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LG-TOPOLOGY

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Abstract. In this paper, we introduce a new model of point free topology for point-set topology. We study the basic concepts in this structure and will find that it is naturally close to point-set topology.
Keywords: Frame, *LGT*-space, open element, closed element, compact element, separation axioms, *ps*-property, sober space, subspace of *LGT*-space, product of *LGT*-spaces.
MSC(2010): Primary: 06D22; Secondary: 54Bxx.

1. Introduction

A lattice L is said to be complete if every subset of L has the supremum. Notice that a complete lattice is necessarily a bounded lattice, i.e., it has the largest element 1 and the least element 0. A frame F is a complete lattice which satisfies the following distributive law: for each $a, b_i \in F$ $(i \in I)$,

$$a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i).$$

In addition, if the dual of the above equation holds, then we say that F is a symmetric frame. A pseudocomplement of an element a of a bounded lattice L is defined by $\max(a^{\perp})$, if there exists, and denoted by a^*, where $a^{\perp} = \{x \in L : \; x \wedge a = 0\}$. Clearly, if F is a frame, then $a^* = \vee(a^{\perp})$. Let F be a frame, then a subset G of F which is closed under finite meets and arbitrary joins is called a subframe. Closure under the empty infimum and supremum implies that subframes inherit top and bottom elements. Let (X, τ) be any topological space, then clearly, τ is a frame and if

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$\mathcal{U} \subseteq \tau$, then $\bigvee_{U\in\mathcal{U}} U = \bigcup_{U\in\mathcal{U}} U$ and $\bigwedge_{U\in\mathcal{U}} U = int_X(\bigcap_{U\in\mathcal{U}} U)$. This example motivates topologist to study the frames as point free topology.

A complete lattice L is said to be completely distributive if whenever $x_{ij} \in X$ for every $i \in I$ and $j \in J$, then $\bigvee_{i\in I} \bigwedge_{j\in J} x_{ij} = \bigwedge_{f\in J^I} \bigvee_{i\in I} x_{if(i)}$; Note that this equality is self-dual. Clearly, every completely distributive lattice is a symmetric frame and consequently is a frame. However, the converses of these implications are not true. For instance, every topology τ on a set X is a frame whereas it is not necessarily a symmetric frame and consequently is not necessarily completely distributive (In fact, if (X, τ) is a T_1-space, then τ is a symmetric frame if and only if (X, τ) is a discrete space). Also, suppose that (X, τ) is an extremally disconnected Hausdorff space without isolated point and $L = \{U \in \tau : \ X \setminus U \in \tau\}$. We can see that L is a complete Boolean algebra and so is a symmetric frame. On the other hand, by [10, 5.16, p 142], X is not a completely distributive.

Since the set of open sets of a topological space is a frame, many important properties of topological spaces may be expressed without referring to the points. The first person who exploit this possibility of applying lattice theory to topology was Henry Wallman. He used the lattice-theoretic ideas to construct what is now called the "Wallman compactification" of a T_1-topological space. This idea was pursued by McKinsey, Tarski, Nöbeling, Lesier, Ehresmann, Bénabou, etc. However, the importance of attention to open sets as a lattice appeared as late as 1962 in [3] and [11]. After that, many authors such as C.H. Dowker, D. Papert, J. Isbell, B. Banaschewski, etc. became interested and developed the field. The pioneering paper [7] by J. Isbell merits particular mention for opening several important topics. In 1983, Johnstone gave an excellent monograph "Stone Spaces" which is still the standard reference book. Until then, all attempts had been about the modeling of topology but not topological space. In a similar method as we deal with general topology, Wang Guo-Jun in [12] and later in [13] construct a model of the topological space on a completely distributive lattice. He, also, introduce a concept, named molecule, which has the point role in this structure. In this article, we will pursue this viewpoint and will introduce a new structure of point free topology as a model of topological space, and review the basic concepts of point-set topology in this structure. Of course, this structure has potentiality for studying more.

Remark 1.1. *The following simple assertions are useful throughout the paper. Let L be a complete pseudocomplemented lattice.*

- *The map $*$ is decreasing and $a \leq a^{**}$ for every $a \in L$.*
- *The map $**$ is identity on L^*, i.e., $a^{***} = a^*$ for all $a \in L$.*
- *For every $a, b \in L$ we have*

$$a \wedge b = 0 \;\Leftrightarrow\; a \leq b^* \;\Leftrightarrow\; b \leq a^* \;\Leftrightarrow\; a^{**} \leq b^* \;\Leftrightarrow\; a^{**} \wedge b = 0.$$

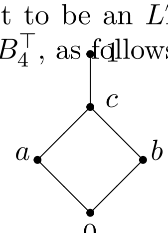
- *If L is a frame and $S \subseteq L$, then $(\vee_{s\in S} s)^* = \wedge_{s\in S} s^*$.*

2. LGT-space and basic properties

Definition 2.1. *Let L be a bounded pseudocomplemented distributive lattice. If τ is a sublattice of L, $\bigvee S \in \tau$ for each $S \subseteq \tau$ and $0, 1 \in \tau$, then τ is called a generalized topology on L and (L, τ) (briefly L) is called an l-generalized topology. Every member of τ is said to be open and any member of $\tau^* = \{t^* : \; t \in \tau\}$ is said to be a closed element. If F is a frame, then $\tau \subseteq F$ is an l-generalized topology on F if and only if τ is a subframe of F. Clearly, the set of closed elements is a $\wedge$-structure, since $(\vee_{\lambda\in\Lambda} t_\lambda)^* = \wedge_{\lambda\in\Lambda} t_\lambda^*$. Furtheremore, if τ^* is a sublattice of F, then we say τ is an l-topology on F and (F, τ) (briefly F) is an l-topological space; for convenience, we denote an l-generalized topological space (resp. l-topological space) by LGT-space (resp. LT-space). Assuming that τ is an l-generalized topology on F and $a \in F$, we define $a^\circ = \vee\{t \in \tau : \; t \leq a\}$ and $\overline{a} = \wedge\{x \in \tau^* : \; a \leq x\}$. Sometimes, we use $int_\tau a$ and $cl_\tau a$ instead of a° and $\overline{a}$, respectively.*

From this point on, any lattice, under which we study LGT-spaces, is a frame.

An LGT-space need not to be an LT-space. For instance, consider the lattice F, denoted by $B_4^\top$, as follows:



It is obvious that $F^* = \{0, a, b, 1\}$ and $a \vee b = c \notin F^*$. Thus (F, F) is an LGT-space while it is not an LG-space.

Note that if (F, τ) is an LGT-space and for every $t_1, t_2 \in \tau$ we have $(t_1 \wedge t_2)^* = t_1^* \vee t_2^*$, then τ^* is a sublattice of F and therefore (F, τ) is an LT-space. We will see in the next proposition that the converse is also true.

We need the following lemma in the next theorem.

Lemma 2.2. *Let F be a frame and $S \subseteq F$ be such that S^* is closed under the join. Then, for every $r, s \in S$, we have $r^* \vee s^* = (r^{**} \wedge s^{**})^*$.*

Proof. Since $r^* \vee s^* \in S^*$, it follows that

$$r^* \vee s^* = (r^* \vee s^*)^{**} = (r^{**} \wedge s^{**})^*.$$

□

The following theorem is an extension of Lemma 212 of [6]

Proposition 2.3. *For any LGT-space (F, τ), the following statements are equivalent:*

(a) $(r \wedge s)^* = r^* \vee s^*$ for every $r, s \in \tau$.

(b) τ^* is a $\vee$-semi sublattice of F.

(c) (F, τ) is an LT-space.

(d) $\overline{a \vee b} = \overline{a} \vee \overline{b}$ for every $a, b \in F$.

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are obvious.

(c) $\Rightarrow$ (d). It is enough to show that $\overline{a \vee b} \leq \overline{a} \vee \overline{b}$. Since (F, τ) is an LT-space and $\overline{a}, \overline{b} \in \tau^*$, it follows that $\overline{a} \vee \overline{b} \in \tau^*$. Clearly, $a \vee b \leq \overline{a} \vee \overline{b}$ and consequently we are done.

(d) $\Rightarrow$ (a). Suppose that $r, s \in \tau$. By (d), it follows that τ^* is closed under the join and so by above lemma, we can write as follow:

$$x \in (r \wedge s)^{\perp} \Leftrightarrow x \wedge r \wedge s = 0 \Leftrightarrow x \wedge r^{**} \wedge s^{**} = 0 \Leftrightarrow x \leq (r^{**} \wedge s^{**})^* = r^* \vee s^*.$$

Therefore, $r^* \vee s^* = (r \wedge s)^*$. □

Recall that a distributive pseudocomplemented lattice L is said to be a Stone algebra if $a^* \vee a^{**} = 1$ for every $a \in L$. In view of this, if S is a sublattice of L and $s^* \vee s^{**} = 1$ for every $s \in S$, then we say L is a S-Stone algebra. Obviously, if $\tau = F$, then the concept of τ-Stone algebra coincide with the concept of Stone algebra.

Proposition 2.4. *Let (F, τ) be an LGT-space. Then, the following statements hold.*

(a) If F is a τ-Stone algebra, then $(r \wedge s)^* = r^* \vee s^*$ for every $r, s \in \tau$. But, the converse is not true.

(b) If $F = \tau$, then the converse of (a) is also true.

Proof. (a). We do, Similar to the proof of Lemma 212 of [6]. Clearly, $r^* \vee s^* \in (r \wedge s)^{\perp}$. Assuming that $x \in (r \wedge s)^{\perp}$, we must show that

$x \leq r^* \vee s^*$. To see this, by Remark 1.2, we can write

$$x \wedge r \wedge s = 0 \;\Leftrightarrow\; x \wedge r^{**} \wedge s = 0 \;\Leftrightarrow\; x \wedge r^{**} \leq s^*$$
$$\Rightarrow\; x = x \wedge 1 = x \wedge (r^* \vee r^{**}) = (x \wedge r^*) \vee (x \wedge r^{**}) \leq r^* \vee s^*.$$

Now, suppose that $F = B_4^{\top}$. Assuming that $\tau = \{0, a, 1\}$, clearly (F, τ) is an LT-space but $a^* \vee a^{**} = b \vee a = c \neq 1$ and hence F is not a τ-Stone algebra.

(b). It follows easily from Lemma 212 of [6]. □

The following proposition is simple to prove and is a basic model for the structure of LGT-spaces.

Proposition 2.5. *Let (X, τ) be a topological space. Then $(P(X), \tau)$ is LT-space and τ^* is the family of closed subsets in X.*

It is obvious that if τ and F are two topology on X and $\tau \subseteq F$, then (F, τ) is an LGT-space. The following proposition shows (F, τ) is not necessarily an LT-space and moreover shows when (τ, τ) is an LT-space, where τ is a topology on a set X. Also, the proposition shows that extremally disconnected spaces have a prominent role in the study of LGT-spaces. For more information about the extremally disconnected spaces, see [4], [5] and [14].

The following proposition can also be deduced from 3.5 of [8]. Recall that if τ is a topology on X, then $U^* = X \setminus \overline{U}$ for every $U \in \tau$.

Proposition 2.6. *Let (X, τ) be a topological space. Then the following statements are equivalent:*

(a) (τ, τ) is an LT-space.

(b) The set $\mathcal{A} = \{\overline{U} : U \in \tau\}$ is closed under the finite intersection.

(c) X is extremally disconnected.

Proof. (a) $\Rightarrow$ (b). Suppose that $\overline{U}, \overline{V} \in \mathcal{A}$. By Proposition 2.4, we have $U^* \vee V^* = (U \wedge V)^*$ and so we can write

$$X \setminus (\overline{U} \cap \overline{V}) = (X \setminus \overline{U}) \cup (X \setminus \overline{V}) = U^* \vee V^* = (U \wedge V)^* = X \setminus \overline{U \cap V}.$$

Therefore, $\overline{U} \cap \overline{V} = \overline{U \cap V} \in \mathcal{A}$.

(b) $\Rightarrow$ (c). Suppose that $U \in \tau$, it is enough to show $\overline{U} \in \tau$. Taking $V = X \setminus \overline{U}$, by hypothesis, there exists $W \in \tau$ such that $\overline{U} \cap \overline{V} = \overline{W}$. Clearly, $(\overline{U} \cap \overline{V})^\circ = \varnothing$ and so $W = \varnothing$. Thus, $\overline{U} \cap \overline{V} = \overline{U} \cap \overline{X \setminus \overline{U}} = \varnothing$ and consequently $\overline{U} = \overline{U}^{\circ} \in \tau$.

(c) $\Rightarrow$ (a). Suppose that $U, V \in \tau$, it is sufficient to prove that $(U \wedge V)^* = U^* \vee V^*$. Since X is extremally disconnected, $\overline{U \cap V} = \overline{U} \cap \overline{V}$.

Now, we do, similar to the implication (a) $\Rightarrow$ (b) and this completes the proof. $\square$

Note that extremally disconnected spaces are not rare, even there exist extremally disconnected spaces which are far from discrete spaces, more precisely every point of which is nonisolated (cf. [1, Proposition 1.6]).

Definition 2.7. *Suppose that F is a frame and $S \subseteq F$. We denote the set of finite meets of elements of S by $Fm(S)$. Set $< S >= \{\vee D : D \subseteq Fm(S)\}$. Clearly, $< S >$ is the smallest subframe of F containing S. If (F, τ) is an LGT-space and $\tau =< S >$ for some $S \subseteq F$, then S is said to be a subbase for the topology τ. A set $B \subseteq \tau$ is called a base for the topology τ if for every $t \in \tau$ there exists $D \subseteq B$ such that $t = \vee D$. Moreover, assuming that F is a frame and $B \subseteq F$, we say B is a base for a topology if for every $b_1, b_2 \in B$ there exists $D \subseteq B$ such that $b_1 \wedge b_2 = \vee D$.*

The proof of the two following propositions is routine.

Proposition 2.8. *Let (F, τ) be an LGT-space. The following statements hold:*

(a) $0^\circ = 0$ *and* $1^\circ = 1$.
(b) $a^\circ \leq a$ *for every* $a \in F$.
(c) If $a, b \in F$ *and* $a \leq b$, *then* $a^\circ \leq b^\circ$.
(d) $\forall a \in F$, $a^\circ = \vee\{t \in B : t \leq a\}$, *where* B *is a base for* τ.
(e) $\forall a \in F$, $a^\circ \in \tau$.
(f) $a = a^\circ$ *if and only if* $a \in \tau$.
(g) $\forall a \in F$, $(a^\circ)^\circ = a^\circ$.
(h) a° *is a greatest element of* τ *that is less than or equal to* a.
(i) If $a_1, ..., a_n \in F$, *then* $(\bigwedge_{i=1}^n a_i)^\circ = \bigwedge_{i=1}^n a_i^\circ$.

Conversely, given a map $\varphi : F \longrightarrow F$ satisfying (a), (b), (c), (g) and (i), if we define $\tau = \{a \in F : \varphi(a) = a\}$, then τ is an LG-topology on F and the interior operator induced by τ coincides with the φ.

Proposition 2.9. *Let (F, τ) be an LGT-space. The following statements hold:*

(a) $\overline{0} = 0$ *and* $\overline{1} = 1$.
(b) $\forall a \in F$, $a \leq \overline{a}$.
(c) If $a, b \in F$ *and* $a \leq b$, *then* $\overline{a} \leq \overline{b}$.
(d) $\overline{a} \in \tau^*$ *for every* $a \in F$.
(e) $a = \overline{a}$ *if and only if* $a \in \tau^*$.

(f) $\overline{\overline{a}} = \overline{a}$ *for all* $a \in F$.

(g) $\overline{a}$ *is a smallest closed element that is greater than or equal to* a.

(h) Assuming that (F, τ) *is an LT-space, if* $a_1, ..., a_n \in F$*, then we have* $\overline{(\bigvee_{i=1}^n a_i)} = \bigvee_{i=1}^n \overline{a_i}$.

Note that the proof of part (h) of the above proposition follows from Proposition 2.4.

Definition 2.10. *A closure operator (resp. interior operator) on a poset* P *is a function* $c : P \longmapsto P$ *(resp.* $i : P \longmapsto P$*) such that*

i) for all $x, y \in P$ *if* $x \leq y$*, then* $c(x) \leq c(y)$ *(resp.* $i(x) \leq i(y)$*);*

ii) $x \leq c(x)$ *(resp.* $i(x) \leq x$*) for all* $x \in P$*;*

iii) $c(c(x)) = c(x)$ *(resp.* $i(i(x)) = i(x)$*) for all* $x \in P$.

Recall that if "c" and "i" are closure and interior operators on a poset P, then clearly ci and ic are order-preserving and idempotent. Therefore, assuming that (F, τ) is an LGT-space, it follows that $a^{\overline{\circ}^{\overline{\circ}}} = a^{\overline{\circ}}$ and $\overline{a}^{\overline{\circ}^{\circ}} = \overline{a}^{\circ}$ for every $a \in F$.

The maps "int", "cl" and "complement" have a close connection. Now, we consider the connection between "int", "cl" and "pseudocomplement" in LGT-spaces.

Proposition 2.11. *Let* (F, τ) *be an LGT-space. For every* $a \in F$*, the following statements hold.*

(a) $\overline{a} = \overline{a}^{**} = a^{*^{\circ^{*}}} = \overline{a^{**}}$.

(b) $a^{*^\circ} \leq \overline{a}^*$ *and if* a^{*° *has complement in* F*, then the equality holds.*

(c) $a^{\overline{*}} \leq a^{\circ^*}$ *and if* a *has complement in* F*, then the equality holds.*

(d) $(Fr(a))^\circ = 0$*, where* $Fr(a) = \overline{a} \wedge a^{\circ^*}$ *(*$Fr(a)$ *is called the frontier of* a *in* F*).*

Proof. (a). Clearly, $\overline{a} \in \tau^*$ and consequently $\overline{a}^{**} = \overline{a}$. To prove the other equality, if we put $A = \{t \in \tau : \ a \leq t^*\}$, $B = \{t \in \tau : \ t \leq a^*\}$ and $C = \{t \in \tau : \ a^{**} \leq t^*\}$, then by Remark 1.2, $A = B = C$. Thus, we can write

$$\overline{a} = \wedge\{t^* : \ t \in \tau, \ a \leq t^*\} = (\vee A)^* = (\vee B)^* = a^{*^{\circ^{*}}},$$

$$\overline{a^{**}} = \wedge\{t^* : \ t \in \tau, \ a^{**} \leq t^*\} = (\vee C)^* = (\vee A)^* = \overline{a}.$$

(b). By (a), it is clear that $a^{*^\circ} \leq a^{*^{\circ^{**}}} = \overline{a}^*$. Now, suppose that a^{*° has a complement. Thus, by part (a) we can write

$$\overline{a}^* = a^{*^{\circ^{**}}} = a^{*^\circ}.$$

(c). Since $a^\circ \leq a$ and every $t^* \in \tau^*$ is closed, we can write

$$a^* \leq a^{\circ^*} \;\Rightarrow\; a^{\overline{*}} \leq \overline{a^{\circ *}} = a^{\circ^*}.$$

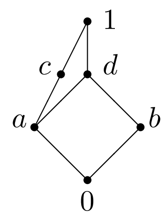
Now, suppose a^c exists, then clearly $a^* = a^c$. Assume that $t \in \tau$ and $a^* \leq t^*$, then $t \leq t^{**} \leq a^{**} = a$ and consequently $t \leq a^\circ$, so $a^{\circ^*} \leq t^*$. Therefore, $a^{\circ^*} \leq a^{\overline{*}}$.

(d). By Proposition 2.9 and part (b) of the proposition, we can write

$$(\overline{a} \wedge a^{\circ^*})^\circ = \overline{a}^\circ \wedge a^{\circ^{*^\circ}} \leq \overline{a}^\circ \wedge \overline{a^\circ}^* \leq \overline{a} \wedge \overline{a}^* = 0.$$

□

Note that for identity in the part (b) of the above proposition if instead of a^{*° we suppose that both a and $\overline{a}$ have complements, then the equality does not necessarily hold. For example, suppose that F be as follows



Clearly, F is a frame, since it has not a copy of N_5 or M_3. Assuming $\tau = \{0, a, 1\}$, it is obvious that (F, τ) is LGT-space. We can easily see that $b = \overline{b}$ has the complement whereas $b^{*^\circ} = c^\circ = a \neq c = b^* = \overline{b}^*$.

Definition 2.12. *Suppose that (F, τ) is an LGT-space. $a \in F$ is said to be a dense element if $\overline{a} = 1$*

In the following proposition, we consider some natural assertions about density.

Proposition 2.13. *Suppose that (F, τ) is an LGT-space and $a \in F$.*

(a) a is a dense element in F if and only if $t \wedge a \neq 0$ for every $t \in \tau \setminus \{0\}$.

(b) $\overline{a} = 1$ if and only if $a^{*^\circ} = 0$.

(c) If $\overline{a^*} = 1$, then $a^\circ = 0$. But the converse is not true.

(d) $a \vee a^*$ is a dense element in F for every $a \in F$.

(e) If a is a dense element, then $\overline{t \wedge a} = \overline{t}$ for every $t \in \tau$.

Proof. (a) and (b) are evident.

(c). The first part of (c) is clear. Now, let F be the topology on $\mathbb{R}$ such that every point $x \in \mathbb{Q}$ is isolated and every point $x \notin \mathbb{Q}$ has the ordinary neighborhood base. Then (F,τ) is an LGT-space, where τ is the ordinary topology on $\mathbb{R}$. If we take $a=\mathbb{Q}$, then $a\in F$ and $a^\circ=0$ whereas $\overline{a^*}=0\neq 1$.

(d). Suppose that $r\in\tau$ and $a\vee a^*\leq r^*$. It is enough to show that $r=0$. To see this, we can write

$$(a\wedge r)\vee(a^*\wedge r)=(a\vee a^*)\wedge r=0 \;\Rightarrow\; r\wedge a=0=r\wedge a^*$$
$$\Rightarrow\; r\leq a^*\;,\; r\wedge a^*=0\;\Rightarrow\; r=0.$$

(e). Suppose that $r\in\tau$ and $\overline{t\wedge a}\leq r^*$, It is enough to show that $t\leq r^*$. Since $\overline{t\wedge a}\leq r^*$ and a is a dense element, it follows that $t\wedge a\wedge r=0$ and so $t\wedge r=0$, consequently $t\leq r^*$. □

3. **Subspace and product space**

Definition 3.1. *Suppose that (F,τ) is an LGT-space and $a\in F$. If we take $F_a=\downarrow a$ and $\tau_a=\{t\wedge a:\ t\in\tau\}$, then clearly (F_a,τ_a) is an LGT-space. We call (F_a,τ_a) as a subspace of (F,τ) (briefly, we say F_a is a subspace of F).*

In the following, we are going to study the basic properties of subspaces in LGT-spaces.

Proposition 3.2. *Suppose that (F,τ) is an LGT-space and $a\in F$.*

(a) If S is a subbase for τ, then $S_a=\{s\wedge a:\ s\in S\}$ is a subbase for τ_a.

(b) If B is a base for τ, then $B_a=\{t\wedge a:\ t\in B\}$ is a base for τ_a.

Proof. The proof is straightforward. □

We need the following lemma for the next proposition.

Lemma 3.3. *Suppose F is a frame, $a,b\in F$ and $b^*=0$. Then $(a\wedge b)^*=a^*$.*

Proof. By the following implication, it is easy.

$(a\wedge b)^\perp=\{x\in F:\ x\wedge a\wedge b=0\}=\{x\in F:\ x\wedge a\leq b^*=0\}=a^\perp.$ □

□

Proposition 3.4. *Suppose that (F,τ) is an LGT-space and $a\in F$. Then, the following statements hold.*

(a) $\{(t\wedge a)^*\wedge a:\ t\in\tau\}$ is the set of closed elements of F_a. In particular, if $a\in F^*$, then $\{t^*\wedge a:\ t\in\tau\}$ is the set of closed elements of F_a.

(b) If $x \leq a$, then $cl_{\tau_a}x = (int_\tau x^* \wedge a)^* \wedge a$.

(c) If $a \in F^*$ and $x \leq a$, then $cl_{\tau_a}x = cl_\tau x \wedge a$. In particular, if a is a closed element in F, then $cl_{\tau_a}x = cl_\tau x$.

(d) If $x \leq a$, then $int_\tau x \leq int_{\tau_a}x$, and the converse of the inclusion is not necessarily true.

(e) If a is an open element in F and $x \leq a$, then $int_\tau x = int_{\tau_a}x$.

Proof. (a). It is enough to show that $(t \wedge a)^*_{F_a} = (t \wedge a)^* \wedge a$ for every $t \in \tau$ (the notion $(t \wedge a)^*_{F_a}$ means the pseodocomplement of $t \wedge a$ with respect of F_a). In fact, if we take $A = \{c \in F_a : \ c \wedge (t \wedge a) = 0\}$, then $A = \{x \wedge a : \ x \in F, \ (x \wedge a) \wedge (t \wedge a) = x \wedge (t \wedge a) = 0\}$ and consequently $(t \wedge a)^*_{F_a} = \vee_{F_a} A = \vee A = (\vee\{x \in F : \ x \wedge (t \wedge a) = 0\}) \wedge a = (t \wedge a)^* \wedge a$. Now, suppose that $a = b^*$ for some $b \in F$. Note that $(a \vee a^*)^* = 0$ for every $a \in F$. Therefore, using above lemma, for every $t \in \tau$, we can write

$$(t \wedge a)^* \wedge a = (t \wedge b^*)^* \wedge b^* = ((t \wedge b^*) \vee b)^*$$
$$= ((t \vee b) \wedge (b^* \vee b))^* = (t \vee b)^* = t^* \wedge b^* = t^* \wedge a.$$

(b). Let $x \in F_a$, then we can write

$$cl_{\tau_a}x = \wedge\{(t \wedge a)^* \wedge a : \ x \leq (t \wedge a)^* \wedge a\} = \wedge\{(t \wedge a)^* : \ x \wedge t \wedge a = 0\} \wedge a$$
$$= (\vee\{t \wedge a : \ x \wedge t = 0\})^* \wedge a = (\vee\{t : \ t \leq x^*\} \wedge a)^* \wedge a = (int_\tau x^* \wedge a)^* \wedge a.$$

(c). Since $a \in F^*$, by part (a) of the proposition and part (a) of Proposition 2.12, we can write

$$cl_{\tau_a}x = (int_\tau x^* \wedge a)^* \wedge a = (int_\tau x^*)^* \wedge a = (cl_\tau x) \wedge a.$$

Now, suppose that $a = t^*$ is a closed element in F and $x \leq a$. Therefore, $x \wedge t = 0$ and so $(cl_\tau x) \wedge t = 0$. Thus, $cl_\tau x \leq t^* = a$ and $cl_{\tau_a}x = (cl_\tau x) \wedge a = cl_\tau x$.

(d). Clearly, $int_\tau x = (int_\tau x) \wedge a$ is an open element in τ_a contained in x and consequently $int_\tau x \leq int_{\tau_a}x$.

(e). Let $a \in \tau$ and $x \in F_a$, then we can write

$$int_{\tau_a}x = \vee\{t \wedge a : \ t \in \tau, \ t \wedge a \leq x\} = \vee\{r \in \tau : \ r \leq x\} = int_\tau x.$$

□

Definition 3.5. *Suppose that (F, τ) is an LGT-space. We say $a \in F$ is τ-compact (briefly, compact) whenever if $S \subseteq \tau$ and $a \leq \vee S$, then there exists a finite subset D of S such that $a \leq \vee D$. We say $a \in F$ is $**$-compact whenever if $S \subseteq \tau$ and $a \leq \vee S$, then there exists a finite subset D of S such that $a \leq (\vee D)^{**}$. We can similarly define Lindelöf,* *countably compact* $**$*-Lindelöf,* $**$*-countably compact etc. If* 1 *(i.e., the top element of* F*) is a compact element in* (F,τ)*, then we say* (F,τ) *(briefly,* F*) is a compact space. Let* L *be a lattice and* $S \subseteq L$*, then we say* S *has the finite meet property if for every finite subset* D *of* S *we have* $\wedge S \neq 0$*.*

The following lemma may be well-known.

Lemma 3.6. *Suppose that* L *is a pseudocomplemented lattice. Then, the following statements are equivalent.*

(a) The map $*$ *is injective.*

(b) $ker(*) = 1$ *where* $ker(*) = \{a \in F : \ a^* = 0\}$*.*

(c) The map of pseudocomplementation coincides with the map of complementation.

(d) L *is a Boolean algebra.*

(e) For every $a, b \in L$ *if* $b^* \wedge a = 0$*, then* $a \leq b$*.*

Proof. (a) $\Rightarrow$ (b). It is evident.

(b) $\Rightarrow$ (c). Assuming $a \in L$, it is enough to show that $a \vee a^* = 1$. To see this, it is clear that $(a \vee a^*)^* = a^* \wedge a^{**} = 0$ and so $a \vee a^* = 1$.

(c) $\Rightarrow$ (d). By Theorem 6.5 of [2, p. 80], it is clear.

(d) $\Rightarrow$ (e). It is clear.

(e) $\Rightarrow$ (a). Suppose that $a, b \in L$ and $b^* = a^*$. Clearly, $b^* \wedge a = a^* \wedge a = 0$ and consequently by assumption $a \leq b$. Similarly, it follows that $b \leq a$. Therefore, $a = b$. □

Inspired by the above lemma, we formulate the following definition.

Definition 3.7. *Assuming* (F,τ) *is an LGT-space, we say the map* $*$ *is topologically injective whenever* $ker(*) \cap \tau = \{1\}$*.*

Evidently, the topologically injectivity does not imply the injectivity. For example, let $F = B_4^{\top}$ and $\tau = \{0, a, 1\}$. Clearly the map $*$ is topologically injective but not injective.

Proposition 3.8. *Suppose that* (F,τ) *is an LGT-space. Then the following statements hold.*

(a) If $a \in F$ *and* $x \in F_a$*, then* x *is compact in* (F,τ) *if and only if it is compact in* (F_a, τ_a)*.*

(b) If the map $*$ *is topologically injective, then* F *is compact if and only if for every* $S \subseteq \tau^*$ *with finite meet property, we have* $\wedge S \neq 0$*.*

(c) If the map $*$ *is topologically injective, then* F *is compact if and only if it is* $**$*-compact.*

(d) If the map $*$ is topologically injective and F is $**$-compact, then every closed element in (F, τ) is $**$-compact.

Proof. (a). The proof is straightforward.

(b $\Rightarrow$). Suppose that $S \subseteq \tau$ and $\wedge_{s\in S} s^* = 0$, then it follows that $(\vee S)^* = 0$ and so, by hypothesis, $\vee S = 1$. Therefore, there exists a finite subset R of S such that $\vee R = 1$. Thus, $\wedge_{r\in R} r^* = 0$.

(b $\Leftarrow$). Suppose that $S \subseteq \tau$ and $1 = \vee S$. Thus, $\wedge_{s\in S} s^* = (\vee S)^* = 0$ and so there exists a finite subset R of S such that $\wedge_{r\in R} r^* = (\vee R)^* = 0$. Therefore, by assumption, $\vee R = 1$.

(c). Clearly, if F is compact, then it is $**$-compact. Conversely, suppose that F is $**$-compact, $S \subseteq \tau$ and $\vee S = 1$. By hypothesis, there exists a finite subset R of S such that $(\vee R)^{**} = 1$. Therefore, $(\vee R)^* = (\vee R)^{***} = 0$ and so, by assumption, $\vee R = 1$.

(d). By part (c), we may suppose that F is compact. Let $x = r^*$ be a closed element in F and $\{t_i\}_{i\in I}$ is an open cover for r^*. Therefore, we can write

$$r^* \leq \vee_{i\in I} t_i \;\Rightarrow r^* \wedge (\vee_{i\in I} t_i)^* = 0 \Leftrightarrow (r \vee (\vee_{i\in I} t_i))^* = 0 \Rightarrow r \vee (\vee_{i\in I} t_i) = 1.$$

Since F is compact there exists $n \in \mathbb{N}$ such that $r \vee (\vee_{i=1}^{n} t_i) = 1$. Thus, $r^* \wedge (\vee_{i=1}^{n} t_i)^* = 0$ and consequently $x = r^* \leq (\vee_{i=1}^{n} t_i)^{**}$. □

Although the following definition is not an accurate model for the ordinary product topology, it is close to it.

Definition 3.9. *Suppose that (F_i, τ_i) is an LGT-space, for every $i \in I$. Clearly, $F = \prod_{i\in I} F_i$ with ordinary order is a frame. Now, we define two topologies on F as follows:*

i) $\tau_p = \{t = (t_i)_{i\in I} :\ t_i \in \tau_i,\ \text{and}\ t_i = 1\ \text{for all except finitely many}\ i \in I\} \cup \{0\}$. This topology is called product topology on F. Sometimes we use τ instead of τ_p. When we deal with $\prod_{i\in I} F_i$ as an LGT-space, we have in view this topology.

ii) $\tau_b = \{t = (t_i)_{i\in I} :\ \forall i \in I,\ t_i \in \tau\}$. This topology is called box topology on F.

Clearly, if π_i is the projection map from F to F_i, then for every $S \subseteq F$ we $\vee S = (\vee_{s\in S} \pi_i(s))_{i\in I}$.

The proof of the following proposition is not difficult and so we left it. Note that for every $x, y \in F = \prod_{i\in I} F_i$, we have $y = x^*$ if and only if $y_i = x_i^*$ for every $i \in I$.

Proposition 3.10. *Suppose that (F_i,τ_i) is an LGT-space, for every $i \in I$, $F=\prod_{i\in I} F_i$, and τ and τ_b are the the product topology and box topology on F, respectively. Then the following statements hold.*

(a) $\tau^* = \{x=(x_i)_{i\in I} \in F : \ x_i \in \tau_i^*,$ *and* $x_i = 0$, *for all except finitely many* $i \in I\} \cup \{1\}$.

(b) $\tau_b^* = \{x=(x_i)_{i\in I} \in F : \ x_i \in \tau_i^*, \ \forall i \in I\}$.

(c) For every $x \in F$, if I is infinite, then we have $int_\tau x \neq 0$ if and only if $x_i = 1$ for all except finitely many $i \in I$, and if I is finite, then we have $int_\tau x \neq 0$ if and only if there exists $i \in I$ such that $x_i^\circ \neq 0$.

(d) For every $x \in F$, if I is finite or $int_\tau x \neq 0$, then $int_\tau x = (x_i^\circ)_{i\in I}$.

(e) For every $x \in F$, if I is infinite, then we have $cl_\tau x \neq 1$ if and only if $x_i = 0$ for all except finitely many $i \in I$, and if I is finite, then we have $cl_\tau x \neq 1$ if and only if there exists $i \in I$ such that $cl_\tau x_i \neq 1$.

(f) For every $x \in F$, if I is finite or $cl_\tau x \neq 1$, then $cl_\tau x = (\overline{x_i})_{i\in I}$.

(g) If $x \in F$, then $int_{\tau_b} x = (x_i^\circ)_{i\in I}$.

(h) If $x \in F$, then $cl_{\tau_b} x = (\overline{x_i})_{i\in I}$.

(i) F is compact with respect to product topology if and only if F_i is compact for every $i \in I$.

(j) F is compact with respect to box topology if and only if I is finite and F_i is compact for every $i \in I$.

4. Separation axioms

The separation axioms are already defined in the literature of point free topology, but we prefer the following definition in this structure.

Definition 4.1. *Let (F,τ) be an LGT-space and $a \in F$. We say (F,τ) is T_0 if for any two different nonzero elements $a,b \in F$ there exist $0 \neq a_1 \leq a$, $0 \neq b_1 \leq b$ and $t \in \tau$ such that $a_1 \leq t$ and $b_1 \wedge t = 0$ or $b_1 \leq t$ and $a_1 \wedge t = 0$. We say (F,τ) is T_1 if for any two different nonzero elements $a,b \in F$ there exist $0 \neq a_1 \leq a$, $0 \neq b_1 \leq b$ and $t \in \tau$ such that $a_1 \leq t$ and $b_1 \wedge t = 0$. We say (F,τ) is T_2 if for any two different nonzero elements $a,b \in F$ there exist $0 \neq a_1 \leq a$, $0 \neq b_1 \leq b$ and $r,s \in \tau$ such that $a_1 \leq r$, $b_1 \leq s$ and $r \wedge s = 0$. We say (F,τ) is regular if for every $0 \neq a \in F$ and every closed element $t^* \in \tau^*$ with $a \nleq t^*$ there exist $0 \neq a_1 \leq a$ and $r,s \in \tau$ such that $a_1 \leq r$, $t^* \leq s$ and $r \wedge s = 0$. A T_0 regular LGT-space is said to be T_3. Using the elements of τ^{**} instead of elements of τ we can define the concepts $**-T_0$, $**-T_1$, $**-T_2$ and $**-$regular.*

Since for every $a,b \in F$, we have $a \wedge b = 0$ if and only if $a^{**} \wedge b^{**} = 0$, we can define $**$-separations as follows. For example, (F,τ) is $**$-regular if and only if for every $0 \neq a \in F$ and every closed element $c \in \tau^*$ with $a \nleq c$ there exist $0 \neq a_1 \leq a$ and $r, s \in \tau$ such that $a_1 \leq r^{**}$, $c \leq s^{**}$ and $r \wedge s = 0$.

It is clear that $T_2 \Rightarrow T_1 \Rightarrow T_0$. Moreover, in the following result it is shown that $T_3 \Rightarrow T_2$.

Proposition 4.2. *Suppose that (F, τ) is an LGT-space. Then the following statements hold.*

(a) If (F, τ) is T_3, then it is T_2.

(b) If (F, τ) is regular, then for every $a \in F$ and $t \in \tau$, if $0 \neq a \leq t$ there exist $0 \neq a_1 \leq a$ and $r \in \tau$ such that $a_1 \leq r \leq \overline{r} \leq t^{**}$.

(c) (F, τ) is $**$-regular if and only if for every $a \in F$ and $t \in \tau$, if $0 \neq a \leq t^{**}$, then there exist $0 \neq a_1 \leq a$ and $r \in \tau$ such that $a_1 \leq r^{**} \leq \overline{r^{**}} \leq t^{**}$.

(d) If τ is a topology on X, then $**$-regularity in $(P(X), \tau)$ coincides with ordinary regularity.

Proof. The proofs of (a) and (b) are routine.

(c $\Rightarrow$). Suppose that $t \in \tau$ and $0 \neq a \leq t$, then $a \wedge t^* = 0$ and by hypothesis there exist $0 \neq a_1 \leq a$ and $r, s \in \tau$ such that $a_1 \leq r^{**}$, $t^* \leq s^{**}$ and $r \wedge s = 0$. Therefore, $a_1 \leq r^{**} \leq s^{***} \leq t^{**}$. On the other hand, Clearly, we have $\overline{r} \wedge s = 0$ and so $\overline{r} \leq s^*$. Therefore, by part (a) of Proposition 2.12, it follows that $\overline{r^{**}} = \overline{r} \leq s^* = s^{***}$.

(c $\Leftarrow$). Suppose that $0 \neq a \in F$, $t^* \in \tau^*$ and $a \nleq t^*$. Therefore, $0 \neq a \wedge t \leq t$ and by hypothesis there exist $0 \neq a_1 \leq a$ and $r \in \tau$ such that $a_1 \leq r^{**} \leq \overline{r^{**}} \leq t$. Clearly, by Proposition 2.12, $\overline{r^{**}} = \overline{r}$ and so there exists $s \in \tau$ such that $\overline{r^{**}} = \overline{r} = s^*$. Therefore, $t^* \leq s^{**}$ and $r^{**} \wedge s^{**} = 0$.

(d). It is trivial, since in any topological space $(P(X), \tau)$ we have $A^* = X \setminus A$ for every $A \in P(X)$. □

Remark 4.3. *It is clear that if one of the separation condition T_0, T_1 and T_2 holds, then for every different nonzero elements $a, b \in F$ there exist $a_1, b_1 \in F$ such that $0 \neq a_1 \leq a$, $0 \neq b_1 \leq b$ and $a_1 \wedge b_1 = 0$. If a frame F has this preliminary separation property, we say F has ps-property. Therefore, in the definition of separation axioms in an LGT-space (F, τ), we can suppose that F has the ps-property and use the phrase "for any two non-comparable nonzero elements $a, b \in F$" or "for any two orthogonal nonzero elements $a, b \in F$" instead of "for any two different nonzero elements $a, b \in F$". Note that, assuming L is a* *lattice and $a, b \in L$, in this paper, when we say a and b are disjoint, we mean $a \wedge b = 0$.*

Proposition 4.4. *Let F and τ be two topology on X such that F has the ps-property and $\tau \subseteq F$. The following statements hold.*

(a) (F, τ) is T_0 if and only if for every two disjoint nonempty elements $A, B \in F$ there exists $U \in \tau$ such that $A \cap U \neq \varnothing$ and $B \nsubseteq cl_F U$, or $B \cap U \neq \varnothing$ and $A \nsubseteq cl_F U$.

(b) (F, τ) is T_1 if and only if for every two disjoint nonempty elements $A, B \in F$ there exists $U \in \tau$ such that $A \cap U \neq \varnothing$ and $B \nsubseteq cl_F U$.

(c) (F, τ) is T_2 if and only if for every two disjoint nonempty elements $A, B \in F$ there exists $U \in \tau$ such that $A \cap U \neq \varnothing$ and $B \nsubseteq cl_\tau U$.

(d) (F, τ) is a regular LGT-space if and only if for every $A \in F$ and every $U \in \tau$ if $A \cap U \neq \varnothing$, then there exist $\varnothing \neq B \in F$ and $V \in \tau$ such that $B \subseteq A \cap U$ and $B \subseteq V \subseteq cl_\tau V \subseteq cl_F U$.

Proof. (a $\Rightarrow$). Let $A, B \in F$ be two disjoint nonempty elements. By hypothesis, without loss of generality, there exist $A_1, B_1 \in F$ with $\varnothing \neq A_1 \subseteq A$, $\varnothing \neq B_1 \subseteq B$ and $U \in \tau$ such that $A_1 \subseteq U$ and $U \cap B_1 = \varnothing$. Clearly, $U \cap A \neq \varnothing$, and $cl_F U \cap B_1 = \varnothing$, consequently $B \nsubseteq cl_F U$.

(a $\Leftarrow$). Let $A, B \in F$ be two disjoint nonempty elements. By hypothesis, without loss of generality, there exists $U \in \tau$ such that $A \cap U \neq \varnothing$ and $B \nsubseteq cl_F U$. Thus, there exists $b \in B \setminus cl_F U$. Clearly, $B_1 \in F$ exists such that $b \in B_1 \subseteq B$ and $B_1 \cap cl_F U = \varnothing$. Therefore, if we set $A_1 = A \cap U$, then $\varnothing \neq A_1 \subseteq A$, $\varnothing \neq B_1 \subseteq B$, $A_1 \subseteq U$ and $B_1 \cap U = \varnothing$.

(b). The proof is similar to (a).

(c $\Rightarrow$). Let $A, B \in F$ be two disjoint nonempty elements. By hypothesis, there exist $A_1, B_1 \in F$ with $\varnothing \neq A_1 \subseteq A$, $\varnothing \neq B_1 \subseteq B$ and $U, V \in \tau$ such that $A_1 \subseteq U$, $B_1 \subseteq V$ and $U \cap V = \varnothing$. Clearly, $A \cap U \neq \varnothing$, $V \cap cl_\tau U = \varnothing$ and consequently $B \nsubseteq cl_\tau U$.

(c $\Leftarrow$). Let $A, B \in F$ be two disjoint nonempty elements. By hypothesis, there exists $U \in \tau$ such that $A \cap U \neq \varnothing$ and $B \nsubseteq cl_\tau U$. Thus, there exists $b \in B \setminus cl_\tau U$. Clearly, $V \in \tau$ exists such that $b \in V$ and $V \cap cl_\tau U = \varnothing$. Therefore, if we set $A_1 = A \cap U$ and $B_1 = V \cap B$, then $\varnothing \neq A_1 \subseteq A$, $\varnothing \neq B_1 \subseteq B$, $A_1 \subseteq U$, $B_1 \subseteq V$ and $U \cap V = \varnothing$.

(d $\Rightarrow$). Assume that $A \in F$, $U \in \tau$ and $A \cap U \neq \varnothing$. Obviously, $(A \cap U) \cap U^* = \varnothing$ and consequently there exist $B \subseteq A \cap U$ and $V, W \in \tau$ such that $\varnothing \neq B \subseteq V$, $U^* \subseteq W$ and $V \cap W = \varnothing$. Clearly, it follows that

$$B \subseteq V \subseteq cl_\tau V \subseteq X \setminus W \subseteq X \setminus U^* = cl_F U.$$

(d $\Leftarrow$). Suppose that $A \in F$, $U \in \tau$ and $\varnothing \neq A \nsubseteq U^*$. Thus, $A \cap cl_F U \neq \varnothing$ and so $A \cap U \neq \varnothing$. Therefore, there exist $\varnothing \neq B \in F$ and $V \in \tau$ such that $B \subseteq A \cap U$ and $B \subseteq V \subseteq cl_\tau V \subseteq cl_F U$. Now, if we set $W = X \setminus cl_\tau V$, then the proof will be completed. □

As we saw, the *ps*-property has a basic role in the studying separation axioms. Hence, in some of the following propositions we pay more attention to this concept.

The following lemma is easy to prove.

Lemma 4.5. *Let F be a frame. Then the following statements are equivalent.*

(a) F has the ps-property.

(b) For every $a \in F$, either a is an atom or there exist nonzero elements $b, c \in F$ such that $b, c \leq a$ and $b \wedge c = 0$.

(c) For every $a, b \in F$ whenever $0 \neq a \lneqq b$, it follows that there exist $a_1, b_1 \in F$ such that $0 \neq a_1 \leq a$, $0 \neq b_1 \leq b$ and $a_1 \wedge b_1 = 0$.

Lemma 4.6. *Let (X, τ) be a topological space.*

(a) If X is T_2, then τ has the ps-property.

(b) If X is a T_1-space, then τ has not necessarily ps-property.

(c) If τ has the ps-property, then X is not necessarily a T_0-space. Moreover, even if X is T_0 and τ has the ps-property, then X is not necessarily a T_1-space.

Proof. (a). Assume that $A, B \in \tau$ and $\varnothing \neq A \subsetneqq B$. Thus, there exist $a \in A$ and $b \in B \setminus A$. Since X is T_2, there exist two points separated by two disjoint nonempty open sets A_1 and B_1 such that $A_1 \subseteq A$ and $B_1 \subseteq B$.

(b). Let X be infinite and τ be the cofinite topology on X. Clearly, X is a T_1-space whereas τ has not the *ps*-property.

(c). Clearly, if τ is the trivial topology on a set X with more than one element, then X is not T_0 whereas τ has the *ps*-property. To complete the proof, suppose (X, τ_1) is a Hausdorff topological space and $a, b \in X$ with $a \neq b$. Set $\tau = \{U \in \tau_1 :\ b \notin U \text{ or } a, b \in U\}$. Clearly, (X, τ) is a T_0-space and τ has the *ps*-property but (X, τ) is not a T_1-space. □

Lemma 4.7. *Let F be a frame. Then F has the ps-property if and only if (F, F) is a $T_2 - LGT$-space.*

Proof. It is straightforward. The following result is an immediate consequence of Remark 4.3, Proposition 4.4 and Lemma 4.7. □

Corollary 4.8. *Suppose that τ is a topology on X. Then τ has the ps-property if and only if for every $U, V \in \tau$, if $\varnothing \neq U \subsetneqq V$, then there exists a $W \in \tau$ such that $W \cap U \neq \varnothing$ and $V \nsubseteq \overline{W}$.*

Definition 4.9. *Let L be a bounded lattice. $1 \neq x \in L$ is called $\wedge$-prime whenever if $a \wedge b \leq x$, then $a \leq x$ or $b \leq x$. Using the duality, we obtain the definition of the $\vee$-prime element. A topological space (X, τ) is called sober if every proper $\wedge$-prime element of τ is of the form $X \setminus \overline{\{x\}}$ for some $x \in X$; equivalently every $\vee$-prime closed subset of X is of the form $\overline{\{x\}}$ for some $x \in X$, for more information about sober spaces see, for example see [9].*

Assume that F is a frame and ΣF is the set of all frame homomorphism from F to the frame $\{0, 1\}$. For every $a \in F$ define $Coz(a) = \{\theta \in \Sigma F : \theta(a) = 1\}$, then $\tau_\Sigma = \{Coz(a) : a \in F\}$ is a topology on ΣF. In the literature, one can see that if (X, τ) is a topological space and $E : X \longrightarrow \Sigma\tau$ is a map such that $E(x)(U) = 1$ if and only if $x \in U$ for every $x \in X$ and every $U \in \tau$, then X is a sober space if and only if the map E is homeomorphism; equivalently, E is bijective, see [**?**]. One can also see that a Hausdorff space is sober and a sober space is T_0. But neither "sober" implies "T_1" nor "T_1" implies "sober".

Remark 4.10. *A natural question at this point is that "what is the connection between sober space and ps-property?". In fact, these two concepts are not comparable. For example, let $\tau = \{(a, 0] : a \in \mathbb{R}\} \cup \{(-\infty, 0]\}$ be a topology on $X = (-\infty, 0]$. Since every nonempty closed subset of X is of the form $(-\infty, a]$. Therefore, every $\vee$-prime closed subset of X is of the form $\overline{\{a\}}$ and consequently X is a sober space (note that, by definition, any $\vee$-prime closed subset is nonempty). But every nonempty element of τ intersects nontrivially the other one and this concludes that τ has not ps-property. In the next proposition we find that even if (X, τ) is a T_1-space and τ has the ps-property, then X is not necessarily a sober space.*

Proposition 4.11. *Let (X, τ) be a T_1-space. If X is a sober space, Then τ has the ps-property. The converse is not true.*

Proof. Let, on the contrary, $U, V \in \tau$ and $\varnothing \neq U \subsetneqq V$ be such that for every $\varnothing \neq U_1 \subseteq U$ and $\varnothing \neq V_1 \subseteq V$ we have $U_1 \cap V_1 \neq \varnothing$. It follows that for every nonempty $A, B \in \tau$ contained in U we have $A \cap B \neq \varnothing$. Define $\theta : \tau \longrightarrow \{0, 1\}$ with $\theta(W) = 1$ if and only if $W \cap U \neq \varnothing$. We show that $\theta \in \Sigma\tau$. Clearly, if $W_i \in \tau$ for every $i \in I$, then

$$\theta(\cup_{i\in I} W_i) = 0 \;\Leftrightarrow\; (\cup_{i\in I} W_i)) \cap U = \varnothing$$

$$\Leftrightarrow\; \forall i \in I,\; \theta(W_i) = 0 \;\Leftrightarrow\; \vee_{i\in I}\theta(W_i) = 0 \;\therefore\; \theta(\cup_{i\in I} W_i) = \vee_{i\in I}\theta(W_i).$$

Also, if $W_1, W_2 \in \tau$, then

$$\theta(W_1 \cap W_2) = 0 \;\Leftrightarrow\; W_1 \cap W_2 \cap U = \varnothing \;\Leftrightarrow\; W_1 \cap U = \varnothing \text{ or } W_2 \cap U = \varnothing$$

$$\Leftrightarrow\; \theta(W_1) \wedge \theta(W_2) = 0 \;\therefore\; \theta(W_1 \cap W_2) = \theta(W_1) \wedge \theta(W_2).$$

Thus, $\theta \in \Sigma\tau$ and since X is sober, there exists $x \in X$ such that $\theta = E(x)$. Clearly, $x \in U$ and $U \neq \{x\}$. Therefore, there exists a point $y \in U \setminus \{x\}$. Suppose W_0 is an arbitrary open neighborhood of y. Clearly, $\theta(W_0) = 1$ and so $x \in W_0$. Thus, X cannot be T_1 and this is a contradiction. Now, we construct a T_1-space (X, τ) such that τ has the ps-property but X is not sober. Let A and B be two infinite disjoint sets. Put $X = A \cup B$ and $\tau = \{U \subseteq X :\; U \subseteq A \text{ or } X \setminus U \text{ is finite }\}$. Clearly, (X, τ) is a T_1-topological space and τ has the ps-property. To complete the proof, we show that X is not sober. It is easy to see that A is $\wedge$-prime element in τ, whereas it is not of the form $X \setminus \overline{\{x\}}$. Therefore, X is not sober. □

Proposition 4.12. *Suppose that (F, τ) is an LGT-space and $a \in F$. Then the following statements hold.*

(a) If F has the ps-property, then F_a has also the ps-property.

(b) If (F, τ) is a T_0 (T_1, T_2) space, then (F_a, τ_a) is too.

(c) If (F, τ) is regular and $a \in F^*$, then (F_a, τ_a) is too.

Proof. *(a) and (b) are straightforward.*

(c). Suppose that $x \in F_a$, y is a closed element in (F_a, τ_a) and $x \nleq y$. By Proposition 3.4, there exists $t \in \tau$ such that $y = t^* \wedge a$. Clearly, $x \nleq t^*$ and by hypothesis there exist $0 \neq x_1 \leq x$ and $r, s \in \tau$ such that $x_1 \leq r$, $t^* \leq s$ and $r \wedge s = 0$. Putting $r_a = r \wedge a$ and $s_a = s \wedge a$, then clearly $r_a, s_a \in \tau_a$ and $x_1 \leq r_a$, $y = t^* \wedge a \leq s_a$ and $r_a \wedge s_a = 0$. □

The following proposition is easy to prove.

Proposition 4.13. *Suppose that (F_i, τ_i) is an LGT-space, for every $i \in I$ and $F = \prod_{i\in I} F_i$. Then F has the ps-property if and only if F_i has the ps-property, for every $i \in I$.*

Proposition 4.14. *Suppose that (F_i, τ_i) is an LGT-space, for every $i \in I$ and $F = \prod_{i\in I} F_i$ and τ is the product topology on F. Then the following statements hold.*

(a) (F, τ) is T_0 if and only if F_i is T_0, for every $i \in I$.

(b) (F,τ) *is* T_1 *if and only if* (F_i,τ_i) *is* T_1*, for every* $i\in I$*.*

(c) If I *is infinite, then* (F,τ) *is not* T_2 $(**-T_2)$ *at all. But, if* I *is finite, then* (F,τ) *is* T_2 *(resp.* $**-T_2$*) if and only if* (F_i,τ_i) *is* T_2 *(resp.* $**-T_2$*), for every* $i\in I$*.*

(d) If I *is infinite, then* (F,τ) *is not regular (∗∗-regular) at all. But, if* I *is finite, then* (F,τ) *is regular (resp. ∗∗-regular) if and only if* (F_i,τ_i) *is regular (resp. ∗∗-regular), for every* $i\in I$*.*

Proof. (a $\Rightarrow$). Suppose that $j\in I$ is arbitrary and $a_j,b_j\in F_j$ are two different nonzero elements. Taking $x,y\in F$ so that $x_i=0=y_i$ for every $i\neq j$, $x_j=a_j$ and $y_j=b_j$, then clearly x and y are two different nonzero elements in F and by assumption, there exist $0\neq c\leq x$, $0\neq d\leq y$ and $t\in\tau$ such that $c\leq t$ and $t\wedge d=0$ or $d\leq t$ and $t\wedge c=0$. Clearly, c_j and d_j are nonzero and $c_j\leq t_j$, $t_j\wedge d_j=0$ or $d_j\leq t_j$, $t_j\wedge c_j=0$.

(a $\Leftarrow$). Suppose that $x,y\in F$ and $0\neq x\neq y\neq 0$. By hypothesis, there exists $j\in I$ such that $x_j\neq y_j$. Without loss of generality, suppose that $x_j\neq 0\neq y_j$. Therefore, there exist $0\neq a_j\leq x_j$, $0\neq b_j\leq y_j$ and $r_j\in\tau_j$ such that $a_j\leq r_j$, $r_j\wedge b_j=0$ or $b_j\leq r_j$, $r_j\wedge a_j=0$. Taking $c,d\in F$ so that $c_i=0=d_i$ for every $i\neq j$, $c_j=a_j$, $d_j=b_j$, $t_i=1$ for every $i\neq j$ and $t_j=r_j$, it is easy to see that $c\leq t$ and $t\wedge d=0$ or $d\leq t$ and $t\wedge c=0$.

We do similarly for the remainder of the proof. Note that if I is infinite, then for every $r,s\in\tau$ we have $r\wedge s\neq 0$. □

Proposition 4.15. *Suppose that* (F_i,τ_i) *is an LGT-space, for every* $i\in I$*,* $F=\prod_{i\in I}F_i$ *and* τ_b *is the box topology on* F*. Then the following statements hold.*

(a) (F,τ_b) *is* T_0 *(resp.* $**-T_0$*) if and only if* F_i *is* T_0 *(resp.* $**-T_0$*), for every* $i\in I$*.*

(d) (F,τ_b) *is* T_1 *(resp.* $**-T_1$*) if and only if* (F_i,τ_i) *is* T_1*, (resp.* $**-T_1$*) for every* $i\in I$*.*

(e) (F,τ_b) *is* T_2 *(resp.* $**-T_2$*) if and only if* (F_i,τ_i) *is* T_2 *(resp.* $**-T_2$*), for every* $i\in I$*.*

(f) (F,τ_b) *is regular (resp. ∗∗-regular) if and only if* (F_i,τ_i) *is regular (resp. ∗∗-regular), for every* $i\in I$*.*

Proof. The proof is straightforward. □

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