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BOUNDED APPROXIMATE CONNES-AMENABILITY OF DUAL BANACH ALGEBRAS

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Abstract. We study the notion of bounded approximate Connes-amenability for dual Banach algebras and characterize this type of algebras in terms of approximate diagonals. We show that bounded approximate Connes-amenability of dual Banach algebras forces them to be unital. For a separable dual Banach algebra, we prove that bounded approximate Connes-amenability implies sequential approximate Connes-amenability.
Keywords: Bounded approximate Connes-amenability, sequential approximate Connes-amenability, multiplier-bounded approximate identity.
MSC(2010): Primary: 43A10; Secondary: 43A20, 46H25.

1. Introduction

The concept of amenability for Banach algebras was introduced by Johnson in 1972 [14]. Several modifications of this notion were introduced by relaxing some of the restrictions on the definition of amenability. Some of the most notable are the concepts of Connes amenability [15] and approximate amenability [10], where the former had been studied previously under different names. We recall the definitions in Definitions 1.1 and 1.2 below. Before proceeding further, we recall some terminology.

Let $\mathcal{A}$ be a Banach algebra. Throughout this paper, the identity element of $\mathcal{A}$, whenever it exists, is denoted by e. The term *unital* Banach algebra refers to a Banach algebra with identity e for which

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$||e|| = 1$. For a Banach algebra $\mathcal{A}$ a *Banach $\mathcal{A}$-bimodule* E, is a Banach space which is algebraically an $\mathcal{A}$-bimodule and there is a constant $C \geq 0$ such that

$$||a \, . \, x|| \leq C||a|| \, ||x|| \quad \text{and} \quad ||x \, . \, a|| \leq C||a|| \, ||x|| \quad (a \in \mathcal{A}, x \in E).$$

We write $\mathcal{L}(\mathcal{A}, E)$ for the Banach space of all bounded linear maps from $\mathcal{A}$ into E. In the case where E is a Banach $\mathcal{A}$-bimodule, its dual E^* is also a Banach $\mathcal{A}$-bimodule. The reader may see [3] for the standard dual module definitions.

Suppose that $\mathcal{A}$ is a Banach algebra and E is a Banach $\mathcal{A}$-bimodule. A *derivation* $D : \mathcal{A} \longrightarrow E$ is a bounded linear map, satisfying

$$D(ab) = Da \, . \, b + a \, . \, Db \quad (a, b \in \mathcal{A}).$$

For $x \in E$, the *inner* derivation $ad_x : \mathcal{A} \longrightarrow E$ is defined by $ad_x(a) = a \, . \, x - x \, . \, a$. A derivation $D : \mathcal{A} \longrightarrow E$ is *inner* if there is $x \in E$ such that $D = ad_x$.

Let $\mathcal{A}$ be a Banach algebra. A Banach $\mathcal{A}$-bimodule E is *dual* if there is a closed submodule E_* of E^* such that $E = (E_*)^*$. We call E_* the *predual* of E. A dual Banach $\mathcal{A}$-bimodule E is *normal* if the module actions of $\mathcal{A}$ on E are w^*-continuous. A Banach algebra $\mathcal{A}$ is *dual* if it is dual as a Banach $\mathcal{A}$-bimodule.

Definition 1.1. A dual Banach algebra $\mathcal{A}$ is *Connes-amenable* if every w^*-continuous derivation from $\mathcal{A}$ into a normal, dual Banach $\mathcal{A}$-bimodule is inner.

Let $\mathcal{A}$ be a Banach algebra. The projective tensor product $\mathcal{A}\hat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule under the operations

$$a \, . \, (b \otimes c) := ab \otimes c, \quad (b \otimes c) \, . \, a := b \otimes ca \quad (a, b, c \in \mathcal{A}),$$

and there is a continuous linear $\mathcal{A}$-bimodule homomorphism $\Pi : \mathcal{A}\hat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ such that $\Pi(a \otimes b) = ab$, for $a, b \in \mathcal{A}$ (see [3]).

Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. We write $\sigma wc(E)$ for the set of all elements $x \in E$ such that the maps

$$\mathcal{A} \longrightarrow E \quad , \quad a \longmapsto \begin{cases} a \, . \, x \\ x \, . \, a \end{cases} ,$$

are w^*-weak continuous. The space $\sigma wc(E)$ ia a closed submodule of E. It is shown in [18, Corollary 4.6], that $\Pi^* \mathcal{A}_* \subseteq \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Taking adjoint, we can extend Π to an $\mathcal{A}$-bimodule homomorphism $\Pi_{\sigma wc}$ from $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ to $\mathcal{A}$.

Let $\mathcal{A}$ be a Banach algebra, and let E be a Banach $\mathcal{A}$-bimodule. A derivation $D : \mathcal{A} \longrightarrow E$ is *approximately inner* if there exists a net $(x_i)_i \subseteq E$, such that

$$Da = \lim_i (a \, . \, x_i - x_i \, . \, a) \qquad (a \in \mathcal{A}).$$

That is, $D = \lim_i ad_{x_i}$ in the strong-operator topology of $\mathcal{L}(\mathcal{A}, E)$.

Definition 1.2. A Banach algebra $\mathcal{A}$ is *approximately amenable* if for each Banach $\mathcal{A}$-bimodule E, every derivation $D : \mathcal{A} \longrightarrow E^*$ is approximately inner.

The reader may see [1, 6, 11–13] for more details. Motivated by Definitions 1.1 and 1.2, the concept of approximate Connes-amenability was introduced and studied in [9].

Definition 1.3. A dual Banach algebra $\mathcal{A}$ is *approximately Connes-amenable* if for each normal, dual Banach $\mathcal{A}$-bimodule E, every w^*-continuous derivation $D : \mathcal{A} \longrightarrow E$, is approximately inner.

The qualifier *sequential* prefixed to the Definitions 1.2 and 1.3 specifies that there is a sequence of inner derivations approximating each given derivation.

In [11], the notion of bounded approximate amenability was also introduced: a Banach algebra $\mathcal{A}$ is *boundedly approximately amenable* if for each Banach $\mathcal{A}$-bimodule E, and every derivation $D : \mathcal{A} \longrightarrow E^*$, there is a net $(\phi_i)_i \subseteq E^*$ such that the net $(ad_{\phi_i})_i$ is norm bounded in $\mathcal{L}(\mathcal{A}, E^*)$ and $Da = \lim_i ad_{\phi_i}(a)$ for $a \in \mathcal{A}$. It leads to the following notion.

Definition 1.4. A dual Banach algebra $\mathcal{A}$ is *boundedly approximately Connes-amenable* if for each normal, dual Banach $\mathcal{A}$-bimodule E, and every w^*-continuous derivation $D : \mathcal{A} \longrightarrow E$, there is a net $(x_i)_i \subseteq E$ such that the net $(ad_{x_i})_i$ is norm bounded in $\mathcal{L}(\mathcal{A}, E)$ and $Da = \lim_i ad_{x_i}(a)$ for $a \in \mathcal{A}$.

In the next section, we continue the investigation of approximate Connes-amenability. We study basic properties of the notion of bounded approximate Connes-amenability. We characterize both approximate and bounded approximate Connes-amenability in terms of approximate diagonals with specified properties. We see that bounded approximate Connes-amenability of a dual Banach algebra is equivalent to that of its unitization. We show that a boundededly approximately Connes-amenable dual Banach algebra must be unital. We prove that any boundedly approximately Connes-amenable dual Banach algebra which is also separable as a Banach space is sequentially approximately Connes-amenable. We conclude by looking at approximate Connes-amenability of the direct sum of two approximately Connes-amenable dual Banach algebras such that one of them has an identity.

2. Bounded approximate connes-amenability

We first state the following which is a combination of [9, Propositions 2.3 and 3.3].

Proposition 2.1. *Let $\mathcal{A}$ be a dual Banach algebra. Then, the following are equivalent:*

(*i*) *$\mathcal{A}$ is approximately Connes-amenable.*

(*ii*) *There is a net $(M_\alpha)_\alpha \subseteq \sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$ such that*

$$a \,.\, M_\alpha - M_\alpha \,.\, a \longrightarrow 0 \quad \text{and} \quad \Pi_{\sigma wc} M_\alpha \longrightarrow e \quad (a \in \mathcal{A}^\sharp)\ .$$

(*iii*) *There is a net $(M'_\alpha)_\alpha \subseteq \sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$ such that*

$$a \,.\, M'_\alpha - M'_\alpha \,.\, a \longrightarrow 0 \quad \text{and} \quad \Pi_{\sigma wc} M'_\alpha = e \quad (a \in \mathcal{A}^\sharp)\ .$$

The following lemma is needed to characterize (bounded) approximate Connes-amenability.

Lemma 2.2. *Let $\mathcal{A}$ be a dual Banach algebra and let E and F be Banach $\mathcal{A}$-bimodules. Then $\sigma wc(E \oplus F) = \sigma wc(E) \oplus \sigma wc(F)$.*

Proof. Let $x + y \in \sigma wc(E \oplus F)$, where $x \in E$ and $y \in F$. Let $a_i \xrightarrow{w^*} a$ in $\mathcal{A}$. Then $a_i \,.\, (x+y) \xrightarrow{wk} a \,.\, (x+y)$ in $E \oplus F$. Then, by the definition of weak topology on $E \oplus F$, we have

$$a_i \,.\, x \xrightarrow{wk} a \,.\, x \quad \text{and} \quad a_i \,.\, y \xrightarrow{wk} a \,.\, y\ ,$$

respectively in E and F. Whence $x \in \sigma wc(E)$ and $y \in \sigma wc(F)$, thus $x + y \in \sigma wc(E) \oplus \sigma wc(F)$.

Conversely, if $x \in \sigma wc(E)$ and $y \in \sigma wc(F)$ and if $a_i \xrightarrow{w^*} a$ in $\mathcal{A}$, then $a_i \,.\, x \xrightarrow{wk} a \,.\, x$ and $a_i \,.\, y \xrightarrow{wk} a \,.\, y$, respectively in E and F. Thus $a_i \,.\, (x + y) \xrightarrow{wk} a \,.\, (x + y)$ in $E \oplus F$, thus $x + y \in \sigma wc(E \oplus F)$. □

The following is an analog of [10, Corollary 2.2].

Theorem 2.3. *Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra. Then $\mathcal{A}$ is approximately Connes-amenable if and only if there are nets $(M_\alpha)_\alpha \subseteq \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$, $(F_\alpha)_\alpha, (G_\alpha)_\alpha \subseteq \sigma wc(\mathcal{A}^*)^*$, such that*

(*i*) $a\ .\ M_\alpha - M_\alpha\ .\ a + F_\alpha \otimes a - a \otimes G_\alpha \longrightarrow 0\ ,\ (a \in \mathcal{A})$.
(*ii*) $a\ .\ F_\alpha \longrightarrow a$ *and* $G_\alpha\ .\ a \longrightarrow a\ ,\ (a \in \mathcal{A})$.
(*iii*) $\Pi_{\sigma wc}(M_\alpha) - F_\alpha - G_\alpha \longrightarrow 0$.

Proof. Let $\mathcal{A}$ be approximately Connes-amenable. Then, by Proposition 2.1, there is a net $(N_\alpha)_\alpha \subseteq \sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$ such that $a\ .\ N_\alpha - N_\alpha\ .\ a \longrightarrow 0$, for all $a \in \mathcal{A}^\sharp$, and $\Pi_{\sigma wc}(N_\alpha) \longrightarrow e$, where e is the identity of $\mathcal{A}^\sharp$. By Lemma 2.2, since $(\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^* = (\mathcal{A} \hat{\otimes} \mathcal{A})^* \oplus (\mathcal{A}^* \otimes e) \oplus (e \otimes \mathcal{A}^*) \oplus (\mathbb{C}e \otimes e)$, we have

$$\sigma wc(\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^* = \sigma wc(\mathcal{A} \hat{\otimes} \mathcal{A})^* \oplus (\sigma wc(\mathcal{A}^*) \otimes e) \oplus (e \otimes \sigma wc(\mathcal{A}^*)) \oplus (\mathbb{C}e \otimes e)\ .$$

Therefore $\sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^* = \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^* \oplus (\sigma wc(\mathcal{A}^*)^* \otimes e) \oplus (e \otimes \sigma wc(\mathcal{A}^*)^*) \oplus (\mathbb{C}e \otimes e)$. Thus we can write $N_\alpha = M_\alpha - F_\alpha \otimes e - e \otimes G_\alpha + c_\alpha e \otimes e$, where $(M_\alpha)_\alpha \subseteq \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$ and $(F_\alpha)_\alpha, (G_\alpha)_\alpha \subseteq \sigma wc(\mathcal{A}^*)^*$ and $(c_\alpha)_\alpha \subseteq \mathbb{C}$.

Applying $\Pi_{\sigma wc}$, we observe that

$$\Pi_{\sigma wc}(M_\alpha) - F_\alpha - G_\alpha + c_\alpha \longrightarrow e\ ,$$

whence $c_\alpha \longrightarrow 1$ and $\Pi_{\sigma wc}(M_\alpha) - F_\alpha - G_\alpha \longrightarrow 0$, that is, we have (*iii*).

Next, for $a \in \mathcal{A}$

$$\begin{aligned} a\ .\ N_\alpha - N_\alpha\ .\ a = a\ .\ M_\alpha - M_\alpha\ .\ a + F_\alpha \otimes a - a \otimes G_\alpha \\ + e \otimes G_\alpha\ .\ a - a\ .\ F_\alpha \otimes e + a \otimes e - e \otimes a \longrightarrow 0\ , \end{aligned}$$

whence we conclude that

$$\lim_\alpha a\ .\ M_\alpha - M_\alpha\ .\ a + F_\alpha \otimes a - a \otimes G_\alpha = 0\ ,\ \text{ and } \lim_\alpha a\ .\ F_\alpha = \lim_\alpha G_\alpha\ .\ a = a,$$

as required.

Conversely, given $(M_\alpha)_\alpha$, $(F_\alpha)_\alpha$ and $(G_\alpha)_\alpha$, set $c_\alpha = 1$ and define $N_\alpha := M_\alpha - F_\alpha \otimes e - e \otimes G_\alpha + e \otimes e$. Then it is easy to check that, for all $a \in \mathcal{A}^\sharp$

$$a\ .\ N_\alpha - N_\alpha\ .\ a \longrightarrow 0 \ \text{ and } \ \Pi_{\sigma wc}(N_\alpha) \longrightarrow e\ ,$$

hence $\mathcal{A}$ is approximately Connes-amenable. □

Theorem 2.4. *Let $\mathcal{A}$ be a boundedly approximately Connes-amenable dual Banach algebra. Then there exists a constant $C > 0$ and nets $(M_\alpha)_\alpha \subseteq \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$, $(F_\alpha)_\alpha, (G_\alpha)_\alpha \subseteq \sigma wc(\mathcal{A}^*)^*$, such that*
(*i*) $\Pi_{\sigma wc}(M_\alpha) = F_\alpha + G_\alpha$.
(*ii*) $a\ .\ F_\alpha \longrightarrow a$, *for all* $a \in \mathcal{A}$.
(*iii*) $||a\ .\ F_\alpha|| \leq C||a||$, *for all* α *and* $a \in \mathcal{A}$.
(*iv*) $G_\alpha\ .\ a \longrightarrow a$, *for all* $a \in \mathcal{A}$.

(v) $||G_\alpha \,.\, a|| \le C||a||$, *for all* α *and* $a \in \mathcal{A}$.
(vi) $\sup_\alpha ||a \,.\, M_\alpha - M_\alpha \,.\, a - a\otimes G_\alpha + F_\alpha \otimes a|| \le C||a||$, *for all* $a \in \mathcal{A}$.
(vii) $a \,.\, M_\alpha - M_\alpha \,.\, a - a \otimes G_\alpha + F_\alpha \otimes a \longrightarrow 0$, *for all* $a \in \mathcal{A}$.

Proof. Regard $\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp$ as an $\mathcal{A}$-bimodule in the usual way. Let $\mathcal{K}$ be the kernel of the map $\Pi_{\sigma wc} : \sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^* \longrightarrow \mathcal{A}^\sharp$, and let $D : \mathcal{A} \longrightarrow \sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$ be the derivation defined by $Da = a \otimes e - e \otimes a$, where e is the identity of $\mathcal{A}^\sharp$. Note that $\sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$ is normal, hence D is w^*-continuous. Clearly, D attains its value in the w^*-closed submodule $\mathcal{K}$. Since $\mathcal{A}$ is boundedly approximately Connes-amenable, there exists a net $(u_\alpha)_\alpha$ in $\mathcal{K}$ such that

$$C := \sup_\alpha \sup_{||a|| \le 1} ||a \,.\, u_\alpha - u_\alpha \,.\, a|| < \infty \ , \quad \text{and} \quad Da = \lim_\alpha a \,.\, u_\alpha - u_\alpha \,.\, a \ ,$$

for all $a \in \mathcal{A}$. Identifying $\sigma wc((\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^*)^*$ with the direct sum

$$\sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^* \oplus (\sigma wc(\mathcal{A}^*)^* \otimes e) \oplus (e \otimes \sigma wc(\mathcal{A}^*)^*) \oplus (\mathbb{C}e \otimes e) \ ,$$

we may write each u_α in the form $u_\alpha = (-M_\alpha) + (F_\alpha \otimes e) + (e \otimes G_\alpha)$, for some $M_\alpha \in \sigma wc((\mathcal{A} \hat{\otimes} \mathcal{A})^*)^*$ and some $F_\alpha,\ G_\alpha \in \sigma wc(\mathcal{A}^*)^*$. We shall show that these nets have the required properties.

First, for all α, we have

$$0 = \Pi_{\sigma wc}(u_\alpha) = -\Pi_{\sigma wc}(M_\alpha) + F_\alpha + G_\alpha \ ,$$

and we obtain (i). Next, since

$$a.u_\alpha - u_\alpha.a = (-a.M_\alpha + M_\alpha.a + a\otimes G_\alpha - F_\alpha \otimes a) + (a.F_\alpha \otimes e) + (-e \otimes G_\alpha.a) \ ,$$

and the left hand side is bounded in norm by $C||a||$, we must have $||a \,.\, F_\alpha|| \le C||a||$, $||G_\alpha \,.\, a|| \le C||a||$ and

$$||a \,.\, M_\alpha - M_\alpha \,.\, a - a \otimes G_\alpha + F_\alpha \otimes a|| \le C||a|| \ ,$$

for all α and $a \in \mathcal{A}$. Whence we have (iii), (v) and (vi).

Finally, for each $a \in \mathcal{A}$,

$$\begin{aligned} a \otimes e - e \otimes a = Da &= \lim_\alpha (a \,.\, u_\alpha - u_\alpha \,.\, a) \\ &= \lim_\alpha (-a \,.\, M_\alpha + M_\alpha \,.\, a + a \otimes G_\alpha - F_\alpha \otimes a) \\ &+ \lim_\alpha (a \,.\, F_\alpha \otimes e) + \lim_\alpha (-e \otimes G_\alpha \,.\, a) \ . \end{aligned}$$

Then, we conclude that

$$a = \lim_\alpha a \,.\, F_\alpha = \lim_\alpha G_\alpha \,.\, a \ \text{ and } \ \lim_\alpha a \,.\, M_\alpha - M_\alpha \,.\, a - a\otimes G_\alpha + F_\alpha \otimes a = 0 \ ,$$

as required. □

Recall that if $\mathcal{A} = (\mathcal{A}_*)^*$ is a dual Banach algebra, then its unitization $\mathcal{A}^\sharp = \mathcal{A} \oplus^1 \mathbb{C}$ is a dual Banach algebra with predual $\mathcal{A}_* \oplus^\infty \mathbb{C}$, where $\oplus^1$ and $\oplus^\infty$ indicate ℓ^1 and ℓ^∞ direct sums, respectively. A Banach $\mathcal{A}$-bimodule E is said to be *neo-unital* if

$$E = \mathcal{A} \ . \ E \ . \ \mathcal{A} = \{a \ . \ x \ . \ b \ : \ a, b \in \mathcal{A}, \ x \in E\} \ .$$

Theorem 2.5. *Let $\mathcal{A}$ be a dual Banach algebra. Then $\mathcal{A}$ is boundedly approximately Connes-amenable if and only if $\mathcal{A}^\sharp$ is boundedly approximately Connes-amenable.*

Proof. Let $\mathcal{A}$ be boundedly approximately Connes-amenable, E be a normal dual Banach $\mathcal{A}^\sharp$-bimodule, and $D : \mathcal{A}^\sharp \longrightarrow E$ be a w^*-continuous derivation. By [10, Lemma 2.3], $D = D_1 + ad_\eta$, where e is the identity of $\mathcal{A}^\sharp$ and $D_1 : \mathcal{A}^\sharp \longrightarrow e \ . \ E \ .e$ is a w^*-continuous derivation into the normal dual Banach $\mathcal{A}^\sharp$-bimodule $e \ . \ E \ .e$, thus $D_1(e) = 0$. Hence, without loss of generality, we may suppose that E is neo-unital and so $D(e) = 0$. By assumption, there is a net $(x_i) \subseteq E$ and $C > 0$ such that $Da = \lim_i(a \ . \ x_i - x_i \ . \ a)$, for all $a \in \mathcal{A}$, and moreover $||a \ . \ x_i - x_i \ . \ a|| \leq C||a||$, for all i. Since $D(e) = 0$ and $e \ . \ x = x \ . \ e$ for all $x \in E$, it follows that

$$D(a + \lambda e) = \lim_i((a + \lambda e) \ . \ x_i - x_i \ . \ (a + \lambda e))$$

and

$$||(a + \lambda e) \ . \ x_i - x_i \ . \ (a + \lambda e)|| \leq C||a|| \leq C||a + \lambda e||$$

hence $\mathcal{A}^\sharp$ is boundedly approximately Connes-amenable.

Conversely, Let E be a normal, dual Banach $\mathcal{A}$-bimodule, and $D : \mathcal{A} \longrightarrow E$ be a w^*-continuous derivation. Setting $e \ . \ x = x \ . \ e = x$, makes E into a normal, dual Banach $\mathcal{A}^\sharp$-bimodule. We extend D to $\mathcal{A}^\sharp$ by setting $D(e) = 0$. Note that this extension is still w^*-continuous. Therefore, there is a net $(x_i) \subseteq E$ and $C > 0$, such that for all $a \in \mathcal{A}$

$$Da = \lim_i(a \ . \ x_i - x_i \ . \ a) \ \text{ with } \ ||a \ . \ x_i - x_i \ . \ a|| \leq C||a|| \ ,$$

as required. □

Theorem 2.6. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras and $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ is a continuous homomorphism with dense range. Then we have the following:*

(i) *Suppose that $\mathcal{A}$ is boundedly approximately amenable, then $\mathcal{B}$ is boundedly approximately amenable.*

(*ii*) *Suppose that $\mathcal{B}$ is a dual Banach algebra. If $\mathcal{A}$ is boundedly approximately amenable, then $\mathcal{B}$ is boundedly approximately Connes-amenable.*

(*iii*) *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are dual Banach algebras. If $\mathcal{A}$ is boundedly approximately Connes-amenable and if θ is w^*-continuous, then $\mathcal{B}$ is boundedly approximately Connes-amenable.*

Proof. We just give the proof of (i). Given a Banach $\mathcal{B}$-bimodule E, then it is also a Banach $\mathcal{A}$-bimodule with actions

$$a \,.\, x := \theta(a) \,.\, x \quad \text{and} \quad x \,.\, a := x \,.\, \theta(a) \quad (a \in \mathcal{A},\ x \in E)\ .$$

If $D : \mathcal{B} \longrightarrow E^*$ is a derivation, then $D\theta : \mathcal{A} \longrightarrow E^*$ is a derivation. Then, there are a net (ϕ_α) in E^* and a constant $C > 0$ such that $D(\theta(a)) = \lim_\alpha \theta(a) \,.\, \phi_\alpha - \phi_\alpha \,.\, \theta(a)$, for every $a \in \mathcal{A}$, and $||ad_{\phi_\alpha}|| < C$, for each α.

For $b \in \mathcal{B}$, there is a net $(a_i) \subseteq \mathcal{A}$ such that $\theta(a_i) \longrightarrow b$. For an arbitrary ϵ, there is an index i such that $||D||\ ||b - \theta(a_i)|| + C||b - \theta(a_i)|| < \dfrac{2\epsilon}{3}$. Then we may choose α such that $||D(\theta(a_i)) - (\theta(a_i) \,.\, \phi_\alpha - \phi_\alpha \,.\, \theta(a_i))|| < \dfrac{\epsilon}{3}$. Now $||Db - (b \,.\, \phi_\alpha - \phi_\alpha \,.\, b)|| < \epsilon$, and we are done. □

Recall that a *multiplier-bounded left approximate identity* for a Banach algebra $\mathcal{A}$ is a left approximate identity (e_i) for $\mathcal{A}$ such that $||e_i a|| \leq K||a||$, for a constant $K > 0$ and for all $a \in \mathcal{A}$. A multiplier-bounded *right* approximate identity is defined similarly.

Theorem 2.7. *Suppose that $\mathcal{A}$ is a boundedly approximately Connes-amenable dual Banach algebra, and has both a multiplier-bounded left approximate identity and a multiplier-bounded right approximate identity. Then $\mathcal{A}$ has an identity.*

Proof. Let (e_α) and (f_β) be, respectively, right and left multiplier-bounded approximate identities for $\mathcal{A}$. Then there exists a constant $K > 0$ such that for all $\alpha,\ \beta$ and $a \in \mathcal{A}$

$$||ae_\alpha|| \leq K||a|| \ \text{ and } \ ||f_\beta a|| \leq K||a||\ .$$

Then, by the definition of the projective tensor norm, we obtain

$$||f_\beta \,.\, m|| \leq K||m|| \ \text{ and } \ ||m \,.\, e_\alpha|| \leq K||m||$$

for every α, β and $m \in \mathcal{A}\hat{\otimes}\mathcal{A}$. Therefore, using the w^*-density of $\mathcal{A}\hat{\otimes}\mathcal{A}$ in $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ and the w^*-continuity of the actions of $\mathcal{A}$ on $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$, we have

$$||f_\beta \,.\, T|| \le K||T|| \ \text{ and } \ ||T \,.\, e_\alpha|| \le K||T|| \ ,$$

for every α, β and $T \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$.

Let (F_i), (G_i), (M_i) and C be the nets and the constant satisfying the conditions of Theorem 2.4. Towards a contradiction, suppose that the net (f_β) is norm unbounded. For every i and every β

$$||f_\beta \,.\, M_i - M_i \,.\, f_\beta - f_\beta \otimes G_i + F_i \otimes f_\beta|| \le C||f_\beta|| \ ,$$

hence by the above estimates we have

$$||(f_\beta \,.\, M_i - M_i \,.\, f_\beta - f_\beta \otimes G_i + F_i \otimes f_\beta) \,.\, e_\alpha|| \le KC||f_\beta|| \ ,$$

for every α, β and i.

Therefore, using left multiplier-boundedness of the net (f_β) we have

$$\begin{aligned} ||f_\beta||\ ||G_i \,.\, e_\alpha|| &\le ||f_\beta \otimes G_i \,.\, e_\alpha - f_\beta \,.\, M_i \,.\, e_\alpha + M_i \,.\, (f_\beta e_\alpha) - F_i \otimes f_\beta e_\alpha|| \\ &\quad + ||f_\beta \,.\, M_i \,.\, e_\alpha - M_i \,.\, (f_\beta e_\alpha) + F_i \otimes f_\beta e_\alpha|| \\ &\le KC||f_\beta|| + K||M_i \,.\, e_\alpha|| + K||M_i||\ ||e_\alpha|| + K||F_i||\ ||e_\alpha|| \end{aligned}$$

for every α, β and i. Hence

$$||G_i \,.\, e_\alpha|| \le KC + \frac{K}{||f_\beta||}(||M_i \,.\, e_\alpha|| + ||M_i||\ ||e_\alpha|| + ||F_i||\ ||e_\alpha||) \ ,$$

for every α, β and i.

For fixed α and i, the assumption that (f_β) is unbounded yields $||G_i \,.\, e_\alpha|| \le KC$. Taking limit with respect to i, we have $||e_\alpha|| \le KC$, for each α. Because (e_α) is a right approximate identity and (f_β) is a left-multiplier bounded set, we obtain

$$||f_\beta|| = \lim_\alpha ||f_\beta e_\alpha|| \le \lim_\alpha K||e_\alpha|| \le K^2C \ ,$$

for all β. This contradicts our assumption that the net (f_β) is unbounded.

Similarly, the net (e_α) is also bounded. Then with an standard argument we obtain a bounded approximate identity for $\mathcal{A}$. Taking a w^*-cluster point yields an identity. □

Theorem 2.8. *Suppose that $\mathcal{A}$ is a boundedly approximately Connes-amenable dual Banach algebra. Then $\mathcal{A}$ has an identity.*

Proof. From Theorem 2.4, there are nets $(F_\alpha)_\alpha, (G_\alpha)_\alpha \subseteq \sigma wc(\mathcal{A}^*)^*$ such that $a \,.\, F_\alpha \longrightarrow a$ and $G_\alpha \,.\, a \longrightarrow a$, for every $a \in \mathcal{A}$, and (F_α) and (G_α) are multiplier-bounded. Since $\mathcal{A} = (\mathcal{A}_*)^*$ is a dual Banach algebra, $\mathcal{A}_* \subseteq \sigma wc(\mathcal{A}^*)$, [18, Corollary 4.6]. Let $\imath : \mathcal{A}_* \longrightarrow \sigma wc(\mathcal{A}^*)$ be the canonical embedding, such that $\imath^*$ is an $\mathcal{A}$-bimodule morphism from $\sigma wc(\mathcal{A}^*)^*$ onto $\mathcal{A}$. Hence $(\imath^*(F_\alpha))_\alpha$ and $(\imath^*(G_\alpha))_\alpha$ are multiplier-bounded right and left approximate identities, respectively, for $\mathcal{A}$. Now, by Theorem 2.7, $\mathcal{A}$ has an identity. □

Let $\mathcal{A} = (\mathcal{A}_*)^*$ and $\mathcal{B} = (\mathcal{B}_*)^*$ be dual Banach algebras, then $\mathcal{A} \oplus^1 \mathcal{B}$ is a dual Banach algebra with predual $\mathcal{A}_* \oplus^\infty \mathcal{B}_*$.

Theorem 2.9. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are dual Banach algebras. Then $\mathcal{A} \oplus^1 \mathcal{B}$ is boundedly approximately Connes-amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are boundedly approximately Connes-amenable.*

Proof. By Theorem 2.8, $\mathcal{A}$ and $\mathcal{B}$ have identities. Hence the proof is similar to [9, Proposition 2.3 (ii)]. □

Theorem 2.10. *Suppose that $M(G)$ is the measure algebra of a locally compact group G. Then*

(i) $M(G)$ is boundedly approximately amenable if and only if G is discrete and amenable.

(ii) $M(G)$ is boundedly approximately Connes-amenable if and only if G is amenable.

Proof. The part (i) is immediate by [4] and [10, Theorem 3.1], and (ii) is obvious by [17] and [9, Theorem 5.2]. □

Example 2.11. Suppose that G is an amenable, non-discrete, locally compact group and $\mathbb{N}_\vee$ is the semigroup $\mathbb{N}$ with product $m \vee n = \max\{m, n\}$. By [2, Theorem 6.1], the dual Banach algebra $\ell^1(\mathbb{N}_\vee)$ is boundedly approximately amenable and so is boundedly approximately Connes-amenable. Hence, by Theorems 2.9 and 2.10 (ii), $\ell^1(\mathbb{N}_\vee) \oplus^1 M(G)$ is boundedly approximately Connes-amenable. However, using the canonical epimorphism $\ell^1(\mathbb{N}_\vee) \oplus^1 M(G) \longrightarrow M(G)$ and Theorems 2.6 and 2.10 (i), $\ell^1(\mathbb{N}_\vee) \oplus^1 M(G)$ is not boundedly approximately amenable.

Theorem 2.12. *Suppose that $\mathcal{A}$ is boundedly approximately Connes-amenable dual Banach algebra. If $\mathcal{A}$ is separable as a Banach space, then it is sequentially approximately Connes-amenable.*

Proof. Let $\{b_n \mid n \in \mathbb{N}\}$ be a countable dense subset of $\mathcal{A}$. Let E be a normal, dual Banach $\mathcal{A}$-bimodule and let $D : \mathcal{A} \longrightarrow E$ be a w^*-continuous derivation. Since $\mathcal{A}$ is boundedly approximately Connes-amenable there exists a $C > 0$ such that for each $n \in \mathbb{N}$, there is $x_n \in E$ such that

$$||Db_k - (b_k \,.\, x_n - x_n \,.\, b_k)|| < \frac{1}{n} \quad (k = 1, ..., n)$$

and

$$||a\ .\ x_n - x_n\ .\ a|| \leq C||a|| \quad (a \in \mathcal{A})\ .$$

This shows that the sequence $(x_n) \subseteq E$ satisfies

$$Db_k = \lim_{n \longrightarrow \infty} (b_k\ .\ x_n - x_n\ .\ b_k) \quad (k = 1, 2, ...)\ ,$$

and the sequence (ad_{x_n}) is bounded in $\mathcal{L}(\mathcal{A}, E)$. Now, for $a \in \mathcal{A}$, without loss of generality, we may assume that $\lim_{k \longrightarrow \infty} b_k = a$. Hence $Da = \lim_{n \longrightarrow \infty} a\ .\ x_n - x_n\ .\ a$, for all $a \in \mathcal{A}$. Whence $\mathcal{A}$ is sequentially approximately Connes-amenable. □

Finally, in the light of [11, Propositions 6.1 and 6.3], we can improve some results in [9] concerning the approximate Connes-amenability of the direct sum of dual Banach algebras. We summarize them in the following remark.

Remark 2.13. (i) Suppose that $\mathcal{A}$ and $\mathcal{B}$ are approximately Connes-amenable dual Banach algebras and that one of them say $\mathcal{B}$ has an identity $e_{\mathcal{B}}$. Then $\mathcal{A} \oplus^1 \mathcal{B}$ is approximately Connes-amenable as well. To see this, suppose that E is a normal, dual Banach $\mathcal{A} \oplus^1 \mathcal{B}$-bimodule, and $D : \mathcal{A} \oplus^1 \mathcal{B} \longrightarrow E$ is a w^*-continuous derivation. We extend the module actions on E by defining $e\ .\ x := x - e_{\mathcal{B}}\ .\ x$ and $x\ .\ e := x - x\ .\ e_{\mathcal{B}}$ for $x \in E$, where e is the identity of $\mathcal{A}^\sharp$. Then E is a normal, dual Banach $\mathcal{A}^\sharp \oplus^1 \mathcal{B}$-bimodule. Next, we extend D to a w^*-continuous derivation from $\mathcal{A}^\sharp \oplus^1 \mathcal{B}$ into E by defining $D(e) := -D(e_{\mathcal{B}})$. Hence D is inner by [9, Proposition 2.3 (ii)].

(ii) Suppose that $\mathcal{A}$ is a dual Banach algebra such that $\mathcal{A} \oplus^1 \mathcal{A}$ is approximately Connes-amenable. A more or less verbatim copy of the argument given in the proof of [11, Proposition 6.3] then shows that $\mathcal{A}$ has a two-sided approximate identity.

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