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MULTIPLE SOLUTIONS FOR A PERTURBED NAVIER BOUNDARY VALUE PROBLEM INVOLVING THE p-BIHARMONIC

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Abstract. The aim of this article is to establish the existence of at least three solutions for a perturbed p-biharmonic equation depending on two real parameters. The approach is based on variational methods.

Keywords: Three solutions; three critical points theorem; p-biharmonic equations; critical point theory.
MSC(2010): Primary: 35J20; Secondary: 47J10, 58E05, 11Y50.

1. Introduction and main results

Consider the Navier boundary value problem involving the p-biharmonic operator

$$(\mathcal{P}_{\lambda,\mu}) \qquad \begin{cases} \Delta\left(|\Delta u|^{p-2}\Delta u\right) = \lambda f(x,u) + \mu g(x,u), & \text{in}\,\Omega, \\ u = \Delta u = 0, & \text{on}\,\partial\Omega, \end{cases}$$

where $\lambda,\, \mu \in]0,+\infty[$, $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a non-empty bounded open set with a sufficient smooth boundary $\partial\Omega$, $p > \max\left\{1, \frac{N}{2}\right\}$. $f,\, g : \Omega\times\mathbb{R} \to \mathbb{R}$ are Carathéodory functions, that is, $f(\cdot, s)$, $g(\cdot, s)$ are measurable in Ω for all $s \in \mathbb{R}$ and $f(x,\cdot)$, $g(x,\cdot)$ are continuous in $\mathbb{R}$ for a.e. $x \in \Omega$. Furthermore, they satisfy the following conditions:

- $\sup_{|s|\leq M} |f(x,s)| \in L^1(\Omega)$ and $\sup_{|s|\leq M} |g(x,s)| \in L^1(\Omega)$ for all $M > 0$.

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Here and in the sequel, X will denote the Sobolev space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and endowed with the norm

$$\|u\| = \left(\int_\Omega |\Delta u|^p \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

As usual, a weak solution of problem $(\mathcal{P}_{\lambda,\mu})$ is any $u \in X$, such that

$$\int_\Omega |\Delta u|^{p-2} \Delta u \Delta \xi \, \mathrm{d}x = \lambda \int_\Omega f(x,u)\xi \, \mathrm{d}x + \mu \int_\Omega g(x,u)\xi \, \mathrm{d}x$$

for every $\xi \in X$.

The fourth-order equation of nonlinearity furnishes a model to study traveling waves in suspension bridges. This fourth-order semilinear elliptic problem can be considered as an analogue of a class of second-order problems which have been studied by many authors. In particular, the deformations of an elastic beam in an equilibrium state, whose two ends are simply supported, can be described by fourth-order boundary value problems and, also for this reason, the existence and multiplicity of solutions for this kind of problems have been widely investigated (see, for instance, [8, 10–23, 25, 26, 29, 30] and references therein).

The same variational methods as were used there, which are based on the seminal paper of Ricceri [28], have been applied to obtain multiple solutions (see, for instance, [1–8, 11, 12, 15, 16, 18–24, 27, 30]). In this paper, precise estimates of parameters λ and μ are given.

Now, we establish the theorem which not only gives the estimate of the λ but also the μ. Before proving the theorem, we give out some notations.

Let

$$k = \sup_{u \in X \setminus \{0\}} \frac{\sup_{x \in \Omega} |u(x)|}{\|u\|}. \tag{1.1}$$

Since $p > \max\left\{1, \frac{N}{2}\right\}$, $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ is compact, and one has $k < +\infty$. For every $x^0 \in \Omega$ and pick r_1, r_2 with $r_2 > r_1 > 0$, such that $B(x^0, r_1) \subset B(x^0, r_2) \subseteq \Omega$, where $B(x^0, r_1)$ denotes the ball with center at x^0 and radius of r_1. Put

$$\sigma = \frac{12(N+2)^2(r_1+r_2)}{(r_2-r_1)^3} \left(\frac{k\pi^{\frac{N}{2}}(r_2^N - r_1^N)}{\Gamma\left(1+\frac{N}{2}\right)} \right)^{\frac{1}{p}}, \tag{1.2}$$

$$\theta = \begin{cases} \frac{3N}{(r_2-r_1)(r_2+r_1)} \left(\frac{k\pi^{\frac{N}{2}}((r_2+r_1)^N-(2r_1)^N)}{2^N\Gamma\left(1+\frac{N}{2}\right)} \right)^{\frac{1}{p}}, & N < \frac{4r_1}{r_2-r_1}, \\ \frac{12r_1}{(r_2-r_1)^2(r_2+r_1)} \left(\frac{k\pi^{\frac{N}{2}}((r_2+r_1)^N-(2r_1)^N)}{2^N\Gamma\left(1+\frac{N}{2}\right)} \right)^{\frac{1}{p}}, & N \geq \frac{4r_1}{r_2-r_1}, \end{cases} \tag{1.3}$$

where $\Gamma(\cdot)$ is the Gamma function. Moreover, put $G^c := \int_\Omega \max_{|s|\leq c} G(x,s)\,\mathrm{d}x$ for all $c > 0$ and $G_d := \inf_{\Omega\times[0,d]} G$ for all $d > 0$, where $G(x,s) = \int_0^s g(x,t)\,\mathrm{d}t$. Clearly, $G^c \geq 0$ and $G_d \leq 0$. We read $\frac{r}{0} = +\infty$ for convenience. Our another result is the following theorem.

Theorem 1.1. *Assume that there exist two positive constants c and d with $c < \theta d$, such that*

(H3) $F(x,s) \geq 0$ *for each* $(x,s) \in \{\overline{\Omega} \setminus B(x^0,r_1)\} \times [0,d]$*;*
(H4) $\frac{\int_\Omega \max_{|s|\leq c} F(x,s)\,\mathrm{d}x}{c^p} \leq \frac{1}{(\sigma d)^p} \int_{B(x^0,r_1)} F(x,d)\,\mathrm{d}x$*;*
(H5) $\limsup\limits_{|s|\to+\infty} \frac{\sup_{x\in\Omega} F(x,s)}{s^p} \leq 0$.

Then, for every $\lambda \in \Lambda := \left] \frac{(\sigma d)^p}{\int_{B(x^0,r_1)} F(x,d)\,\mathrm{d}x}, \frac{c^p}{\int_\Omega \max_{|s|\leq c} F(x,s)\,\mathrm{d}x} \right[$ *and for every Carathéodory function* $g : \Omega \times \mathbb{R} \to \mathbb{R}$ *such that*

(H6) $\limsup\limits_{|s|\to+\infty} \frac{\sup_{x\in\Omega} G(x,s)}{s^p} < +\infty$,

there exists $\overline{\delta}_{\lambda,g} > 0$ *such that, for each* $\mu \in [0, \overline{\delta}_{\lambda,g}[$*, problem* $(\mathcal{P}_{\lambda,\mu})$ *has at least three solutions, where*

$$\delta_{\lambda,g} := \min\left\{ \frac{c^p - \lambda p k^p \int_\Omega \max_{|s|\leq c} F(x,s)\,\mathrm{d}x}{pk^p G^c}, \right.$$
$$\left. \frac{\sigma^p d^p - \lambda p k^p \int_{B(x^0,r_1)} F(x,d)\,\mathrm{d}x}{pk^p\,|\Omega|\,G_d} \right\},$$

$$\overline{\delta}_{\lambda,g} := \min\left\{ \delta_{\lambda,g}, \frac{1}{\max\left\{0, pk^p\,|\Omega| \limsup\limits_{|s|\to+\infty} \frac{\sup_{x\in\Omega} G(x,s)}{s^p} \right\}} \right\},$$

and $|\Omega|$ is the Lebesgue measure of Ω.

Remark 1.2. *Here, we prove that the perturbed problem $(\mathcal{P}_{\lambda,\mu})$ has at least three solutions by choosing μ in a suitable way but under a more* *general growth condition on the nonlinear term. In particular, we require on the primitive of the function f both a growth more than quadratic in a suitable interval and a growth less than quadratic at infinity, moreover on g an asymptotic condition is requested.*

If $N = 1$ and the function f is dependent on u only, we can get better result than Theorem 1.1. For simplicity, fixing $\Omega = [0, 1]$, $p > 1$, consider the following equation,

$$(\mathcal{P}'_{\lambda,\mu}) \qquad \begin{cases} (|u''|^{p-2}\, u'')'' = \lambda f(u) + \mu g(x, u), & \text{in }]0, 1[, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}.$$

Now, we present another result.

Theorem 1.3. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist two positive constants c, d with $c < d$, such that*

(H7) $f(s) \geq 0$ *for each* $s \in [-c, d]$*;*
(H8) $F(c) < p\left(\frac{c}{16d}\right)^p F(d)$*;*
(H9) $\limsup\limits_{|s|\to+\infty} \frac{F(s)}{s^p} \leq 0$.

Then, for every $\lambda \in \Lambda := \left]\frac{(16d)^p}{pF(d)}, \frac{c^p}{F(c)}\right[$ and for every Carathéodory function $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ such that

(H10) $\limsup\limits_{|s|\to+\infty} \frac{\sup_{x\in[0,1]} G(x,s)}{s^p} < +\infty$,

there exists $\overline{\delta}_{\lambda,g} > 0$, for each $\mu \in [0, \overline{\delta}_{\lambda,g}[$, the problem $(\mathcal{P}'_{\lambda,\mu})$ has at least three solutions, where

$$\delta_{\lambda,g} := \min\left\{ \frac{c^p - \lambda F(c)}{G^c}, \frac{\lambda F(d) - \frac{(16d)^p}{p}}{G_d} \right\}$$

and

$$\overline{\delta}_{\lambda,g} := \min\left\{ \delta_{\lambda,g}, \frac{p}{\max\left\{0, \limsup\limits_{|s|\to+\infty} \frac{\sup_{x\in[0,1]} G(x,s)}{s^p}\right\}} \right\}.$$

Remark 1.4. *In this situation, Theorem 1.3 gives out a location of the set $\Lambda \in [0, +\infty[$ and an estimate of μ which is more piecewise than [20].*

Now, we give an application of previous result.

Example 1.5. Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(s) = \begin{cases} s^2 & s \le 1 \\ \frac{1}{s^2} & s > 1 \end{cases}$$

and

$$g(s) = \frac{2s}{1+s^2}.$$

The following problem

$$\begin{cases} u^{(4)} = \lambda f(u) + \mu g(x,u), & x \in]0,1[, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

admits at least three solutions, for each $\lambda \in]384, 3000[$, $\mu < \frac{3\times 10^{-6} - \lambda \times 10^{-9}}{3}$.

Indeed, bearing in mind that in this case we have $p = 2$,

$$F(u) = \begin{cases} \frac{u^3}{3} & u \le 1 \\ -\frac{1}{u} + \frac{4}{3} & u > 1 \end{cases}$$

and

$$G(u) = \ln(1+u^2),$$

for every $u \in \mathbb{R}$, by choosing $c = 10^{-3}$ and $d = 1$, an easy computation shows that all assumptions of Theorem 1.3 are satisfied and our claim follows.

2. Preliminary

Our main tools are three critical points theorems that we recall here in a convenient form. The theorem was obtained in [9].

Theorem 2.1 (see [9]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that:

- (a_1) $\frac{\sup_{\Phi(x)\le r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$,
- (a_2) *for each* $\lambda \in \Lambda_r := \left] \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(x)} \right[$ *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

3. Proof the main results

Proof of Theorem 1.1. Fix λ, g and μ as in the conclusion. For each $u \in X$,

$$\Phi(u) = \frac{1}{p} \|u\|^p$$

and

$$\Psi(u) = \int_\Omega \left[F(x, u(x)) + \frac{\mu}{\lambda} G(x, u(x)) \right] \mathrm{d}x.$$

On the space $C^0(\overline{\Omega})$, we consider the norm $\|u\|_\infty = \sup_{x\in\Omega} |u(x)|$. Due to (1.1), we have

$$\|u\|_\infty \le k \|u\| . \tag{3.1}$$

Since the critical points of the functional $\Phi(u) - \lambda\Psi(u)$ on X are exactly the weak solutions of problem $(\mathcal{P}_{\lambda,\mu})$, our aim is to apply Theorem 2.1 to $\Phi(u)$ and $\Psi(u)$. To this end, taking into account that the regularity assumptions of Theorem 2.1 on $\Phi(u)$ and $\Psi(u)$ are satisfied and the assumptions of f and g, so we will only verify (a_1) and (a_2).

Put $r = \frac{1}{p}(\frac{c}{k})^p$ taking into account (3.1), one has

$$\begin{aligned} \sup_{\Phi(u)\le r} \Psi(u) &= \sup_{\Phi(u)\le r} \int_\Omega \left[F(x, u(x)) + \frac{\mu}{\lambda} G(x, u(x)) \right] \mathrm{d}x \\ &\le \int_\Omega \max_{|s|\le c} F(x, s)\, \mathrm{d}x + \frac{\mu}{\lambda} G^c. \end{aligned} \tag{3.2}$$

Now, let $\bar{u}(x)$ be the function defined by

$$\bar{u}(x) = \begin{cases} 0, & x \in \Omega \backslash B(x^0, r_2), \\ \frac{d\left(3(l^4-r_2^4)-4(r_1+r_2)(l^3-r_2^3)+6r_1r_2(l^2-r_2^2)\right)}{(r_2-r_1)^3(r_1+r_2)}, & x \in B(x^0, r_2)\backslash B(x^0, r_1), \\ d, & x \in B(x^0, r_1), \end{cases}$$

where $l = \mathrm{dist}(x, x^0) = \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}$. We have

$$\frac{\partial \overline{u}(x)}{\partial x_i} = \begin{cases} 0, & x \in \Omega\backslash B(x^0, r_2) \cup B(x^0, r_1), \\ \frac{12d\left(l^2(x_i-x_i^0)-(r_1+r_2)l(x_i-x_i^0)+r_1r_2(x_i-x_i^0)\right)}{(r_2-r_1)^3(r_1+r_2)}, & x \in B(x^0, r_2)\backslash B(x^0, r_1), \end{cases}$$

$$\frac{\partial^2 \overline{u}(x)}{\partial^2 x_i} = \begin{cases} 0, & x \in \Omega\backslash B(x^0, r_2) \cup B(x^0, r_1), \\ \frac{12d\left(r_1r_2+(2l-r_1-r_2)(x_i-x_i^0)^2/l-(r_2+r_1-l)l\right)}{(r_2-r_1)^3(r_1+r_2)}, & x \in B(x^0, r_2)\backslash B(x^0, r_1), \end{cases}$$

$$\sum_{i=1}^{N} \frac{\partial^2 \overline{u}(x)}{\partial^2 x_i} = \begin{cases} 0, & x \in \Omega \backslash B(x^0, r_2) \cup B(x^0, r_1), \\ \frac{12d\left((N+2)l^2-(N+1)(r_1+r_2)l+Nr_1r_2\right)}{(r_2-r_1)^3(r_1+r_2)}, & x \in B(x^0, r_2) \backslash B(x^0, r_1). \end{cases}$$

It is easy to verify that $\overline{u} \in X$, and in particular, one has

(3.3)

$$\|\overline{u}\|^p = \frac{(12d)^p 2\pi^{\frac{N}{2}}}{(r_2-r_1)^{3p}(r_1+r_2)^p\Gamma(\frac{N}{2})} \int_{r_1}^{r_2} |(N+2)r^2 - (N+1)(r_1+r_2)r + Nr_1r_2|^p r^{N-1}\,\mathrm{d}r.$$

Here, we obtain from (1.2), (1.3) and (3.3) that

$$\frac{\theta^p d^p}{k^p} < \|\overline{u}\|^p < \frac{\sigma^p d^p}{k^p}. \tag{3.4}$$

By the assumption

$$d\theta > c,$$

it follows from (3.4) that

$$\frac{\|\overline{u}\|^p}{p} > \frac{d^p\theta^p}{k^p p} > \frac{1}{p}\left(\frac{c}{k}\right)^p = r.$$

Now, by (3.2), we have

$$\begin{aligned} &\frac{\Psi(\bar{u})}{\Phi(\bar{u})} \geq \frac{pk^p \int_{B(x^0,r_1)} F(x,d)\,\mathrm{d}x}{\sigma^p d^p} + \frac{\mu}{\lambda}\frac{pk^p\,|\Omega|\,G_d}{\sigma^p d^p}. \\ &\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} \leq pk^p \frac{\int_\Omega \max_{|s|\leq c} F(x,s)\,\mathrm{d}x}{c^p} + pk^p\frac{\mu}{\lambda}\frac{G^c}{c^p}. \end{aligned} \tag{3.5}$$

Since $\mu < \delta$, one has

$$\mu < \frac{c^p - \lambda pk^p \int_\Omega \max_{|s|\leq c} F(x,s)\,\mathrm{d}x}{pk^p G^c},$$

this means

$$pk^p \frac{\int_\Omega \max_{|s|\leq c} F(x,s)\,\mathrm{d}x}{c^p} + pk^p\frac{\mu}{\lambda}\frac{G^c}{c^p} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{\sigma^p d^p - \lambda pk^p \int_{B(x^0,r_1)} F(x,d)\,\mathrm{d}x}{pk^p\,|\Omega|\,G_d},$$

this means

$$\frac{pk^p \int_{B(x^0,r_1)} F(x,d)\,\mathrm{d}x}{\sigma^p d^p} + \frac{\mu}{\lambda}\frac{pk^p\,|\Omega|\,G_d}{\sigma^p d^p} > \frac{1}{\lambda}.$$

We obtain

$$pk^p \frac{\int_\Omega \max_{|s|\le c} F(x,s)\,\mathrm{d}x}{c^p} + pk^p \frac{\mu}{\lambda}\frac{G^c}{c^p} < \frac{1}{\lambda} < \tag{3.6}$$
$$\frac{pk^p \int_{B(x^0,r_1)} F(x,d)\,\mathrm{d}x}{\sigma^p d^p} + \frac{\mu}{\lambda}\frac{pk^p\,|\Omega|\,G_d}{\sigma^p d^p}.$$

Hence, from (3.5) and (3.6), condition (a_1) of Theorem 2.1 is verified.

Finally, since $\mu < \overline{\delta}_g$ we can fix $l > 0$ such that $\limsup\limits_{|s|\to+\infty} \frac{\sup_{x\in\Omega} G(x,s)}{s^p} < l$ and $\mu l < \frac{1}{pk^p|\Omega|}$. Therefore, there exists a function $h \in L^1(\Omega)$ such that

$$G(x,s) \le ls^p + h(x)$$

for each $(x,s) \in \Omega \times \mathbb{R}$.

Now, fix $0 < \varepsilon < \frac{1-\mu lk^p|\Omega|p}{\lambda k^p|\Omega|p}$. From (H5) there is a function $h_\varepsilon(x) \in L^1(\Omega)$ such that

$$F(x,s) \le \varepsilon s^p + h_\varepsilon(x)$$

for each $(x,s) \in \Omega \times \mathbb{R}$. It follows that, by (3.1), for each $u \in X$,

$$\Phi(u) - \lambda\Psi(u) \ge \left(\frac{1}{p} - \lambda\varepsilon k^p\,|\Omega| - \mu lk^p\,|\Omega|\right)\|u\|^p - \lambda\,\|h_\varepsilon\|_{L^1} - \mu\,\|h\|_{L^1}.$$

This leads to the coercivity of $\Phi(u) - \lambda\Psi(u)$ and condition (a_2) of Theorem 2.1 is verified. Since from (3.5) and (3.6), we can get

$$\lambda \in \left]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r}\Psi(x)}\right[.$$

Now, Theorem 2.1 ensures the existence of three critical points for the functional $\Phi(u) - \lambda\Psi(u)$ and the proof is complete. □

Proof of Theorem 1.3. Fix λ, g and μ as in the conclusion. Take $X = W_0^{1,p}(0,1) \cap W^{2,p}(0,1)$ endowed with the usual norm

$$\|u\| = \left(\int_0^1 \left|u''(t)\right|^p\,\mathrm{d}t\right)^{1/p}$$

and, for each $u \in X$,

$$\Phi(u) = \frac{1}{p}\,\|u\|^p$$

and

$$\Psi(u) = \int_0^1 \left[F(u(x)) + \frac{\mu}{\lambda}G(x,u(x))\right]\,\mathrm{d}x.$$

Since the critical points of the functional $\Phi(u)-\lambda\Psi(u)$ on X are exactly the weak solutions of problem $(\mathcal{P}'_{\lambda,\mu})$, our aim is to apply Theorem 2.1 to $\Phi(u)$ and $\Psi(u)$. To this end, taking into account that the regularity assumptions of Theorem 2.1 on $\Phi(u)$ and $\Psi(u)$ are satisfied, we will verify (a_1) and (a_2) only. Let $\bar{u}(x)$ be the function defined by

$$\bar{u}(x) = \begin{cases} d - 16d(1/4 - |x-1/2|)^2, & x \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1], \\ d, & x \in]\frac{1}{4}, \frac{3}{4}[, \end{cases}$$

and put $r = (2c)^p$. Clearly, $\bar{u} \in X$,

$$\Phi(\bar{u}) = \frac{(32d)^p}{2p},$$

and

$$\Psi(\bar{u}) \geq \frac{1}{2}F(d) + \frac{\mu}{\lambda}\int_0^1 G(x, \bar{u}(x))\,\mathrm{d}x \geq \frac{1}{2}F(d) + \frac{\mu}{\lambda}\inf_{[0,1]\times[0,d]} G.$$

Since

$$\max_{x\in[0,1]} |u(x)| \leq \frac{1}{2\sqrt[p]{p}}\,\|u\| \tag{3.7}$$

for all $u \in X$ (see Lemma 2 of [18]), one has $\max_{x\in[0,1]}|u(x)| \leq c$ for all $u \in \Phi^{-1}(]-\infty, r])$. Therefore, we have

$$\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} \leq \frac{\int_0^1 [F(c) + \frac{\mu}{\lambda}G(x,c)]\,\mathrm{d}x}{(2c)^p} = \frac{F(c) + \frac{\mu}{\lambda}\int_0^1 \max_{|s|\leq c} G(x,s)\,\mathrm{d}x}{(2c)^p}$$

that is,

$$\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} \leq \frac{F(c)}{(2c)^p} + \frac{\lambda}{\mu}\frac{G^c}{(2c)^p}, \tag{3.8}$$

and

$$\frac{\Psi(\bar{u})}{\Phi(\bar{u})} \geq \frac{\frac{1}{2}F(d) + \frac{\mu}{\lambda}\int_0^1 G(x,\overline{u})\,\mathrm{d}x}{\frac{(32d)^p}{2p}} \geq \frac{F(d)}{\frac{(32d)^p}{2p}} + \frac{\mu}{\lambda}\frac{G(d)}{\frac{(32d)^p}{2p}},$$

which is

$$\frac{\Psi(\bar{u})}{\Phi(\bar{u})} \geq \frac{F(d)}{\frac{(32d)^p}{2p}} + \frac{\mu}{\lambda}\frac{G(d)}{\frac{(32d)^p}{2p}}. \tag{3.9}$$

Since $\mu < \delta_g$, one has

$$\mu < \frac{(2c)^p - \lambda F(c)}{G^c}, \qquad \frac{F(c)}{(2c)^p} + \frac{\lambda}{\mu}\frac{G^c}{(2c)^p} < \frac{1}{\lambda}$$

and

$$\mu < \frac{\lambda F(d) - \frac{(32d)^p}{2p}}{G_d}, \qquad \frac{\Phi(\bar{u})}{\Psi(\bar{u})} \geq \frac{F(d)}{\frac{(32d)^p}{2p}} + \frac{\mu}{\lambda}\frac{G(d)}{\frac{(32d)^p}{2p}} > \frac{1}{\lambda},$$

we obtain that,

$$\frac{F(c)}{(2c)^p} + \frac{\lambda}{\mu}\frac{G^c}{(2c)^p} < \frac{1}{\lambda} < \frac{F(d)}{\frac{(32d)^p}{2p}} + \frac{\mu}{\lambda}\frac{G(d)}{\frac{(32d)^p}{2p}}. \tag{3.10}$$

Hence, from (3.8), (3.9) and (3.10), condition (a_1) of Theorem 2.1 is verified.

Finally, since $\mu < \overline{\delta}_g$ we can fix $l > 0$ such that $\limsup_{|s|\to+\infty} \frac{\sup_{x\in[0,1]} G(x,s)}{s^p} < l$ and $\mu l < p$. Therefore, there exists a positive constant k such that

$$G(x,s) \leq ls^p + k$$

for each $(x,s) \in [0,1] \times \mathbb{R}$. Now, fix $0 < \varepsilon < \frac{2^p - \mu l}{\lambda}$. From (H9) there is a positive constant k_ε such that

$$F(s) \leq \varepsilon s^p + k_\varepsilon$$

for each $s \in \mathbb{R}$. It follows that, by (3.7), for each $u \in X$,

$$\Phi(u) - \lambda\Psi(u) \geq \left(\frac{1}{p} - \frac{\lambda}{2^p p}\varepsilon - \frac{l}{2^p p}\mu\right)\|u\|_X - \lambda k_\varepsilon - \mu k.$$

This leads to the coercivity of $\Phi(u) - \lambda\Psi(u)$ and condition (a_2) of Theorem 2.1 is verified. Since from (3.8), (3.9) and (3.10), we can get

$$\lambda \in \left]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\leq r}\Psi(x)}\right[.$$

Now, Theorem 2.1 ensures the existence of three critical points for the functional $\Phi(u) - \lambda\Psi(u)$ and the proof is complete. □

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