such that

$$x \in \Omega \Rightarrow x^{r^n} \in \Omega$$

holds true for all $x \in A$. Now for every $x \in A$, the Ransford spectrum $sp^{\Omega}(x)$ of x with respect to the Ransford set Ω is defined by

$$sp^{\Omega}(x) = \{\lambda \in \mathbb{C} : \lambda - x \notin \Omega\}.$$

Since $\sigma_\delta(x)$ is bounded and $sp^{\Omega}(x) \subseteq \sigma_\delta(x)$, $sp^{\Omega}(x)$ is bounded for all $x \in A$. Put $\phi = \frac{f}{f(1)}$. Since $f \in \Psi_r$ and $f(\phi(x) - x) = \frac{f(x)}{f(1)} f(1) - f(x) = 0$ for every $x \in A$, $\phi(x) - x \notin \Omega$ and so $\phi(x) \in sp^{\Omega}(x)$ for all $x \in A$. Now it follows from Theorem 4.1 that the linear functional ϕ is multiplicative. Hence $f(xy) = \frac{f(x)f(y)}{f(1)}$ for all $x, y \in A$. $\square$

Corollary 4.3. *Suppose that $f : A \to \mathbb{C}$ is a unital δ-almost multiplicative linear functional such that*

$$f(x) \neq 0 \Rightarrow f(x^2) \neq 0$$

holds true for all $x \in A$. Then f is multiplicative.

Corollary 4.4. *There is $\delta > 0$ such that if ϕ is a unital linear functional on A with*

$$|\phi(a^2) - (\phi(a))^2| < \delta \quad (a \in A, \|a\| = 1), \tag{4.1}$$

and furthermore, satisfying

$$\phi(a^2) = 0 \Rightarrow \phi(a) = 0 \quad (a \in A, \|a\| = 1), \tag{4.2}$$

then ϕ is multiplicative.

Proof. Let $0 < \epsilon < \frac{1}{4}$. By [1, Corollary 3.6], there is $\delta > 0$ such that the linear functional ϕ satisfying (4.1) is ϵ-almost multiplicative. Now if ϕ satisfies (4.2), then by Corollary 4.3, ϕ is multiplicative. $\square$

By [9, Lemma 4], the linear functional ϕ on commutative Banach algebra A satisfying (4.1) is 2δ-almost multiplicative. Hence, if A is commutative, then $\delta = \frac{1}{8}$ in Corollary 4.4.

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(Ehsan Anjidani) Department of Mathematics, University of Neyshabur, P.O. Box 91136-899, Neyshabur, Iran

E-mail address: `anjidani@neyshabur.ac.ir`