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THE SYMMETRIC TWO-STEP P-STABLE NONLINEAR PREDICTOR-CORRECTOR METHODS FOR THE NUMERICAL SOLUTION OF SECOND ORDER INITIAL VALUE PROBLEMS

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Abstract. In this paper, we propose a modification of the second order method introduced in [Q. Li and X. Y. Wu, A two-step explicit P-stable method for solving second order initial value problems, *Appl. Math. Comput.* 138 (2003), no. 2-3, 435–442] for the numerical solution of IVPs for second order ODEs. The numerical results obtained by the new method for some problems show its superiority in efficiency, accuracy and stability.
Keywords: Hybrid methods, P-stable, off-step points, predictor-corrector.
MSC(2010): Primary: 65L05; Secondary: 65L06.

1. Introduction

Let us consider the initial value problems of second order ordinary differential equations

$$y'' = f(x, y), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \tag{1.1}$$

where we presume that $f(x, y)$ is sufficiently differentiable and that the first derivative does not appear explicitly in $f(x, y)$. The numerical methods have been paid much attention to in recent years because the problems are usually encountered in celestial mechanics, quantum mechanical scattering theory, theoretical physics and chemistry, and electronics. Generally, the solution of (1.1) is periodic, so it is expected

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that the results produced by some numerical methods be of the periodicity of the analytic solution. In 1976, Lambert and Watson [11] proposed the concepts of periodicity interval and P-stability which can be used to discuss the stability of the numerical method for second order initial value problems. Although many P-stable methods have been proposed, such as linear multistep methods, high order hybrid P-stable methods, implicit Runge-Kutta-Nystrom and so on [10], these methods are implicit, so an iteration subprocess is needed in each step. The numerical integration methods for (1.1) can be divided into two distinct classes: (a) problems for which the solution period is known (even approximately) in advance (see [9, 15]); (b) problems for which the period is not known. There is a vast literature available for the numerical solution of this problem. Computational methods involving a parameter proposed by Sesappa Rai et al [16, 17], Shokri [18–20], Sommeijer [27], Vanden Berghe et al [30], Van Deale et al [28, 29] and Xiang [33] yield the numerical solution to the problem of the first class. Numerical treatment to the problems of the second class have been presented by Chawla et al [5, 6], Simos [21], Hollevoet et al [9], Ananthakrishnaiah [1, 2], and Chawla and Neta [4]. In 2001, the nonlinear explicit A-stable methods for the numerical solution of first order IVPs introduced by Wu and Xia [31]. Li and Wu [13] in 2003 gives the nonlinear explicit P-stable methods for second order IVPs, then Li and Wu [14], Stavroyiannis and Simos [25], Stavroyiannis and Simos [26] and Li [12] also have presented some modifications for these methods which have low algebraic orders, or have high computational costs. In this paper, we study an another modification of Wu's method with high algebraic order and simple structure.

Lambert and Watson [11], have developed linear, symmetric multistep methods of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+1-j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+1-j}, \quad k \geq 2, \tag{1.2}$$

where $h(> 0)$ is the step length of integration and $\alpha_j = \alpha_{k-j}$, $\beta_j = \beta_{k-j}$, $j = 0(1)k$, on the discrete point set $\{x_n : x_n = nh, n = 0, 1, \ldots\}$, for finding the numerical solution of the special initial value problem (1.1). They derive methods for $k = 2, 4$ and 6. Motivated by the idea, we will present the new two-step explicit P-stable methods of orders four and six for solving (1.1).

In recent years a class of explicit methods at high order for stiff problems is presented by some authors (see [7, 8, 22–25]) in which with the aid of a special vector operation, these methods can be extended to be vector-applicable [13]. Motivated by the idea, we have presented a class of two-step explicit symmetric P-stable methods for solving (1.1) [20]. This method has convergence of orders four and six.

2. **Preliminaries**

To investigate the stability properties of methods for solving the initial value problem (1.1), Lambert and Watson [11] introduced the scalar test equation

$$y'' = -\omega^2 y, \quad \omega \in \mathbb{R}. \tag{2.1}$$

When applying a symmetric two-step method to the test equation (2.1), one obtains the following difference equation of the form:

$$y_{n+1} - 2C(H)y_n + y_{n-1} = 0, \tag{2.2}$$

where $H = \omega h$ and h is a fixed step length, $C(H)$ is a rational polynomial with respect to H. The characteristic equation and polynomial are defined by the following respectively:

$$\xi^2 - 2C(H)\xi + 1 = 0, \tag{2.3}$$

$$Q(z, \xi) = Q_0(z^2)\xi^2 + Q_1(z^2)\xi + Q_2(z^2), \tag{2.4}$$

where $z = iwh$ and $Q_0(z^2)$, $Q_1(z^2)$, and $Q_2(z^2)$ are determined by the left side of (2.2) (see [7]).

Definition 2.1. *Let ξ_1, ξ_2 be the two roots of* (2.2)*, the method* (1.2) *is unconditionally stable if $|\xi_1| \leq 1$, $|\xi_2| \leq 1$ for all values of wh.*

Definition 2.2. *The interval $(0, H_0^2)$ is the periodicity interval of the method* (1.2) *if the roots of* (2.2) *satisfy $\xi_1 = \overline{\xi}_2 = e^{ig(H)}$, for all $H^2 \in (0, H_0^2)$, where $g(H)$ is a real function of H.*

Definition 2.3. *The method* (1.2) *is P-stable if the periodicity interval of the method is* $(0, +\infty)$.

Definition 2.4. *The order of the root of* (2.3) *(say ξ_1) is p if ξ_1 satisfies*

$$e^z - \xi_1 = Cz^{p+1} + O(z^{p+2}), \quad z \to 0, \tag{2.5}$$

where C is the error constant of ξ_1.

Theorem 2.5. *Suppose* (2.2) *is the characteristic equation of some method, and* $\mid C(H) \mid < 1$ *for all* $H^2 \in (0, H_0^2)$*, then the interval of periodicity of the method is* $(0, H_0^2)$*.*

Proof. See [11]. □

Theorem 2.6. *Set* $p \geq 1$*, the root of the characteristic polynomial of some method is of order p if and only if*

$$Q(z, e^z) = C\frac{\partial^2 Q}{\partial \xi^2}(0,1)z^{p+2} + O(z^{p+3}), \quad z \to 0. \tag{2.6}$$

Proof. See [8]. □

Lambert and Watson [11] have proved that the method described by (1.2) has a nonvanishing interval of periodicity only if it is symmetric and for P-stability the order cannot exceed 2. Further, the method is implicit. Later Chawla and Rao [6] noted that Numerov method has phase-lag error of $\frac{H^6}{480}$ and derived a Numerov type method of algebraic order four with minimal phase-lag $\frac{H^6}{12096}$ and having an periodicity interval (0, 2.71). This method is implicit and its implementation involves the computations of Jacobians and solution of nonlinear systems of equations. So subsequently, many authors proposed explicit modifications of Numerov method.

3. The new two-step nonlinear predictor-corrector methods

For the numerical integration of (1.1), we consider two-step, symmetric, methods of the form

$$y_{n+1} + y_{n-1} = 2y_n \exp\left(\frac{h^2 f_n + h^4 b\left(f''_{n+\alpha} + f''_{n-\alpha}\right)}{2y_n}\right), \tag{3.1}$$

where α, b are two arbitrary parameters, $0 < \alpha < 1$ and presume $y_n \neq 0$. Our aim in this work, is a modification of Wu's methods by finding nonlinear P-stable methods of higher-order with high derivatives and off-step points. The condition $y_n \neq 0$, is introduced for the first time in Wu's paper [13], because of the structure of the method, and it also exists in the modified papers [25, 26] by Stavroyiannis and Simos. Our method which is another modification of [13] also needs this condition. Indeed, this singularity inherits from the original method.

Formula (3.1) can only be used if we know the values of the solution $y(x)$ and $y''(x)$ at two successive points. These two values will be assumed to be given. Further, this equation is referred to as an explicit or predictor formula since y_{n+1} occurs only on one side of the equation. In other words, the unknown y_{n+1} can be calculated directly.

Now with the difference equation (3.1), we can associate the difference operator L defined next.

Definition 3.1. *Let the differential equation* (1.1) *have a unique solution* $y(x)$ *on* $[a,b]$ *and suppose that* $y(x) \in C^{(p+2)}[a,b]$ *for* $p \geq 1$. *Then the difference operator* L *for method* (3.1) *can be written as*

$$L[y(x),h]=y(x+h)+y(x-h) -2y(x)\exp\left(\frac{h^2y''(x)+h^4b\big(y^{(4)}(x+\alpha h)+y^{(4)}(x-\alpha h)\big)}{2y(x)}\right).$$

In order for the difference equation (3.1) to be useful for numerical integration, it is necessary that it would be satisfied to high accuracy by the solution of the differential equation $y''=f(x,y)$, when h is small for an arbitrary function $f(x,y)$. This imposes restrictions on the coefficients α and b. We assume that the function $y(x)$ has continuous derivatives at least of order 8.

We firstly use the Taylor series expansion to determine all the coefficients of (9) which can be written as

$$L[y(x),h]=C_0y(x_n)+C_1hy^{(1)}(x_n)+\cdots+C_ph^qy^{(p)}(x_n)+\cdots$$

where $C_0=C_1=C_2=C_3=0$, $C_4=\frac{1}{12}-2b$, $C_5=0$, $C_6=\frac{1}{360}-b\alpha^2$, $C_7=0$ and $C_8=\frac{1}{20160}-\frac{1}{12}b\alpha^4$.

Definition 3.2. *The method* (3.1) *is said to be of order* p *if,*

$$C_0=C_1=C_2=\cdots=C_{p+1}=0,\quad C_{p+2}\neq 0,$$

thus for any function $y(x)\in C^{(p+2)}$ *and for some nonzero constant* C_{p+2}*, we have*

$$L[y(x),h]=C_{p+2}h^{p+2}y^{(p+2)}(x_n)+O(h^{p+3}), \tag{3.2}$$

where C_{p+2} *is called the error constant.*

In particular, $L[y(x),h]$ vanishes identically when $y(x)$ is a polynomial whose degree is less than or equal to p.

Applying (3.1) to the scalar test equation (2.1), one gets its characteristic equation (2.3), where $H^2=(\omega h)^2$ and

$$C(H)=\exp(-\frac{1}{2}H^2+bH^4-\frac{1}{2}b\alpha^2H^6).$$

Theorem 3.3. *The method presented in* (3.1) *is P-stable if* $\sqrt{b} < \alpha < 1$.

Proof. In order to prove this theorem, we must provide conditions such that $| C(H) |< 1$ for every H^2. Therefore we discuss the behavior of

$$-\frac{1}{2}H^2 + bH^4 - \frac{1}{2}b\alpha^2H^6 < 0,$$

with considering α and b. That is, the restriction for α and b should be calculated in a way that P-stability is warranted. Then we have

$$-\frac{1}{2}H^2 + bH^4 - \frac{1}{2}b\alpha^2H^6 \quad = \quad H^2\Big(- \frac{1}{2} + bH^2 - \frac{1}{2}b\alpha^2H^4\Big) < 0,$$

since $H^2 > 0$, then we have

$$-\frac{1}{2} + bH^2 - \frac{1}{2}b\alpha^2H^4 < 0.$$

For this propose by assuming $H^2 = x$, the coefficient of x^2 and Δ from quadratic polynomial $\varphi(x) = -\frac{1}{2} + bx - \frac{1}{2}b\alpha^2x^2$ should be negative. Then we have

$$-\frac{1}{2} + bx - \frac{1}{2}b\alpha^2x^2 < 0, \quad \Rightarrow -\frac{1}{2}b\alpha^2 < 0 \quad \texttt{and} \quad \Delta < 0,$$

where $\Delta = b^2 - b\alpha^2 < 0$ and this means that b is positive and then we can write $\alpha^2 > b$ for all $0 < \alpha < 1$ and $-\frac{1}{2}b\alpha^2 < 0$. So $\alpha > \sqrt{b}$. Therefore, $\sqrt{b} < \alpha < 1$ and we will have $-\frac{1}{2} + bx - \frac{1}{2}b\alpha^2x^2 < 0$. That is say $| C(H) |< 1$, which warranties the P-stability of the method (3.1) and completes the proof. □

Theorem 3.4. *Method* (3.1) *is of order* 4 *if* $b = \frac{1}{24}$ *and* $\alpha \neq \sqrt{\frac{1}{15}}$ *and it is of order* 6 *if* $b = \frac{1}{24}$ *and* $\alpha = \sqrt{\frac{1}{15}}$.

Proof. Since $C_0 = C_1 = C_2 = C_3 = C_5 = 0$, $C_4 = \frac{1}{12} - 2b$, $C_5 = 0$, $C_6 = \frac{1}{360} - \frac{\alpha^2}{24}$, $C_7 = 0$ and $C_8 = \frac{1}{20160} - \frac{2\alpha^4}{576}$. Now if we take $b = \frac{1}{24}$ then $C_4 = 0$ and this means that our new method (3.1) has order at least 4. Furthermore, since $C_7 = 0$, if we take $b = \frac{1}{24}$, $\alpha = \sqrt{\frac{1}{15}}$ is only root of C_6 in $(0, 1)$. Then, if $\alpha \neq \sqrt{\frac{1}{15}}$, the order of the new method (3.1) is exactly 4 and if $\alpha = \sqrt{\frac{1}{15}}$, then $C_i = 0$, $i = 0, 1, \ldots, 7$ and $c_8 = \frac{31}{907200}$, so the order of (3.1) is 6 and in this case the local truncation error is

$$E_8 = \frac{31}{907200}h^8y^{(8)}(\zeta)y^{(6)}(\zeta),$$

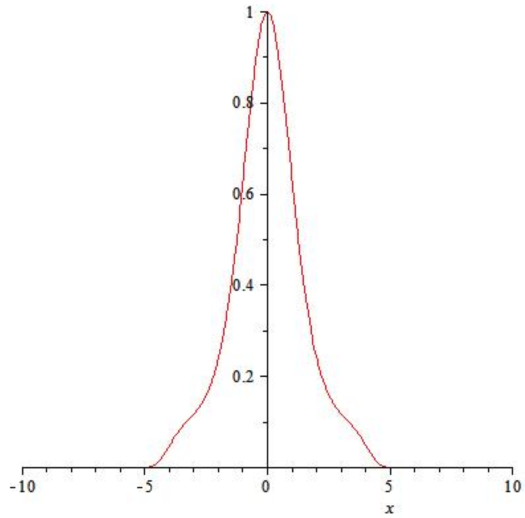

FIGURE 1. Behavior of the stability function $C(H)$, for $b = \frac{1}{24}$ and $\alpha = \sqrt{\frac{1}{15}}$.

and this completes the proof and the behaviour of the $C(H)$, in this case, is shown in Fig. 1. $\square$

By choosing $b = \frac{1}{24}$, we can write the new two-step, symmetric, P-stable method (3.1) as follow

(3.3)
$$y_{n+1} + y_{n-1} = 2y_n \exp\left(\frac{h^2 f_n + \frac{h^4}{24}\left(f''_{n+\alpha} + f''_{n-\alpha}\right)}{2y_n}\right), \quad \frac{1}{2\sqrt{6}} < \alpha < 1.$$

If we take $\alpha = \frac{1}{2}$, we have

$$y_{n+1} + y_{n-1} = 2y_n \exp\left(\frac{h^2 f_n + \frac{h^4}{24}(f''_{n+\frac{1}{2}} + f''_{n-\frac{1}{2}})}{2y_n}\right), \tag{3.4}$$

is the explicit two-step P-stable method of order 4. In the numerical experiment for (3.4), one obtains two more unknowns, $y_{n+\frac{1}{2}}$ and $y_{n-\frac{1}{2}}$, to be solved beside y_{n+1}. For this purpose, Simos [23] has used the $O(h^6)$ differentiation formula given by

$$\overline{y}_{n+1} = 2y_n - y_{n-1} + h^2 f_n, \quad f_n = f(x_n, y_n),$$

$$\overline{\overline{y}}_{n+1} = 2y_n - y_{n-1} + \frac{h^2}{12}\left(\overline{f}_{n+1} + 10f_n + f_{n-1}\right),$$

$$\widetilde{y}_{n+1} = 2y_n - y_{n-1} + \frac{h^2}{60}\left(\overline{\overline{f}}_{n+1} + 26f_n^{(k)} + f_{n-1} + 16\left(\overline{f}_{n+1/2} + \overline{f}_{n-1/2}\right)\right),$$

$$f_i = f(x_i, y_i), \quad i = n-1, n, \quad \overline{f}_{n+1} = f(x_{n+1}, \overline{y}_{n+1}),$$

$$y_{n+\frac{1}{2}} = \frac{1}{104}\left(5\overline{\overline{y}}_{n+1} + 146y_n - 47y_{n-1}\right) + \frac{h^2}{4992}\left(-59\overline{f}_{n+1} + 1438f_n + 253f_{n-1}\right),$$

$$y_{n-\frac{1}{2}} = \frac{1}{52}\left(3\overline{\overline{y}}_{n+1} + 20y_n + 29y_{n-1}\right) + \frac{h^2}{4992}\left(41\overline{f}_{n+1} - 682f_n - 271f_{n-1}\right).$$

Moreover its local truncation error is

$$E_6 = -\frac{11}{1440}h^6 y^{(6)}(\zeta).$$

If we take $\alpha = \sqrt{\frac{1}{15}}$, we have

$$y_{n+1} + y_{n-1} = 2y_n \exp\left(\frac{h^2 f_n + \frac{h^4}{24}\left(f''_{n+\sqrt{\frac{1}{15}}} + f''_{n-\sqrt{\frac{1}{15}}}\right)}{2y_n}\right), \tag{3.5}$$

is the explicit two-step P-stable method of order 6. In the numerical experiment for (3.5), one obtains two more unknowns, $y_{n+\sqrt{\frac{1}{15}}}$ and $y_{n-\sqrt{\frac{1}{15}}}$, to be solved beside y_{n+1}. For this purpose, Simos [23] has used the $O(h^6)$ differentiation formula given by

$$\begin{aligned}
y_{n+\sqrt{\frac{1}{15}}} \;=\; & \frac{1}{171360}\Bigg[112\sqrt{15}y_{n+1} + \frac{191195648}{1125}y_n + 112\sqrt{15}y_{n-1} \\
& - \frac{h^2}{\sqrt{15}}\Bigg[\left(\frac{509404}{\sqrt{15}} + \frac{503132}{25}\right)\widetilde{f}_{n+1} - \left(\frac{85479296}{16875}\sqrt{15}\right)f_n \\
& + \left(\frac{509404}{\sqrt{15}} - \frac{503132}{25}\right)f_{n-1} \\
& - 9\sqrt{3}\left(\frac{868}{16875}\sqrt{3}\sqrt{15} + \frac{8092}{75}\sqrt{7}\right)\overline{f}_{n+s} \\
& - 9\sqrt{3}\left(\frac{868}{16875}\sqrt{3}\sqrt{15} - \frac{8092}{75}\sqrt{7}\right)\overline{f}_{n-s}\Bigg]\Bigg],
\end{aligned}$$

and

$$
\begin{aligned}
y_{n-\sqrt{\frac{1}{15}}} &= \frac{1}{171360}\Bigg[112\sqrt{15}y_{n+1} + \frac{191195648}{1125}y_n + 112\sqrt{15}y_{n-1} \\
&- \frac{h^2}{\sqrt{15}}\Bigg[\left(\frac{509404}{\sqrt{15}} - \frac{503132}{25}\right)\widetilde{f}_{n+1} - \left(\frac{85479296}{16875}\sqrt{15}\right)f_n \\
&+ \left(\frac{509404}{\sqrt{15}} + \frac{503132}{25}\right)f_{n-1} \\
&- 9\sqrt{3}\left(\frac{868}{16875}\sqrt{3}\sqrt{15} - \frac{8092}{75}\sqrt{7}\right)\overline{f}_{n+s} \\
&- 9\sqrt{3}\left(\frac{868}{16875}\sqrt{3}\sqrt{15} + \frac{8092}{75}\sqrt{7}\right)\overline{f}_{n-s}\Bigg]\Bigg],
\end{aligned}
$$

where $s = \frac{\sqrt{21}}{3}$ and

$$
\begin{aligned}
\overline{y}_{n+s} &= \frac{1}{486}\Big[3\sqrt{7}y_{n+1}\left(7\sqrt{7} + 27\sqrt{3}\right) + 192y_n \\
&+ 3\sqrt{7}y_{n-1}\left(7\sqrt{7} - 27\sqrt{3}\right) + h^2\Big[2\sqrt{7}\widetilde{f}_{n+1}\left(9\sqrt{3} + 7\sqrt{7}\right) \\
&+ 224f_n + 2\sqrt{7}f_{n-1}\left(7\sqrt{7} - 9\sqrt{3}\right)\Big]\Big],
\end{aligned}
$$

$$
\begin{aligned}
\overline{y}_{n+s} &= \frac{1}{486}\Big[3\sqrt{7}y_{n+1}\left(7\sqrt{7} - 27\sqrt{3}\right) + 192y_n \\
&+ 3\sqrt{7}y_{n-1}\left(7\sqrt{7} + 27\sqrt{3}\right) + h^2\Big[2\sqrt{7}\widetilde{f}_{n+1}\left(9\sqrt{3} - 7\sqrt{7}\right) \\
&+ 224f_n + 2\sqrt{7}f_{n-1}\left(7\sqrt{7} + 9\sqrt{3}\right)\Big]\Big].
\end{aligned}
$$

4. **Numerical examples**

In this section, we present some numerical results obtained by our new nonlinear methods and compare them with those of other multistep methods.

Example 4.1. *Consider the initial value problem*

$$
\begin{cases} y'' = -\omega^2 y, \\ y(0) = 1, \quad y'(0) = 0, \end{cases}
$$

with the exact solution $y = \cos(\omega x)$. Set $\omega = 10$. Absolute errors in $y(x)$, with $h = \pi/200$, $\pi/400$, $\pi/800$ and $\pi/1600$, obtained by the new method (3.5), are listed in Tables 1 and 2 for comparison, where the other numerical results are from Li and Wu [13].

	New method (3.5)		Wu's method	
Point	$h = \frac{\pi}{200}$	$h = \frac{\pi}{400}$	$h = \frac{\pi}{200}$	$h = \frac{\pi}{400}$
5π	1.2898e-005	3.3073e-007	5.1541e-002	3.2539e-003
10π	6.5789e-005	4.4419e-007	2.0104e-001	1.3001e-002
15π	3.8678e-004	9.5241e-007	4.3307e-001	2.9117e-002
20π	7.1661e-004	1.2743e-006	7.2365e-001	5.1678e-002

Table 1. Absolute errors for the example 4.1, with $h = \pi/200$ and $h = \pi/400$, are calculated for comparison among two methods: Li and Wu [13] and our new method (3.5).

	New Method (3.5)		Wu's Method	
Point	$h = \frac{\pi}{800}$	$h = \frac{\pi}{1600}$	$h = \frac{\pi}{800}$	$h = \frac{\pi}{1600}$
5π	2.1930e-008	3.3093e-010	2.0362e-004	1.2731e-005
10π	5.1391e-008	7.4533e-010	8.1460e-004	5.0928e-005
15π	1.7547e-007	4.6950e-009	1.8327e-003	1.1459e-004
20π	8.6830e-007	8.9078e-009	3.2575e-003	2.0372e-004

Table 2. Absolute errors for the example 4.1, with $h = \pi/800$ and $h = \pi/1600$, are calculated for comparison among two methods: Li and Wu [13] and our new method (3.5).

Example 4.2. *Consider the initial value problem*

$$\begin{cases} y'' = 50y^3, \\ y(1) = 1/6, \quad y'(1) = -5/36, \end{cases}$$

with the exact solution $y(x) = 1/(1 + 5x)$. In the numerical experiment, we take the step length $h = 0.1$, 0.01, 0.001, and for simplicity, the true value at $x = 1 + h$ is taken as the second starting value. In Tables 3 and 4, we present the absolute errors at the points $x = 5$, 10, 15, 20.

Example 4.3. *Consider the two-body problem*

$$\begin{cases} y_1'' = -\frac{y_1}{r^3}, \quad y_1(0) = 1, \quad y_1'(0) = 0, \\ y_2'' = -\frac{y_2}{r^3}, \quad y_2(0) = 0, \quad y_2'(0) = 1, \end{cases}$$

Method (3.5)			
x	$h = 0.1$	$h = 0.01$	$h = 0.001$
5	1.3021e-006	3.3698e-008	1.2136e-011
10	5.1254e-006	9.2586e-008	8.2149e-011
15	2.0114e-005	3.2158e-007	3.3274e-010
20	7.2365e-005	6.3655e-007	7.1023e-010

Table 3. Absolute errors for the example 4.2, with $h = 0.1$, $h = 0.01$ and $h = 0.001$, are calculated for comparison among two methods: Li and Wu [13] and our new method (3.5).

Wu's Method			
x	$h = 0.1$	$h = 0.01$	$h = 0.001$
5	2.4119e-003	3.0515e-005	3.1215e-007
10	1.6102e-002	2.3060e-004	2.3630e-006
15	4.0043e-002	7.5739e-004	7.8201e-006
20	1.5401e-001	1.7428e-003	1.8350e-005

Table 4. Absolute errors for the example 4.2, with $h = 0.1$, $h = 0.01$ and $h = 0.001$, are calculated for comparison among two methods: Li and Wu [13] and our new method (3.5).

where $r = \sqrt{y_1^2 + y_2^2}$. The true solution is $y_1(x) = \cos(x)$ and $y_2(x) = \sin(x)$. In the numerical experiment, we take the step length $h = \pi/200$, $\pi/400$, $\pi/800$, $\pi/1600$, and for simplicity, the true value at $x = h$ is taken as the second starting value. In Tables 5 and 6, we present the absolute errors (infinite norm) at the points $x = 50\pi$, 100π, 150π, 200π.

Example 4.4. *We consider the following almost periodic problem studied by Stiefel and Bettis*

$$z''(t) + z(t) = 0.001e^{it}, \quad z(0) = 1, \quad z'(0) = 0.9995i, \quad z \in \mathbb{C}.$$

Its exact solution is $z(t) = (1.0.0005it)e^{it}$. If we set $z(t) = u(t)+iv(t)$, $u, v \in \mathbb{R}$, then the problem can be rewritten in the equivalent form

$$u'' + u = 0.001\cos(t), \quad u(0) = 1, \quad u'(0) = 0,$$

	New Method (3.5)		Wu's Method	
Point	$h = \frac{\pi}{200}$	$h = \frac{\pi}{400}$	$h = \frac{\pi}{200}$	$h = \frac{\pi}{400}$
50π	2.2361e-006	1.2320e-008	1.2898e-002	3.3073e-003
100π	6.1425e-006	5.2036e-008	2.5789e-002	6.4419e-003
150π	2.1231e-005	3.2148e-007	3.8678e-002	9.5241e-003
200π	5.1210e-005	6.2598e-007	5.1561e-002	1.2743e-002

Table 5. Absolute errors for the example 4.3, with $h = \pi/200$ and $h = \pi/400$ are calculated for comparison among two methods: Li and Wu [13] and our new method (3.5).

	New Method (3.5)		Wu's Method	
Point	$h = \frac{\pi}{800}$	$h = \frac{\pi}{1600}$	$h = \frac{\pi}{800}$	$h = \frac{\pi}{1600}$
50π	2.1245e-010	2.1489e-011	5.1930e-004	3.3093e-004
100π	6.2578e-010	8.1436e-011	1.1391e-003	1.4533e-004
150π	2.1478e-009	1.0236e-010	1.7547e-003	4.6950e-004
200π	8.1002e-009	9.2589e-010	2.6830e-003	3.9078e-004

Table 6. Absolute errors for the example 4.3, with $h = \pi/800$ and $h = \pi/1600$ are calculated for comparison among two methods: Li and Wu [13] and our new method (3.5).

$$v'' + v = 0.001\sin(t), \quad v(0) = 0, \quad v'(0) = 0.9995,$$

with the exact solution

$$u(t) = \cos(t) + 0.0005t\sin(t),$$

and

$$v(t) = \sin(t) - 0.0005t\cos(t).$$

The solution for z_k was computed with the step sizes of $h = \pi/10$, $\pi/20$ and $\pi/40$ and in the range of $0 \leq x \leq 100\pi$. The absolute errors are listed in table 7 for comparison with the eighth-order Runge-Kutta-Nyström of Dormand [7], the eighth-order hybrid method of Simos and et al [3] and our new method (3.5).

h	Dormand	Simos	New Method (3.5)
$\pi/10$	2.6e-2	6.8e-3	4.6e-9
$\pi/20$	1.1e-4	9.8e-6	2.1e-10
$\pi/40$	4.2e-7	1.3e-8	3.2e-12

TABLE 7. Comparison of the absolute errors in the approximations using the new method (3.5), eighth-order Runge-Kutta method of Dormand [7] and eighth-order method of Simos and et al [3] for example 4.4.

5. **Conclusions**

In this paper, we have presented the new two-step P-stable nonlinear predictor-corrector method of orders 4 and 6. The details of the procedure adapted for the applications have been given in Section 3. With high derivatives and two symmetric off-step points, we have improved the algebraic order of Wu's method [13] up to four and six. The numerical results obtained by the new method for some problems show its superiority in efficiency, accuracy and stability.

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