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WEIGHTED COMPOSITION OPERATORS ON MEASURABLE DIFFERENTIAL FORM SPACES

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Abstract. In this paper, we consider weighted composition operators between measurable differential forms and then some classic properties of these operators are characterized.
Keywords: Riemann surfaces, differential forms, (weighted) composition operators, adjoint operator.
MSC(2010): Primary 47B38; Secondary 30F30.

1. Introduction and preliminaries

A two dimensional manifold M is a connected Haussdorf topological space such that every x in M has a neighborhood homeomorphic to an open disc in the plane. If M is a two dimensional manifold, a complex chart on M is a homeomorphism $\alpha : U_\alpha \to V_\alpha$ of an open subset $U_\alpha \subset M$ onto an open subset $V_\alpha \subset \mathbb{C}$. Two charts α and β are analytically compatible if transition map

$$\tau_{\alpha\beta} = \beta \circ \alpha^{-1} : \alpha(U_\alpha \cap U_\beta) \to \beta(U_\alpha \cap U_\beta)$$

is biholomorphic. A complex atlas on M is a collection of analytically equivalent compatible charts $\mathcal{A} = \{(\alpha, U_\alpha)\}$ whose domains cover M, i.e. $M = \bigcup_\alpha U_\alpha$. Two complex atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are analitycally equivalent if $\mathcal{A}_1 \cup \mathcal{A}_2$ is a complex atlas. An analytic structure on a two dimensional manifold M is an equivalence class of analytically equivalent atlases. A Riemann surface is a two dimensional manifold with an analytic structure.

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A 0-form on N is a complex valued function on N. A 1-form ω on N is an ordered assignment of two function f^{β} and g^{β} to each local coordinate chart (β, V_{β}) on N such that the expression $f^{\beta}d\beta + g^{\beta}d\bar{\beta}$ is invariant under coordinate changes. A 2-form Ω on N is an assignment of a function f^{β} to each local coordinate β such that the expression $f^{\beta}d\beta \wedge d\bar{\beta}$ is invariant under coordinate changes.

Put $A = \{\beta^{-1}(V) : V \subseteq \mathbb{C}$ is open and $\{(\beta, V_{\beta})\}_{\beta\in B}$ is any local chart$\}$. Let $\mathcal{A}_B$ be the σ-algebra generated by A and let Σ be the Borel σ-algebra on N. Then $\mathcal{A}_B \subseteq \Sigma$. Take $\mathcal{A}_{\beta} = \{A \cap V_{\beta} : A \in \mathcal{A}_B\}$. Then $\mathcal{A}_{\beta}$ is a σ-algebra on V_{β}. Note that every $\mathcal{A}_B$-measurable complex valued function on N is also $\mathcal{A}_{\beta}$-measurable function on V_{β} for each β. Let $\omega = f^{\beta}d\beta + g^{\beta}d\bar{\beta}$. Then ω is called measurable if the coefficient functions $f^{\beta}, g^{\beta} : (V_{\beta}, \mathcal{A}_{\beta}) \to \mathbb{C}$ are measurable. By $\Lambda^0(N)$ we denote the set of all $\mathcal{A}_B$-measurable complex valued functions such as $f : N \to \mathbb{C}$ that $f_{|V_{\beta}}$ is an $\mathcal{A}_{\beta}$-measurable function for each β. The support of f is defined by $\mathrm{supp}(f) = \{x \in N : f(x) \neq 0\}$. Let m be the Lebesgue measure on the complex plan $\mathbb{C}$. For each $K \in \mathcal{A}_B$, put $\mu_{\beta}(K) := m(\beta(K \cap V_{\beta}))$. Then μ_{β} is a measure on $\mathcal{A}_{\beta}$. Now, suppose that $\{V_{\beta}\}_{\beta\in B}$ be a countable triangulation of N such that each V_{β} is contained in the domain of some chart on N (see [11]). Define $\mu_B := \Sigma_{\beta\in B}\ \mu_{\beta}$. Then for each $K \in \mathcal{A}_B$, $\mu_B(K) = \Sigma_{\beta\in B}\ m(\beta(K \cap V_{\beta}))$, and so μ_B is a measure on $\mathcal{A}_B$. Throughout this paper, equalities of sets or functions, set inclusions, and inequalities between functions are interpreted as being valid up to an μ-null set.

Let M and N be Riemann surfaces. A continuous map $\rho : M \to N$ is analytic if for any chart α on M and for any chart β on N with $\rho(U_{\alpha}) \subset V_{\beta}$ the function

$$\rho_{\alpha\beta} = \beta \circ \rho \circ \alpha^{-1} : \alpha(U_{\alpha}) \to \beta(V_{\beta})$$

is analytic. Throughout this paper $\rho : M \to N$ will denote analytic map between the Riemann surfaces M and N. For a 1-form $\omega = f^{\beta}d\beta + g^{\beta}d\bar{\beta}$, define ${}^*\omega = -if^{\beta}d\beta + ig^{\beta}d\bar{\beta}$ and $\omega_{\beta} = f^{\beta}_{\beta}dz + g^{\beta}_{\beta}d\bar{z}$, where $f^{\beta}_{\beta} = f^{\beta} \circ \beta^{-1}$ and $g^{\beta}_{\beta} = g^{\beta} \circ \beta^{-1}$. Let $\Lambda^1_2(N)$ be the space of measurable 1-forms on the Riemann surface N which satisfy the following:

$$\|\omega\|^2_N = \int_N \omega \wedge^* \overline{\omega} < \infty.$$

It is well known that $\Lambda^1_2(N)$ is a Hilbert space, with inner product given by $\langle \omega_1, \omega_2 \rangle := \int_N \omega_1 \wedge^* \overline{\omega_2}$ (see [6]). Note that $\mu_B(\{x \in U_{\alpha} :$

$\rho'_{\alpha\beta}(\alpha(x)) = 0\}) = 0$, for each α and β with $\rho(U_\alpha) \subset V_\beta$. Thus $\|\eta\|_{\Lambda_2^1(M)} = \|\eta\|_{\Lambda_2^1(M\setminus Z)}$, where $Z = \{\alpha(x) : \rho'_{\alpha\beta}(\alpha(x)) = 0\}$ and $\eta \in \Lambda_2^1(M)$.

For an analytic map $\rho : M \to N$ and $u \in \Lambda^0(M)$, the weighted composition operator $uC_\rho : \Lambda_2^1(N) \to \Lambda_2^1(M)$ defined as $uC_\rho(\omega) = u\rho^*(\omega)$, where $\rho^*(\omega) = f^\beta \circ \rho d\beta \circ \rho + g^\beta \circ \rho d\bar{\beta} \circ \rho$ is the pullback of the form $\omega = f^\beta d\beta + g^\beta d\bar{\beta}$. Note that $uC_\rho = M_u C_\rho$, where M_u is the multiplication operator and C_ρ is a classic composition operator. The texts [4] and [10] are excellent sources for these operators.

The study on weighted composition operators on various function spaces has received considerable attention in past decades. Characterizations, which usually involved interplay of symbol functions, for certain types of weighted composition operators have been obtained. These operators on Riemann surfaces were first studied by Mihaila in [8]; she obtained some results on composition operators on Riemann surfaces and posed some questions on these operators. Subsequently, Guang Fu Cao in [1] characterized invertibility and Fredholmness of these type operators on Riemann surfaces. Also in [2], composition and the Toeplitz operators on analytic differential forms for Riemann surfaces are defined and some classic properties of these operators are discussed.

The first question which we shall try to answer in this paper is when uC_ρ defines a bounded operator between the spaces $\Lambda_2^1(N)$ and $\Lambda_2^1(M)$. One of the most fundamental questions relating to weighted composition operators is how to obtain a reasonable representation for their adjoints (see [3, 5, 7]). In Theorem 2.6 a problem of Ioana Mihaila [9] is answered on the computation of adjoint of composition operators on measurable differential form spaces for Riemann surfaces. Also, some other properties of these operators are discussed.

2. Main results

Suppose that U and V are bounded domains in $\mathbb{C}$ and $\varphi : U \to V$ is an analytic and nonconstant function. For each $w \in \varphi(U)$, let $\{z_j(w)\}$ be the collection of zeros of $\varphi(z) - w$, including multiplicities. Let f and g be non-negative measurable functions defined on U and V respectively. By the area formula [4, Theorem 2.32], we have

$$\int_U g(\varphi(z))|\varphi'(z)|^2 f(z)dA(z) = \int_{\varphi(U)} g(w)(\Sigma_{j\geq 1} f(z_j(w)))dA(w).$$

Definition 2.1. Let $f \in \Lambda^0(M)$ and $\rho : M \to N$ be an analytic map. The generalized counting function $N(f,\rho) : N \to \mathbb{C} \cup \{\infty\}$ is defined by

$$N(f,\rho)(w) = \sum_{z \in M : \rho(z) = w} f(z)$$

where the number of z above is counted with appropriate multiplicity.

It follows from Definition 2.1 that $N(1,\rho)(w) = \operatorname{card}\{z \in M : \rho(z) = w\} := N_\rho(w)$ and for every $f \in \Lambda^0(N)$ we obtain $N(f \circ \rho, \rho)(w) = f(w)N_\rho(w)$. In the following theorem, we give a necessary and sufficient condition for boundedness of weighted composition operators on differential 1-form spaces.

Theorem 2.2. *Let M and N be Riemann surfaces, $u \in \Lambda^0(M)$ and let $\rho : M \to N$ be an analytic map. Then, the weighted composition operator uC_ρ is a bounded operator from $\Lambda^1_2(N)$ to $\Lambda^1_2(M)$ if and only if the generalized counting function $N(|u|^2, \rho)$ is bounded.*

Proof. To prove the theorem, we use the method which is inspired by Mihaila [9]. First, we assume that $\{U_\alpha\}_{\alpha \in A}$ and $\{V_\beta\}_{\beta \in B}$ are finite triangulations of M and N such that each U_α and V_β are contained in the domain of some charts on M and N respectively with $\rho(U_\alpha) \subseteq V_\beta$. For $w \in \beta(V_\beta)$, put $z_j(w) = \{z \in \alpha(U_\alpha) : \rho_{\alpha\beta}(z) = w\}$. Now let $\omega = f^\beta d\beta + g^\beta d\bar{\beta} \in \Lambda^1_2(N)$. Then $\omega_\beta = f^\beta_\beta dz + g^\beta_\beta d\bar{z}$ and $(uC_\rho(\omega))_\alpha = (u.f^\beta \circ \rho) \circ \alpha^{-1} \rho'_{\alpha\beta} dz + (u.g^\beta \circ \rho) \circ \alpha^{-1} \overline{\rho'_{\alpha\beta}} d\bar{z} = u_\alpha (C_\rho(\omega))_\alpha$. Therefore, by the area formula we have

$$\begin{aligned}
\|uC_\rho(\omega)\|^2 &= 2 \sum_{\alpha \in A} \int_{\alpha(U_\alpha)} |u_\alpha(z)|^2 (|f^\beta_\beta|^2 + |g^\beta_\beta|^2) \circ \rho_{\alpha\beta}(z) |\rho'_{\alpha\beta}(z)|^2 dA(z) \\
&= 2 \sum_{\beta \in B} \int_{\rho_{\alpha\beta}(\alpha(U_\alpha))} (|f^\beta_\beta|^2 + |g^\beta_\beta|^2)(w) (\sum_{j \geq 1} |u_\alpha(z_j(w))|^2) dA(w) \\
&= 2 \sum_{\beta \in B} \sum_{\alpha : \rho(U_\alpha) \subset V_\beta} \int_{\beta(V_\beta)} (|f^\beta_\beta|^2 + |g^\beta_\beta|^2)(w) (\sum_{j \geq 1} |u_\alpha(z_j(w))|^2) dA(w) \\
&= 2 \sum_{\beta \in B} \int_{\beta(V_\beta)} (|f^\beta_\beta|^2 + |g^\beta_\beta|^2)(w) (\sum_{\alpha : \rho(U_\alpha) \subset V_\beta} \sum_{z \in \alpha(U_\alpha) : \beta \circ \rho \circ \alpha^{-1}(z) = w} |u_\alpha(z)|^2) dA(w) \\
&= 2 \sum_{\beta \in B} \int_{\beta(V_\beta)} (|f^\beta_\beta|^2 + |g^\beta_\beta|^2)(w) (\sum_{\alpha : \rho(U_\alpha) \subset V_\beta} \sum_{k \in U_\alpha : \rho(k) = \beta^{-1}(w)} |u(k)|^2) dA(w)
\end{aligned}$$

$$= 2\sum_{\beta\in B}\int_{\beta(V_\beta)}(|f_\beta^\beta|^2 + |g_\beta^\beta|^2)(w)(\sum_{k\in M:\rho(k)=\beta^{-1}(w)} |u(k)|^2)dA(w)$$

$$= 2\sum_{\beta\in B}\int_{\beta(V_\beta)}(|f_\beta^\beta|^2 + |g_\beta^\beta|^2)(w)N(|u|^2,\rho)(\beta^{-1}(w))\ dA(w)$$

$$= 2\sum_{\beta\in B}\int_{\beta(V_\beta)}(|f_\beta^\beta|^2+|g_\beta^\beta|^2)(w)(N(|u|^2,\rho))_\beta(w)\ dA(w) = \int_N N(|u|^2,\rho)\ \omega\wedge^*\bar{\omega}.$$

Now, suppose that $\{U_\alpha\}_{\alpha\in A}$ and $\{V_\beta\}_{\beta\in B}$ are countable triangulations of M and N instead of the finite ones. By the same method used in the proof of Theorem 2.2 in [9], we again get that $\|uC_\rho(\omega)\|^2 = \int_N N(|u|^2,\rho)\ \omega\wedge^*\bar{\omega}$. Therefore, $uC_\rho : \Lambda_2^1(N)\to\Lambda_2^1(M)$ is bounded if and only if there exist a positive constant C such that $N(|u|^2,\rho)(w)\le C$ for all $w\in N$. In this case $\|uC_\rho\|\le\sqrt{N(|u|^2,\rho)}$. □

Corollary 2.3. Corollary 2.3. *(a) The composition operator C_ρ is a bounded operator from $\Lambda_2^1(N)$ to $\Lambda_2^1(M)$ if and only if the counting function N_ρ is bounded.*

(b) The multiplication operator $M_u : \Lambda_2^1(M)\to\Lambda_2^1(M)$ defined as $M_u(\omega) = uf_\alpha d\alpha + ug^\alpha d\bar{\alpha}$ is bounded if and only if $u\in L^\infty(M)$.

Theorem 2.4. *Let $N(|u|^2,\rho)$ is a bounded below function on the set $\{\rho(x) : u(x)\neq 0\}$. Then, the weighted composition operator $uC_\rho : \Lambda_2^1(N)\to\Lambda_2^1(M)$ is compact if and only if it is the zero operator.*

Proof. Put $A = \{x\in M : u(x)\neq 0\}$ and let $S = intA$ be the interior points of A. Let $N(|u|^2,\rho)\ge k$ on $\rho(A)$ for some $k>0$ and let $uC_\rho : \Lambda_2^1(N)\to\Lambda_2^1(M)$ be a nonzero compact operator. Then $\mu(S)>0$. Since ρ is analytic, we can choose an open set $G\subseteq\rho(S)$ such that $G\subseteq V_\beta$ for some local chart (V_β,β) on N. Let L be the subspace of $\Lambda_2^1(N)$ consisting of square integrable measurable 1-forms on G extended to be zero outside G. Let $\omega_n = f^n d\beta + g^n d\bar{\beta}$ be an arbitrary orthonormal sequence in L. Bessel's inequality implies that $\omega_n\to 0$ as weakly. Since a compact operator maps weakly convergent sequences into norm convergent ones, it follows that $\|uC_\rho(\omega_n)\|\to 0$. On the other hand, by Theorem 2.2 we have

$$\|uC_\rho(\omega_n)\|^2 = \int_G N(|u|^2,\rho)\ (\omega_n\wedge^*\bar{\omega}_n)\ge k\|\omega_n\|^2 = k>0.$$

But this is a contradiction. □ □

Remark 2.5. Recall that an atom of the measure μ_B is an element $A \in \mathcal{A}_B$ with $\mu_B(A) > 0$ such that for each $F \in \mathcal{A}_B$, if $F \subseteq A$ then either $\mu_B(F) = 0$ or $\mu_B(F) = \mu_B(A)$. Now, we show that μ_B is a non-atomic measure on $\mathcal{A}_B$. Let $A \in \mathcal{A}_B$ is an atom with respect to the measure μ_B. Then, there is $\beta_0 \in B$ such that $m(\beta_0(A \cap V_{\beta_0})) > 0$. Since the Lebesgue measure m on the complex plan $\mathbb{C}$ is non-atomic, there exists a $K \subset \beta_0(A \cap V_{\beta_0})$ with $0 < m(K) < m(\beta_0(A \cap V_{\beta_0}))$. Put $F = \beta_0^{-1}(K)$. Then we have

$$0 < m(K) = m(K \cap \beta_0(V_{\beta_0})) = m(\beta_0(F) \cap V_{\beta_0}) \leq \mu_B(F)$$

$$= \sum_{\beta \in B} m(\beta(F) \cap V_\beta) = \sum_{\beta_0 \neq \beta \in B} m(\beta(F) \cap V_\beta) + m(\beta_0(F) \cap V_{\beta_0})$$

$$< \sum_{\beta_0 \neq \beta \in B} m(\beta(A \cap V_\beta) + m(\beta_0(A \cap V_{\beta_0}) = \sum_{\beta \in B} m(\beta(A \cap V_\beta) = \mu_B(A).$$

But this is a contradiction. Thus, a weighted composition operator on $L^2(N, \mathcal{A}_B, \mu_B)$ is compact if and only if it is a zero operator (see [12]).

Proposition 2.6. *Let $u \in \Lambda^0(N)$ and let the multiplication operator $M_u : \Lambda_2^1(N) \to \Lambda_2^1(N)$ be bounded. Then M_u has closed range if and only if u be bounded away from zero on $supp(u)$.*

Proof. Let $|u| \geq k$ a.e. on $\mathrm{supp}(u)$ for some $k > 0$. Then, for all $\omega \in \Lambda_2^1(supp(u))$ we have $\|M_u(\omega)\|_{\Lambda_2^1(\mathrm{supp}(u))} \geq k\|\omega\|_{\Lambda_2^1(\mathrm{supp}(u))}$, and so M_u has closed range. Now, suppose u is not bounded away from zero on $\mathrm{supp}(u)$. Since N is non-atomic, we can find a $\beta \in B$ and a sequence of open subsets $\{O_n\}_n$ with $\mu_B(O_n) > 0$ such that $O_n \subseteq V_\beta \cap \mathrm{supp}(u)$ and $|u| < \frac{1}{n}$ a.e., on O_n, for each $n \in \mathbb{N}$. Choose $\omega_n := f_n^\beta d\beta + g_n^\beta d\bar{\beta}$ to be a sequence of 1-forms such that $\omega_n \in \Lambda_2^1(O_n)$, for each $n \in \mathbb{N}$. Then we have

$$\|M_u(\omega_n)\|^2 = \int_{O_n} |u|^2(\omega_n \wedge^* \bar{\omega}_n) \leq \frac{1}{n^2}\ \|\omega_n\|^2.$$

Thus, the range of M_u is not closed. □

It is known that the bounded weighted composition operator $uC_\rho : \Lambda_2^1(N) \to \Lambda_2^1(M)$ has closed range if and only $(uC_\rho)^*(uC_\rho)$ has closed range on $\Lambda_2^1(N)$. Then, we have the following corollary.

Corollary 2.7. *Let the weighted composition operator $uC_\rho : \Lambda_2^1(N) \to \Lambda_2^1(M)$ be bounded. Then, uC_ρ has closed range if and only if $N(|u|^2, \rho)$ be bounded away from zero on its support.*

In [9], Mihaila considered composition operators acting on measurable and analytic differential form spaces for Riemann surfaces and asked the following question: *Can the adjoints of these operators be computed, at least in some of the cases?* In the following theorem, we compute an explicit formula on the measurable differential form spaces for the adjoint of a weighted composition operator with symbol ρ and measurable function u.

Theorem 2.8. *Let M and N be Riemann surfaces, $u \in \Lambda^0(M)$ and let $\rho : M \to N$ be an analytic map. Then, whenever $uC_\rho : \Lambda^1_2(N) \to \Lambda^1_2(M)$ is a bounded operator, $(uC_\rho)^*$ is given by the formula*

$$(uC_\rho)^*(\eta) = N(\frac{\bar{u}k^\alpha}{\rho'_{\alpha\beta} \circ \alpha}, \rho)d\beta + N(\frac{\bar{u}l^\alpha}{\overline{\rho'_{\alpha\beta}} \circ \alpha}, \rho)d\bar{\beta},$$

where $\eta = k^\alpha d\alpha + l^\alpha d\bar{\alpha} \in \Lambda^1_2(M)$ and $\rho(U_\alpha) \subset V_\beta$.

Proof. Let $\omega = f^\beta d\beta + g^\beta d\bar{\beta} \in \Lambda^1_2(N)$. Then, for each local chart (U_α, α) in M we get that $(uC_\rho(\omega))_\alpha \wedge ({}^*\bar{\eta})_\alpha = 2[(u_\alpha \overline{k^\alpha_\alpha} \rho'_{\alpha\beta})(f^\beta_\beta \circ \rho_{\alpha\beta}) + (u_\alpha \overline{l^\alpha_\alpha \rho'_{\alpha\beta}})(g^\beta_\beta \circ \rho_{\alpha\beta})]dA$, where we have used the fact that $idz \wedge d\bar{z} = 2dA$. Also, since for each α, $\rho_{\alpha\beta}$ is analytic function on U_α, then the set $U_{\alpha 0} := \{u \in U_\alpha : \rho'_{\alpha\beta}(\alpha(u)) = 0\}$ is at most countable for each α and so $\mu(U_{\alpha 0}) = 0$. Then by the area formula we have

$$\langle \omega, (uC_\rho)^*(\eta) \rangle = \langle uC_\rho(\omega), \eta \rangle = \int_M uC_\rho(\omega) \wedge^* \bar{\eta} = \sum_\alpha \int_{\alpha(U_\alpha)} (uC_\rho(\omega))_\alpha \wedge ({}^*\bar{\eta})_\alpha$$

$$= 2 \sum_\alpha \int_{\alpha(U_\alpha)} [(u_\alpha \overline{k^\alpha_\alpha} \rho'_{\alpha\beta})(f^\beta_\beta \circ \rho_{\alpha\beta}) + (u_\alpha \overline{l^\alpha_\alpha \rho'_{\alpha\beta}})(g^\beta_\beta \circ \rho_{\alpha\beta})](z) dA(z)$$

$$= 2 \sum_\alpha \int_{\alpha(U_\alpha)} [\frac{u_\alpha \overline{k^\alpha_\alpha}}{\overline{\rho'_{\alpha\beta}}}(f^\beta_\beta \circ \rho_{\alpha\beta})|\rho'_{\alpha\beta}|^2 + \frac{u_\alpha \overline{l^\alpha_\alpha}}{\rho'_{\alpha\beta}}(g^\beta_\beta \circ \rho_{\alpha\beta})|\rho'_{\alpha\beta}|^2](z) dA(z)$$

$$= 2 \sum_\beta \int_{\rho_{\alpha\beta}(\alpha(U_\alpha))} [f^\beta_\beta(w)(\sum_{j \geq 1}(\frac{u_\alpha \overline{k^\alpha_\alpha}}{\overline{\rho'_{\alpha\beta}}})(z_j(w)) + g^\beta_\beta(w)(\sum_{j \geq 1}(\frac{u_\alpha \overline{l^\alpha_\alpha}}{\rho'_{\alpha\beta}})(z_j(w))]dA(w)$$

$$= 2 \sum_\beta \int_{\beta(V_\beta)} [f^\beta_\beta(w)(\sum_{\alpha : \rho(U_\alpha) \subset V_\beta} \sum_{j \geq 1} (\frac{u\overline{k^\alpha}}{\overline{\rho'_{\alpha\beta}} \circ \alpha})_\alpha)(z_j(w))$$

$$+ g^\beta_\beta(w)(\sum_{\alpha : \rho(U_\alpha) \subset V_\beta} \sum_{j \geq 1} (\frac{u\overline{l^\alpha}}{\rho'_{\alpha\beta} \circ \alpha})_\alpha)(z_j(w))]dA(w)$$

$$= 2\sum_{\beta}\int_{\beta(V_\beta)}[f^{\beta}_{\beta}(w)(\sum_{y\in M:\rho(y)=\beta^{-1}(w)}\frac{\bar{u}k^{\alpha}}{\rho'_{\alpha\beta}\circ\alpha}(y))\overline{)_{\beta}}$$

$$+g^{\beta}_{\beta}(w)(\sum_{y\in M:\rho(y)=\beta^{-1}(w)}\frac{\bar{u}l^{\alpha}}{\overline{\rho'_{\alpha\beta}}\circ\alpha}(y))\overline{)_{\beta}}\,]dA(w)$$

$$= 2\sum_{\beta}\int_{\beta(V_\beta)}[f^{\beta}_{\beta}(w)N(\frac{\bar{u}k^{\alpha}}{\rho'_{\alpha\beta}\circ\alpha},\rho)_{\beta}(w)+g^{\beta}_{\beta}(w)N(\frac{\bar{u}l^{\alpha}}{\overline{\rho'_{\alpha\beta}}\circ\alpha},\rho)_{\beta}(w)]dA(w)$$

$$= \langle\omega, N(\frac{\bar{u}k^{\alpha}}{\rho'_{\alpha\beta}\circ\alpha},\rho)d\beta + N(\frac{\bar{u}l^{\alpha}}{\overline{\rho'_{\alpha\beta}}\circ\alpha},\rho)d\bar{\beta}\rangle.$$

Thus $(uC_\rho)^*(\eta) = N(\frac{\bar{u}k^{\alpha}}{\rho'_{\alpha\beta}\circ\alpha},\rho)d\beta + N(\frac{\bar{u}l^{\alpha}}{\overline{\rho'_{\alpha\beta}}\circ\alpha},\rho)d\bar{\beta}$. □

Now, let $\omega = f^{\beta}d\beta + g^{\beta}d\bar{\beta} \in \Lambda^1_2(N)$, $\eta = k^{\alpha}d\alpha + l^{\alpha}d\bar{\alpha} \in \Lambda^1_2(M)$ and uC_ρ is a bounded operator from $\Lambda^1_2(N)$ to $\Lambda^1_2(M)$. Then by Theorem 2.8 we have

$$(uC_\rho)^*(uC_\rho)(\omega) = N(\frac{|u|^2}{\rho'_{\alpha\beta}\circ\alpha}\rho'_{\alpha\beta}\circ\alpha.f^{\beta}\circ\rho,\rho)d\beta + N(\frac{|u|^2}{\overline{\rho'_{\alpha\beta}}\circ\alpha}\overline{\rho'_{\alpha\beta}}\circ\alpha.g^{\beta}\circ\rho,\rho)d\bar{\beta}$$

$$= f^{\beta}N(|u|^2,\rho)d\beta + g^{\beta}N(|u|^2,\rho)d\bar{\beta} = N(|u|^2,\rho)\omega$$

and

$$(uC_\rho)(uC_\rho)^*(\eta) = uN(\frac{\bar{u}k^{\alpha}}{\rho'_{\alpha\beta}\circ\alpha},\rho)\circ\rho\; d(\beta\circ\rho) + uN(\frac{\bar{u}l^{\alpha}}{\overline{\rho'_{\alpha\beta}}\circ\alpha},\rho)\circ\rho\; d(\bar{\beta}\circ\rho),$$

where $d(\beta\circ\alpha) = \rho'_{\alpha\beta}\circ\alpha d\alpha$ and

$$(N(\frac{\bar{u}k^{\alpha}}{\rho'_{\alpha\beta}\circ\alpha},\rho)\circ\rho)(k) = \sum_{x\in\rho^{-1}(\rho(k))}\frac{\bar{u}(x)k^{\alpha}(x)}{\rho'_{\alpha\beta}\circ\alpha(x)}, \quad \text{for all } k\in M.$$

Also, by using Theorem 2.8, it is easy to check that $ker(uC_\rho) = ker(uC_\rho)^*(uC_\rho) = \{\omega\in\Lambda^1_2(N) : N(|u|^2,\rho)\omega = 0\} = \Lambda^1_2([suppN(|u|^2,\rho)]^c)$. It follows that $\Lambda^1_2(N) = \Lambda^1_2(suppN(|u|^2,\rho))\bigoplus\Lambda^1_2([suppN(|u|^2,\rho)]^c)$. These observations establish the following corollary.

Corollary 2.9. *Let $\omega\in\Lambda^1_2(N)$, $\eta = k^{\alpha}d\alpha + l^{\alpha}d\bar{\alpha}\in\Lambda^1_2(M)$ and let uC_ρ be a bounded operator from $\Lambda^1_2(N)$ to $\Lambda^1_2(M)$. Then*

(a) $(uC_\rho)^*(uC_\rho)(\omega) = N(|u|^2,\rho)\omega$.

(b) $(uC_\rho)(uC_\rho)^*(\eta) = uN(\frac{\bar{u}k^\alpha}{\rho'_{\alpha\beta}\circ\alpha}, \rho) \circ \rho.\rho'_{\alpha\beta} \circ \alpha d\alpha + uN(\frac{\bar{u}l^\alpha}{\overline{\rho'_{\alpha\beta}}\circ\alpha}, \rho) \circ \rho.\overline{\rho'_{\alpha\beta}} \circ \alpha d\bar{\alpha}$, *where* $\rho(U_\alpha) \subset V_\beta$.

(c) uC_ρ *is one-to-one if and only if* $N(|u|^2, \rho) \geq \alpha$ *on* N, *for some* $\alpha > 0$.

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