ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 41 (2015), No. 1, pp. 87–100

Title:

The influence of S-embedded subgroups on the structure of finite groups

Author(s):

J. Li, W. Shi and D. Yu

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 1, pp. 87–100
Online ISSN: 1735-8515

THE INFLUENCE OF S-EMBEDDED SUBGROUPS ON THE STRUCTURE OF FINITE GROUPS

J. LI, W. SHI* AND D. YU

(Communicated by Ali Reza Ashrafi)

Abstract. Let H be a subgroup of a group G. H is said to be S-embedded in G if G has a normal subgroup T such that HT is an S-permutable subgroup of G and $H \cap T \leq H_{sG}$, where H_{sG} denotes the subgroup generated by all those subgroups of H which are S-permutable in G. In this paper, we investigate the influence of minimal S-embedded subgroups on the structure of finite groups. We determine the structure of finite groups with some minimal S-embedded subgroups. We also give some new characterizations of p-nilpotency of finite groups in terms of the S-embedding property. As applications, some previously known results are generalized.
Keywords: Finite groups, S-embedded subgroups, the generalized Fitting subgroups, soluble groups, p-nilpotent groups.
MSC(2010): Primary: 20D10; Secondary: 20D15, 20D20, 20D25.

1. Introduction

Throughout this paper, all groups considered are finite.

Recall that a minimal subgroup of a group is a subgroup of prime order. It is an interesting topic in finite group theory to determine the structure of a group G whose minimal subgroups are well-situated in G. The following theorem due to Gaschütz and Itô [13, Theorem 5.7] shows that groups whose minimal subgroups are normal are soluble of a special nature: *Let G be a group such that all minimal subgroups of G are normal in G. Then G is soluble and its commutator group G' has a normal Sylow 2-subgroup with nilpotent factor group.* Furthermore,

Article electronically published on February 15, 2015.
Received: 2 April 2013, Accepted: 15 December 2013.
*Corresponding author.

Buckley in [3] proved that if every minimal subgroup of a group G of odd order is normal in G, then G is supersoluble. Later on, many authors have investigated the structure of groups whose minimal subgroups have some embedding properties. For example, in [19–21], Skiba gives some new characterizations of hypercyclically embedded subgroups in terms of minimal subgroups which possess some given embedding properties. The present paper is a contribution to the study of groups with some minimal subgroups which have the following S-embedding property.

In recent years, Guo, Shum and Skiba in [6–10, 18] have introduced a series of generalized permutable subgroups and gave many new characterizations about the solubility and the supersolubility of finite groups. One example is the concept of S-embedded subgroups ([10]). Let G be a group and let H be a subgroup of G. H is said to be S-permutable in G if H permutes with every Sylow subgroup of G. By [10], H is said to be S-embedded in G if G has a normal subgroup T such that HT is an S-permutable subgroup of G and $H \cap T \leq H_{sG}$, where H_{sG} denotes the subgroup generated by all those subgroups of H which are S-permutable in G. Some important results have been obtained by Guo, Shum and Skiba in [10, 11].

In this paper, we mainly focus our attention on groups with some S-embedded minimal subgroups and prove the solubility of these groups. In particular, we determine the structure of a group G with some minimal subgroups which are S-embedded in G. On the other hand, we give some new characterizations of p-nilpotency of finite groups by means of S-embedded subgroups.

2. **Preliminaries**

A class of groups $\mathfrak{F}$ is said to be a formation if $\mathfrak{F}$ is a homomorph and every group G has a smallest normal subgroup (denoted by $G^{\mathfrak{F}}$) whose quotient is still in $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be s-closed if every subgroup of G belongs to $\mathfrak{F}$ whenever $G \in \mathfrak{F}$. A formation $\mathfrak{F}$ is said to be saturated if $G/\Phi(G) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$. A chief factor H/K of a group G is said to be $\mathfrak{F}$-central (or $\mathfrak{F}$-eccentric) in G if $[H/K](G/C_G(H/K)) \in \mathfrak{F}$ (or $[H/K](G/C_G(H/K)) \notin \mathfrak{F}$, respectively). In this paper, $Z_\infty^{\mathfrak{F}}(G)$ denotes the $\mathfrak{F}$-hypercenter of a group G, that is, the product of all such normal subgroups H of G whose G-chief factors are $\mathfrak{F}$-central. Let $\mathfrak{U}$ and $\mathfrak{N}$ denote the class of all supersoluble groups and the class of all nilpotent groups respectively. As usual, $Z_\infty(G)$ denotes the hypercenter of a group G. The other notation and terminologies are standard and the reader is referred to [12, 14] if necessary.

The following two lemmas are well known.

Lemma 2.1. *Let G be a group and $A \leq G$. Let $\mathfrak{F}$ be a non-empty saturated formation and $Z = Z_\infty^{\mathfrak{F}}(G)$. Then*

(1) *If A is normal in G, then $AZ/A \leq Z_\infty^{\mathfrak{F}}(G/A)$.*

(2) *If $\mathfrak{F}$ is s-closed, then $Z \cap A \leq Z_\infty^{\mathfrak{F}}(A)$.*

(3) *If $G \in \mathfrak{F}$, then $Z = G$.*

Lemma 2.2. [11, Corollary 3.2.9] *Let $\mathfrak{F}$ be a saturated formation and G a group. Then*

$$[G^{\mathfrak{F}}, Z_\infty^{\mathfrak{F}}(G)] = 1.$$

Lemma 2.3. [9, Lemma 2.1] *Let G be a group and $H \leq K \leq G$.*

(1) *If H is S-embedded in G, then H is S-embedded in K.*

(2) *Suppose that H is normal in G. Then the subgroup HE/H is S-embedded in G/H for every S-embedded subgroup E in G satisfying $(|H|, |E|) = 1$.*

(3) *If H is S-embedded in G and K is normal in G, then G has a normal subgroup T such that $HT \leq K$ is S-permutable in G and $H \cap T \leq H_{sG}$.*

Lemma 2.4. [9, Lemma 2.2] *Let P be a normal p-subgroup of a group G. Suppose that P is of exponent p and every minimal subgroup of P is S-embedded in G. Then $P \leq Z_\infty^{\mathfrak{U}}(G)$.*

Lemma 2.5. *Let p be an odd prime and P be a normal p-subgroup of a group G such that every minimal subgroup of P is S-embedded in G. Then $P \leq Z_\infty^{\mathfrak{U}}(G)$.*

Proof. Assume that the result is false and consider a counterexample (G, P) for which $|G||P|$ is minimal. Let P/R be a chief factor of G. Then $R \neq 1$ by Lemma 2.4. By Lemma 2.3(3), (G, R) satisfies the hypothesis and so $R \leq Z_\infty^{\mathfrak{U}}(G)$ and P/R is not cyclic by the choice of G. Let N be any normal subgroup of G with $N < P$. Similarly, $N \leq Z_\infty^{\mathfrak{U}}(G)$. If N is not contained in R, then $P/R = NR/R \simeq N/N \cap R$ and therefore $P \leq Z_\infty^{\mathfrak{U}}(G)$, a contradiction. Hence $N \leq R$. Now, we conclude that G has a normal subgroup R such that P/R is a non-cyclic chief factor of G, $R \leq Z_\infty^{\mathfrak{U}}(G)$ and $N \leq R$ for any normal subgroup N of G contained in P with $N \neq P$. By [4, Ch.5, Theorem 3.13], P possesses a characteristic subgroup D of exponent p such that every nontrivial

p'-automorphism of P induces a nontrivial automorphism of D. This implies that $C_G(D)/C_G(P)$ is a p-group. Hence, by Lemma 2.4, we have $D < P$. Thus $D \leq Z_\infty^{\mathfrak{U}}(G)$. Let $1 = D_0 < D_1 < \cdots < D_t = D$ be a G-chief series of D. Let $C_i = C_G(D_i/D_{i-1})$ and $C = C_1 \cap C_2 \cap \cdots \cap C_t$. Then $C_G(D) \leq C$ and $C/C_G(D)$ is a p-group (see [4, Ch.5, Theorem 3.2]). Furthermore, since D_i/D_{i-1} is of order p, G/C is abelian of exponent $p-1$. Now we conclude that $G/C_G(P/R)$ is abelian of exponent dividing $p-1$ as $C_G(P) \leq C_G(P/R)$ and $O_p(G/C_G(P/R)) = 1$ (see [11, Lemma 1.7.11] or [3, Ch.A, Lemma 13.6]). Hence, by [24, Ch.1, Theorem 1.4], $|P/R| = p$, a contradiction. Thus, the proof is completed. □

Lemma 2.6. [13, Ch.6, Theorem 14.3] *Let P be an abelian Sylow subgroup of a group G. Then $G' \cap Z(G) \cap P = 1$.*

A group G is called quasinilpotent if for any chief factor H/K of G, every automorphism of H/K induced by an element of G is inner. The generalized Fitting subgroup $F^*(G)$ of a group G is the product of all normal quasinilpotent subgroups of G. The following well known facts about the generalized Fitting subgroup of a group G will be used in our proofs (see [14, Chapter X]).

Lemma 2.7. *Let G be a group. Then*

(1) *If N is a normal subgroup of G, then $F^*(N) = N \cap F^*(G)$.*

(2) *$F(G) \leq F^*(G) = F^*(F^*(G))$. If $F^*(G)$ is soluble, then $F^*(G) = F(G)$.*

(3) *$C_G(F^*(G)) \leq F^*(G)$.*

(4) *G is quasinilpotent if and only if $G/Z_\infty(G)$ is semisimple.*

Lemma 2.8. [18, Theorem B] *Let $\mathfrak{F}$ be any formation and G a group. If N is a normal subgroup of G and $F^*(N) \leq Z_\infty^{\mathfrak{F}}(G)$, then $N \leq Z_\infty^{\mathfrak{F}}(G)$.*

Lemma 2.9. *Let P be a nontrivial 2-group and H a nontrivial automorphism group of P fixing the involutions of P. If H is cyclic of odd order and H acts irreducibly on $P/\Phi(P)$, then $|P| = 2^{3s}$, $|\Phi(P)| = 2^s$ with $s \geq 1$, $P' = \Phi(P) = Z(P) = \Omega_1(P)$ and $|H|$ divides $2^s + 1$.*

Proof. See Theorems 1.3 and 2.2 in [13]. □

3. **Main results**

In this section, we first characterize the structure of groups whose minimal subgroups are S-embedded.

Theorem 3.1. *Let G be a group. If every minimal subgroup of G is S-embedded in G, then G is soluble.*

Proof. Suppose that the result is false and let G be a counterexample of minimal order. We proceed the proof by the following steps.

(1) Every proper subgroup of G is soluble.

Let M be a proper subgroup of G. Then, by Lemma 2.3(1), every minimal subgroup of M is S-embedded in M and therefore M satisfies the hypothesis. The minimal choice of G yields that M is soluble.

(2) G is not a non-abelian simple group.

Assume that G is a non-abelian simple group and let L be a minimal subgroup of G with $|L| = p$. Then, by the hypothesis, G has a normal subgroup T such that LT is S-permutable in G and $L \cap T \leq L_{sG}$. Since G is simple, $T = 1$ or G, which implies that L is S-permutable in G. Hence $L \leq O_p(G)$, a contradiction. Thus, (2) holds.

(3) $\overline{G} = G/\Phi(G)$ is a minimal simple group.

By (2), suppose that N is any nontrivial proper normal subgroup of G. Let M be any maximal subgroup of G. By (1), both N and M are soluble. If N is not contained in M, then $G = MN$ and so $G/N \simeq M/M \cap N$ is soluble. It follows that G is soluble, a contradiction. Hence $N \leq \Phi(G)$ and therefore (3) holds.

(4) Final contradiction.

By (3) and the well-known result of Thompson, $\overline{G}$ is isomorphic to one of the following groups:

(i) $L_2(p)$, $p > 3$ is a prime, and 5 does not divide $p^2 - 1$;

(ii) $L_2(3^r)$, r is an odd prime;

(iii) $L_2(2^r)$, r is a prime;

(iv) $Sz(2^r)$, r is an odd prime;

(v) $L_3(3)$.

By [13, Ch.II, Theorem 8.10], [25, p.117, Theorem 4.1] and the order of $L_3(3)$, we know that for some odd prime $t \in \pi(\overline{G})$, the Sylow t-subgroups of $\overline{G}$ are cyclic. We assert that $t \notin \pi(\Phi(G))$. Otherwise, let P be a Sylow t-subgroup of $\Phi(G)$ and G_t be a Sylow t-subgroup of G. Then G_t/P is cyclic. By Lemma 2.5, we see that $P \leq Z_{\infty}^{\mathfrak{U}}(G)$. Since $G/C_G(P)$ is supersoluble by Lemma 2.2, $C_G(P) = G$ by (1). Therefore $P \leq Z(G)$ and so G_t is abelian. Moreover, by (3), we have that $G = G'$. It follows that $P \cap Z(G) \cap G' = P$, which contradicts Lemma 2.6. Hence $\Phi(G)$ is a t'-group. Let $L/\Phi(G)$ be a minimal subgroup of $\overline{G}$ of order t. Then, by the preceding argument, we have that $L/\Phi(G) = \langle x \rangle \Phi(G)/\Phi(G)$ for some $x \in G$ with $|x| = t$. By Lemma 2.3(2), $L/\Phi(G)$ is S-embedded in

$\overline{G}$. Similar to (2), we derive a contradiction, finishing the proof of this part. □

Remark 3.2. (1) The converse of Theorem 3.1 is not true in general. For example, let $G = A_4$, the alternating group of degree 4. Then G is soluble. But any minimal subgroup of G is not S-embedded in G because G has no subgroup of order 6.

(2) The condition in Theorem 3.1 that "every minimal subgroup of G is S-embedded in G" can not be replaced by "every minimal subgroup of each non-cyclic Sylow subgroup of G is S-embedded in G". Let $G = SL(2,5)$. Then every minimal subgroup of each non-cyclic Sylow subgroup of G is normal in G. But G is a quasisimple group.

With respect to this example, the following question seems interesting.

Question 3.3. *Let G be a group such that every minimal subgroup of each non-cyclic Sylow subgroup of G is S-embedded in G. What can we say about the structure of G?*

Remark 3.4. The condition in Theorem 3.1 cannot guarantee the supersolubility of G. Let $G = [Q_8]Z_3$, the semi-direct product of Q_8 by Z_3, where Q_8 is the quaternion group of order 8 and Z_3 is cyclic of order 3. Then every minimal subgroup of G is S-embedded in G, but G is not supersoluble.

Note that in the above example G is a minimal non-nilpotent group. In fact, we obtain the following more general result.

Theorem 3.5. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and N be a normal subgroup of a group G such that $G/N \in \mathfrak{F}$. If every minimal subgroup of $F^*(N)$ is S-embedded in G, then either $G \in \mathfrak{F}$ or G contains a minimal non-nilpotent subgroup K satisfying the following properties:*

(i) *K has a nontrivial normal Sylow 2-subgroup K_2 such that $K_2 \leq O_2(G)$;*

(ii) *$|K_2| = 2^{3s}$ and $|\Phi(K_2)| = 2^s$, where $s \geq 1$;*

(iii) *$K_2' = \Phi(K_2) = Z(K_2) = \Omega_1(K_2)$;*

(iv) *If $2 \neq p \in \pi(K)$, then p divides $2^s + 1$.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. Then

(1) $F = F^*(N) = F(N)$.

By Lemma 2.3(1), every minimal subgroup of $F^*(N)$ is S-embedded in $F^*(N)$. It follows from Theorem 3.1 that $F^*(N)$ is soluble. Hence $F^*(N) = F(N)$ by Lemma 2.7(2).

(2) $2 \in \pi(F)$.

If not, then F is of odd order. By Lemma 2.5, $F^*(N) = F(N) \leq Z_\infty^{\mathfrak{U}}(G)$ and consequently $F^*(N) \leq Z_\infty^{\mathfrak{F}}(G)$ by the hypothesis. Applying Lemma 2.8, we have that $N \leq Z_\infty^{\mathfrak{F}}(G)$. Hence $G \in \mathfrak{F}$ by the hypothesis, a contradiction. Thus, (2) holds.

(3) Conclusion.

We first claim that there exists a Sylow p-subgroup P of G such that F_2P is not 2-nilpotent, where $p \neq 2$ and F_2 is the Sylow 2-subgroup of F. Suppose this is false. Then $F_2 \leq Z_\infty(G) \leq Z_\infty^{\mathfrak{U}}(G)$. By Lemma 2.5, every Sylow subgroup of F of odd order is also contained in $Z_\infty^{\mathfrak{U}}(G)$. Therefore $F^*(N) = F(N) \leq Z_\infty^{\mathfrak{U}}(G)$. As in (2), we have that $G \in \mathfrak{F}$, a contradiction. Hence G has a Sylow p-subgroup P such that F_2P is not 2-nilpotent, where p is an odd prime. Then F_2P contains a minimal non-2-nilpotent subgroup K. By [13, Ch.IV, Theorem 5.4], K is a minimal non-nilpotent group. It follows from [11, Theorem 3.4.11] that K satisfies

(i) $K = [K_2]K_p$, where K_2 is normal in K with $K_2 = K^{\mathfrak{N}}$ and K_p is a cyclic Sylow p-subgroup of K with $p \neq 2$;

(ii) $K_2/\Phi(K_2)$ is a chief factor of K.

Obviously $K_2 \leq F_2 \leq O_2(G)$. By (ii), K_p acts irreducibly on $K_2/\Phi(K_2)$. Now we show that K_p fixes the involutions of K_2. Let x be any involution of K_2. Let $L = \langle x \rangle$. Then, by Lemma 2.3, L is S-embedded in K and so K has a normal subgroup T such that LT is S-permutable in K and $L \cap T \leq L_{sG}$. Set $V = K_2 \cap T$. Then LV is S-permutable in K and $L \cap V \leq L_{sG}$. If $V = 1$, then L is S-permutable in K and so K_p fixes x. Suppose that $V \neq 1$. If V is not contained in $\Phi(K_2)$, then, by (ii), $V = K_2$ and therefore L is S-permutable in K. As above, $K_p \leq C_G(x)$. Assume that $V \leq \Phi(K_2)$. If $LVK_p < K$, then K_p fixes x as LVK_p is nilpotent. If $LVK_p = K$, then $LV = K_2$, which implies that K_2 is cyclic, a contradiction. Hence, by the preceding argument, we see that K_p fixes all the involutions of K_2. By Lemma 2.9, $|K_2| = 2^{3s}$, $|\Phi(K_2)| = 2^s$, where $s \geq 1$, and $K_2' = \Phi(K_2) = Z(K_2) = \Omega_1(K_2)$; in addition, $|K_p/C_{K_p}(K_2)|$ divides $2^s + 1$ and so does p. This contradiction completes the proof. □

Recall that a subgroup H of a group G is said to be c-normal in G if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$, where H_G denotes the largest normal subgroup of G contained in H (see [22]). It is easy to see from the definition of S-embedded subgroups that all the normal subgroups, the S-permutable subgroups and the c-normal subgroups are S-embedded subgroups.

Corollary 3.6. (Buckley, [3]). *Let G be a group of odd order. If every minimal subgroup of G is normal in G, then G is supersoluble.*

Corollary 3.7. (Li, Wang, [16]). *Let G be a groups with a normal subgroup N such that G/N is supersoluble. If every cyclic subgroup of $F^*(N)$ of prime order or order 4 is S-permutable in G, then G is supersoluble.*

Proof. Assume that G is not supersoluble. Then, G contains a minimal non-nilpotent group K satisfying the properties (i)-(iii) in Theorem 3.5. Let L be any cyclic subgroup of K_2 of order 4. Then L is not contained in $\Omega_1(K_2)$. By the hypothesis, $LK_p = K_pL$, where K_p is a Sylow p-subgroup of K with $p \neq 2$. Clearly $LK_p < K$, so $K_p \leq C_G(L)$. This shows that K_p acts trivially on $\Omega_2(K_2)$. By the well-known Theorem Blackburn, $K_p \leq C_K(K_2)$, a contradiction. Hence G is supersoluble. □

Corollary 3.8. (Wei, Wang, Li, [24]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that G is a group with a normal subgroup N such that $G/N \in \mathfrak{F}$. If all minimal subgroups of $F^*(N)$ and all cyclic subgroups of $F^*(N)$ of order 4 are c-normal in G, then $G \in \mathfrak{F}$.*

Proof. Assume that $G \notin \mathfrak{F}$. Then G contains a minimal non-nilpotent group K with properties (i)-(iii) in Theorem 3.5. Let L be a cyclic subgroup of K_2 of order 4. Then L is not contained in $Z(K_2)$ by (iii). By the hypothesis, K has a normal subgroup T such that $K = LT$ and $L \cap T \leq L_G$. Note that $T < K$. If not, L is normal in K and since $C_K(L)$ does not contain K_2, $C_K(L)K_p$ is a proper subgroup of K, where K_p is a Sylow p-subgroup of K with $p > 2$. Therefore $K_p \leq C_K(L)$ and so K_p is normal in K, a contradiction. Hence $T < K$. But, since $K_p \leq T$, K_p is normal in K, also a contradiction. Thus, $G \in \mathfrak{F}$, as desired. □

The following part is devoted to investigating the influence of S-embedded subgroups on the p-nilpotency of groups.

Lemma 3.9. *Let G be a group with a normal subgroup N such that G/N is p-nilpotent, where $p \in \pi(G)$. Assume that every subgroup of N with order p is contained in $Z_\infty(G)$ and every cyclic subgroup of N of order 4 (if $p = 2$) not contained in $Z_\infty(G)$ is S-embedded in G. Then G is p-nilpotent.*

Proof. Assume that the assertion is false and let G be a counterexample of minimal order.

First, we claim that every proper subgroup of G is p-nilpotent. Let L be a proper subgroup of G. Since G/N is p-nilpotent, $L/L\cap N \simeq LN/N$ is p-nilpotent. On the other hand, if R is a cyclic subgroup of $L \cap N$ of order p, then $R \leq L \cap Z_\infty(G) \leq Z_\infty(L)$ by the hypothesis and Lemma 2.1. Besides, if R is a cyclic subgroup of $L\cap N$ of order 4 not contained in $Z_\infty(L)$, then R is S-embedded in L by the hypothesis and Lemmas 2.1 and 2.3. Thus L satisfies the hypothesis and so it is p-nilpotent by the minimality of G. Therefore every proper subgroup of G is p-nilpotent. Hence G is a minimal non-p-nilpotent group. By [13, Ch.IV, Theorem 5.4], G is a minimal non-nilpotent group. Then, by [11, Theorem 3.4.11], G has the following properties:

(i) $G = [P]Q$, where $P = G^{\mathfrak{N}}$ is the Sylow p-subgroup of G and Q is a Sylow q-subgroup of G with $p, q \in \pi(G)$ and $p \neq q$;

(ii) $P/\Phi(P)$ is a chief factor of G;

(iii) P is of exponent p or 4;

(iv) $\Phi(P) = P \cap \Phi(G)$ and $\Phi(G) = Z_\infty(G)$.

Write $\Phi = \Phi(P)$. Note that $P \leq N$. Otherwise, $P \cap N$ is a proper subgroup of P which is normal in G. Therefore $P \cap N \leq \Phi$ since by (ii), P/Φ is a chief factor of G. As the class of all p-nilpotent groups is a saturated formation, we have that $G/P \cap N$ is p-nilpotent. It follows that G is p-nilpotent [11, Lemma 1.8.1], which violates our initial assumption on G. Hence $P \leq N$. If P is of exponent p, then P is contained in $Z_\infty(G)$ by the hypothesis, from which we deduce that G is nilpotent, a contradiction. Hence $p = 2$ and P is of exponent 4 by (iii). If all cyclic subgroups of G of order 4 are contained in $Z_\infty(G)$, then P is also contained in $Z_\infty(G)$ by the hypothesis, a contradiction. Thus, there must exist a cyclic subgroup H of P of order 4 such that $H \nsubseteq Z_\infty(G)$. Note that H is also not contained in Φ since $\Phi \leq Z_\infty(G)$. By the hypothesis, H is S-embedded in G. First, if H is S-permutable in G, then $H\Phi/\Phi$ is S-permutable in G/Φ and so it is a normal subgroup of $(H\Phi/\Phi)(Q\Phi/\Phi)$. Since P/Φ is elementary abelian, $H\Phi/\Phi$ is also normalized by P/Φ and so $H\Phi/\Phi$ is normal in G/Φ. This induces that $P = H$, by which we have that G is nilpotent, a contradiction. Therefore, by the hypothesis and Lemma 2.3, G has a normal subgroup T such that $HT \leq P$ is S-permutable in G and $H \cap T \leq H_{sG} \neq H$. Clearly, $T \neq P$ and so $T\Phi \neq P$. This implies that $T \leq \Phi$. But then $H\Phi/\Phi = HT\Phi/\Phi$ is an S-permutable subgroup of G/Φ. Similarly as above, we have that $P = H$, which implies that G is nilpotent, a final contradiction finishing the proof. $\square$

Lemma 3.10. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{N}$. Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup N satisfying that:*

(i) *$G/N \in \mathfrak{F}$ and*

(ii) *for each $p \in \pi(N)$, every subgroup of N of order p is contained in $Z_\infty^{\mathfrak{F}}(G)$ and every cyclic subgroup of N with order 4 (if $p = 2$) not contained in $Z_\infty^{\mathfrak{F}}(G)$ is S-embedded in G.*

Proof. The necessity is evident and we only prove the sufficiency. Assume that the assertion is false let G be a minimal counterexample.

Since $G/N \in \mathfrak{F}$, $G^{\mathfrak{F}} \subseteq N$. By Lemma 2.2, we have that $Z_\infty^{\mathfrak{F}}(G) \cap G^{\mathfrak{F}} \subseteq Z(G^{\mathfrak{F}}) \subseteq Z_\infty(G^{\mathfrak{F}})$. Thus, every subgroup of $G^{\mathfrak{F}}$ of order prime is contained in $Z_\infty(G^{\mathfrak{F}})$ by the hypothesis. If $p = 2$, then every cyclic subgroup of $G^{\mathfrak{F}}$ of order 4 not contained in $Z_\infty(G^{\mathfrak{F}})$ is S-embedded in $G^{\mathfrak{F}}$ by the hypothesis and Lemmas 2.1 and 2.3. Lemma 3.9 suggests that $G^{\mathfrak{F}}$ is nilpotent. Let M be a maximal subgroup of G such that $G^{\mathfrak{F}} \nsubseteq M$. Then $G = MG^{\mathfrak{F}}$. Let $Z = Z_\infty^{\mathfrak{F}}(G) \cap M$. Since $[Z_\infty^{\mathfrak{F}}(G), G^{\mathfrak{F}}] = 1$ by Lemma 2.2, every G-chief factor H/K below Z is still an M-chief factor and $G^{\mathfrak{F}} \subseteq C_G(H/K)$. Hence $M/C_M(H/K) \simeq MC_G(H/K)/C_G(H/K) = G/C_G(H/K) \in \mathfrak{F}$. Therefore $Z \subseteq Z_\infty^{\mathfrak{F}}(M)$. Now, it is easy to see that M satisfies the hypothesis by Lemmas 2.1 and 2.3. Hence $M \in \mathfrak{F}$ by the choice of G. By [11, Theorem 3.4.2], G possesses the following properties:

(i) $G^{\mathfrak{F}}$ is a p-group, for some prime $p \in \pi(G)$;

(ii) $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is $\mathfrak{F}$-eccentric;

(iii) If $p > 2$, then $G^{\mathfrak{F}}$ is of exponent p, and if $p = 2$, then $G^{\mathfrak{F}}$ is of exponent 2 or 4.

Set $\Phi = \Phi(G^{\mathfrak{F}})$. Let A/Φ be a subgroup of $G^{\mathfrak{F}}/\Phi$ of order p which is normal in some Sylow p-subgroup of G/Φ. Then $A/\Phi = H\Phi/\Phi$, where H is a cyclic subgroup of $G^{\mathfrak{F}}$ of order p or 4. If $H \subseteq Z_\infty^{\mathfrak{F}}(G)$, then $A/\Phi = H\Phi/\Phi \leq G^{\mathfrak{F}}/\Phi \cap Z_\infty^{\mathfrak{F}}(G)\Phi/\Phi \leq G^{\mathfrak{F}}/\Phi \cap Z_\infty^{\mathfrak{F}}(G/\Phi)$ by Lemma 2.1. It follows that $G^{\mathfrak{F}}/\Phi \leq Z_\infty^{\mathfrak{F}}(G/\Phi)$ and so $G^{\mathfrak{F}}/\Phi$ is $\mathfrak{F}$-central, a contradiction. Thus, by the hypothesis and (iii), $p = 2$ and H is a cyclic subgroup of order 4 not contained in $Z_\infty^{\mathfrak{F}}(G)$. Therefore H is S-embedded in G by our assumption on G. By Lemma 2.3, G has a normal subgroup T contained in $G^{\mathfrak{F}}$ such that HT is S-permutable in G and $H \cap T \leq H_{sG}$. If H is S-permutable in G, then $A/\Phi = H\Phi/\Phi$ is S-permutable in G/Φ and so $O^2(G/\Phi) \leq N_G(A/\Phi)$. Since A/Φ is normal in some Sylow 2-subgroup of G/Φ, we obtain that A/Φ is normal in G/Φ, so that $A/\Phi = G^{\mathfrak{F}}/\Phi$ is an $\mathfrak{F}$-central chief factor, contrary to (ii) above. Hence $1 \neq T \leq \Phi$. But $H\Phi/\Phi = HT\Phi/\Phi$, which shows that A/Φ is S-permutable in G/Φ since HT is S-permutable in G. As above, one derive a contradiction. Thus, the proof is complete. □

Lemma 3.11. *A group G is nilpotent if and only if G has a normal subgroup N such that:*

(i) *G/N is nilpotent and*

(ii) *for each $p \in \pi(F^*(N))$, every subgroup of $F^*(N)$ of order p is contained in $Z_\infty(G)$ and every cyclic subgroup of $F^*(N)$ of order 4 (if $p = 2$) not contained in $Z_\infty(G)$ is S-embedded in G.*

Proof. The necessity part is obvious. Now we prove the sufficiency part.

Assume that this is not true and let G be a minimal counterexample. Suppose that M is a maximal normal subgroup of G. Clearly, $M/M \cap N \simeq MN/N$ is nilpotent because G/N is nilpotent. Since $F^*(M \cap N) \leq F^*(N)$ by Lemma 2.7, M satisfies the hypothesis by Lemmas 2.1 and 2.3. Hence M is nilpotent by the choice of G. It follows that $F(G)$ is the unique maximal normal subgroup of G and $G/F(G)$ is a non-abelian simple chief factor of G. Therefore, if $N < G$, then N is nilpotent and so $F^*(N) = F(N) = N$. Thus, by Lemma 3.9, G is nilpotent. This contradiction shows that $N = G$. If $F^*(N) = F^*(G) = G$, then G is nilpotent by Lemma 3.9 again, a contradiction. Thereby $F^*(G) < G$ and so $F^*(G) = F(G)$. Now let $G^{\mathfrak{N}}$ denote the nilpotent residual of G. Suppose that $G^{\mathfrak{N}} < G$. Then, since $F^*(G^{\mathfrak{N}}) = G^{\mathfrak{N}} \leq F^*(G)$, G is nilpotent by Lemma 3.9, a contradiction. This induces that $G^{\mathfrak{N}} = G$, especially $G = G'$. By Lemma 2.2, we have that $Z_\infty(G) \cap G^{\mathfrak{N}} \subseteq Z(G^{\mathfrak{N}}) = Z(G)$ and so $Z_\infty(G) = Z(G)$.

Now suppose that p is a prime dividing the order of $F^*(G)$ and let P be a Sylow p-subgroup of $F^*(G)$. Then P is normal in G. Let Q be a Sylow q-subgroup of G, where $q \neq p$. Put $L = PQ$. Then L is p-nilpotent by Lemma 3.9 and the hypothesis, and so $L = P \times Q$, i.e. $Q \leq C_G(P)$, from which we conclude that $O^p(G) \leq C_G(P)$. Hence $C_G(P) = G$ since $G^{\mathfrak{N}} = G$ and therefore $P \leq Z(G)$. Consequently $F^*(G) = F(G) \leq Z(G) = Z_\infty(G)$, which implies that $F(G) = Z_\infty(G)$ by the above arguments. It follows that $G/Z_\infty(G)$ is a non-abelian simple group. By Lemma 2.7, G is quasinilpotent and therefore $F^*(G) = G$, a contradiction. Thus, the proof of this lemma is complete. □

Theorem 3.12. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{N}$. Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup N satisfying that:*

(i) *$G/N \in \mathfrak{F}$ and*

(ii) *for each $p \in \pi(F^*(N))$, every subgroup of $F^*(N)$ of order p is contained in $Z_\infty^{\mathfrak{F}}(G)$ and every cyclic subgroup of $F^*(N)$ with order 4 (if $p = 2$) not contained in $Z_\infty^{\mathfrak{F}}(G)$ is S-embedded in G.*

Proof. The necessity is clear and we need only prove the sufficiency. Obviously, $G^{\mathfrak{F}} \subseteq N$. Hence $F^*(G^{\mathfrak{F}}) \subseteq F^*(N)$ by Lemma 2.7(1). Besides, $Z_\infty^{\mathfrak{F}}(G) \cap G^{\mathfrak{F}} \subseteq Z(G^{\mathfrak{F}}) \subseteq Z_\infty(G^{\mathfrak{F}})$ by Lemma 2.2. Therefore every subgroup of $F^*(G^{\mathfrak{F}})$ of prime order is contained in $Z_\infty(G^{\mathfrak{F}})$ and every cyclic subgroup of $F^*(G^{\mathfrak{F}})$ of order 4 (if $p = 2$) not contained in $Z_\infty(G^{\mathfrak{F}})$ is S-embedded in $G^{\mathfrak{F}}$ by the hypothesis and Lemmas 2.1 and 2.3. Hence $G^{\mathfrak{F}}$ is nilpotent by Lemma 3.11 and therefore $F^*(G^{\mathfrak{F}}) = G^{\mathfrak{F}}$. It follows from Lemma 3.10 that $G \in \mathfrak{F}$, as desired. □

Now, we present some applications of Theorem 3.12.

Corollary 3.13. (Ballester-Bolinches, Wang, [1]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{N}$. Suppose that every cyclic subgroup of $G^{\mathfrak{F}}$ of order 4 is c-normal in G. Then G belongs to $\mathfrak{F}$ if and only if every cyclic subgroup of $G^{\mathfrak{F}}$ with prime order is contained in $Z_\infty^{\mathfrak{F}}(G)$.*

Corollary 3.14. (Ballester-Bolinches, Wang, [1]). *Let G be group such that every cyclic subgroup of $F^*(G)$ of order 4 is c-normal in G, where $F^*(G)$ is the generalized Fitting subgroup of G. If every cyclic subgroup of $F^*(G)$ of prime order is contained in $Z_\infty(G)$, then G is nilpotent.*

Corollary 3.15. (Wang, [23]). *Let G be a group and N be a normal subgroup of G such that G/N is nilpotent. Suppose that every cyclic subgroup of $F^*(N)$ of order 4 is c-normal in G. Then G is nilpotent if and only if every cyclic subgroup of $F^*(N)$ of prime order is contained in $Z_\infty(G)$.*

Corollary 3.16. (Li, Wang, [17]). *Suppose that N is a normal subgroup of a group G such that G/N is p-nilpotent, where p is a fixed prime number. Assume that every cyclic subgroup of N with order p is contained in $Z_\infty(G)$. If $p = 2$, in addition, suppose that every cyclic subgroup of order 4 of N is S-permutable in G or lies in $Z_\infty(G)$. Then G is p-nilpotent.*

Corollary 3.17. (Li, Wang, [17]). *Suppose that N is a normal subgroup of a group G such that G/N is nilpotent. Suppose that every cyclic* *subgroup of $F^*(N)$ of order 4 is S-permutable in G. Then G is nilpotent if and only if every cyclic subgroup of $F^*(N)$ of prime order is contained in $Z_\infty(G)$.*

Acknowledgments

The authors would like to express their sincere thanks to the referees, who read the manuscript carefully and made a number of constructive comments and helpful suggestions which have significantly affected its final form. This work was supported by the National Natural Science Foundation of China (Grant Nos. 11171364, 11271301), the Scientific Research Foundation of Yongchuan Science and Technology Commission (Grant No. Ycstc, 2013nc8006), the Scientific Research Foundation of Chongqing University of Arts and Sciences (Grant No. R2012SC21) and the Program for Innovation Team Building at Institutions of Higher Education in Chongqing (Grant No. KJTD201321).

References

[1] A. Ballester-Bolinches and Y. M. Wang, Finite groups with some C-normal minimal subgroups, *J. Pure and Applied Algebra* **153** (2000), no. 2, 121–127.
[2] H. G. Bray, W. E. Deskins, D. Johnson, J. F. Humphreys, B. M. Puttaswamaiah, P. Venzke and G. L. Walls, Between Nilpotent and Solvable, M. Weinstein (Ed.), Polygonal Publ. House, Washington, 1982.
[3] J. Buckley, Finite groups whose minimal subgroups are normal, *Math. Z.* **116** (1970) 15–17.
[4] K. Doerk and T. Hawkes, Finite Soluble Groups, Walter de Gruyter & Co., Berlin, 1992.
[5] D. Gorenstein, Finite Groups, Chelsea Publishing Company, New York, 1980.
[6] W. B. Guo, K. P. Shum and A. N. Skiba, X-semipermutable subgroups of finite groups, *J. Algebra* **315** (2007), no. 1, 31–41.
[7] W. B. Guo, K. P. Shum and A. N. Skiba, Conditionally permutable subgroups and supersolubility of finite groups, *Southeast Asian Bull. Math.* **29** (2005), no. 3, 493–510.
[8] W. B. Guo, A. N. Skiba and K. P. Shum, X-permutable subgroups, *Siberian Math. J.* **48** (2007), no. 4, 593–605.
[9] W. B. Guo, K. P. Shum and A. N. Skiba, Criterions of supersolubility for products of supersoluble groups, *Publ. Math. Debrecen* **68** (2006), no. 3-4, 433–449.
[10] W. B. Guo, K. P. Shum and A. N. Skiba, On solubility and supersolubility of some classes of finite groups, *Sci. China Ser. A* **52** (2009), no. 2, 272–286.
[11] W. B. Guo, Y. Lu and W. J. Niu, On S-embedded subgroups of finite groups, *Algebra Logic* **49** (2010), no. 4, 293–304.
[12] W. B. Guo, The Theory of Classes of Groups, Kluwer Academic Publishers Group, Dordrecht, Science Press, Beijing, 2000.

[13] T. Hawkes, On the automorphism group of a 2-group, *Proc. London Math. Soc. (3)* **26** (1973) 207–225.
[14] B. Huppert, Endliche Gruppen I (German), Springer-Verlag, Berlin-New York 1967.
[15] B. Huppert and N. Blackburn, Finite Groups III, Springer-Verlag, Berlin-New York, 1982.
[16] Y. M. Li and Y. M. Wang, The influence of minimal subgroups on the structure of a finite group, *Proc. American Math. Soc.* **131** (2003), no. 2, 337–341.
[17] Y. M. Li and Y. M. Wang, Finite groups with some π-quasinormal minimal subgroups *Acta Math. Hungar.* **102** (2004), no. 3, 213–222.
[18] A. N. Skiba, On weakly s-permutable subgroups of finite groups, *J. Algebra* **315** (2007), no. 1, 192–209.
[19] A. N. Skiba, On two questions of L. A. Shemetkov concerning hypercyclically embedded subgroups of finite groups, *J. Group Theory* **13** (2010), no. 6, 841–850.
[20] A. N. Skiba, Cyclicity conditions for G-chief factors of a normal subgroup of a group G, *Sibirsk. Mat. Zh. (Russian)* **52** (2011), no. 1, 127–130.
[21] A. N. Skiba, A characterization of the hypercyclically embedded subgroups of finite groups, *J. Pure Appl. Algebra* **215** (2011), no. 3, 257–261.
[22] Y. M. Wang, c-normality of groups and its properties, *J. Algebra* **180** (1996), no. 3, 954–965.
[23] Y. M. Wang, The influence of minimal subgroups on the structure of finite groups, *Acta Mathematica Sinica (English Acta Math. Sin. (Engl. Ser.)* **16** (2000), no. 1, 63–70.
[24] H. Q. Wei, Y. M. Wang and Y. M. Li, On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups (II), *Comm. Algebra* **31** (2003), no. 10, 4807–4816.
[25] R. A. Wilson, The Finite Simple Groups, Springer-Verlag London, London, 2009.

(Jinbao Li) Key Laboratory of Group & Graph Theories and Applications, Chongqing University of Arts and Sciences, Chongqing 402160, P. R. China
E-mail address: `leejinbao25@163.com`

(Wujie Shi) Key Laboratory of Group & Graph Theories and Applications, Chongqing 402160, P. R. China
E-mail address: `wjshi@suda.edu.cn`

(Dapeng Yu) Key Laboratory of Group & Graph Theories and Applications, Chongqing 402160, P. R. China
E-mail address: `yudapeng0@sina.com`