ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 41 (2015), No. 1, pp. 281–289

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 1, pp. 281–289
Online ISSN: 1735-8515

ON MEROMORPHIC SOLUTIONS OF CERTAIN TYPE OF DIFFERENCE EQUATIONS

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(Communicated by)

Abstract. We mainly discuss the existence of meromorphic (entire) solutions of certain type of non-linear difference equation of the form: $f(z)^m + P(z)f(z+c)^n = Q(z)$, which is a supplement of previous results in [K. Liu, L. Z. Yang and X. L. Liu, Existence of entire solutions of nonlinear difference equations, *Czechoslovak Math. J.* **61** (2011), no. 2, 565–576, and X. G. Qi, Value distribution and uniqueness of difference polynomials and entire solutions of difference equations, *Ann. Polon. Math.* **102** (2011), no. 2, 129–142].
Keywords: Difference equation; Meromorphic functions; Order.
MSC(2010): Primary: 39B32; Secondary: 30D35.

1. Introduction

In what follows, a meromorphic function will mean meromorphic in the whole complex plane. We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna Theory [7,14]. In particular, we denote the order and lower hyper order of growth of a meromorphic $f(z)$ by $\sigma(f)$ and $\mu_2(f)$, respectively. For a set $E \subset R_+$, let $\lambda(E)$ be the logarithmic measure of E. The upper logarithmic densities of E is defined by

$$\overline{\log\operatorname{dens}}\,(E) = \limsup_{r\to\infty} \frac{\lambda(E\cap[1,r])}{\log r}.$$

We note that E may be different each time it occurs.

Article electronically published on February 15, 2015.
Received: 8 February 2013, Accepted: 19 January 2014.
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Recently, many articles focused on complex difference equations [1–3, 9]. The background for these considerations lies in the recent difference counterparts of Nevanlinna theory. The key result here is the difference analogue of the lemma on the logarithmic derivative obtained by Halburd-Korhonen [5, 6] and Chiang-Feng [3], independently.

Yang and Laine considered the existence of entire solutions with finite order of the following difference equation, they obtained:

Theorem A [15, Theorem 3.4]. *Let $P(z)$, $Q(z)$ be polynomials. Then a non-linear difference equation*

$$f(z)^2 + P(z)f(z+1) = Q(z)$$

has no transcendental entire solutions with finite order.

One of the present authors improved Theorem A, and got the following result.

Theorem B [13, Corollary 2]. *Let $P(z)$, $Q(z)$ be polynomials, and let n, m be two positive integers such that $n \neq m$. Then, the equation*

$$f(z)^m + P(z)f(z+c)^n = Q(z), \tag{1.1}$$

has no transcendental entire solutions with finite order.

In [10], Liu, Yang and Liu considered the existence of entire solutions of equation (1.1), where $Q(z)$ is a nonzero rational function.

It is natural to ask what happens if $P(z)$ and $Q(z)$ in equation (1.1) are transcendental. Corresponding to this question, we give the following results:

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function with finite order, m and n be two positive integers such that $m \geq n+4$ (or $n \geq m+4$), $Q(z)$ be a meromorphic function satisfying $\overline{N}(r, \frac{1}{Q}) = S(r, f)$ and $P(z)$ be a non-zero meromorphic function satisfying that $T(r, P) = S(r, f)$. Then, $f(z)$ is not a solution of equation (1.1).*

Remark 1.2. *The proof of Theorem 1.1 is based on some ideas from [4].*

Using a similar reasoning as in Theorem 1.1, we conclude that:

Corollary 1.3. *Let $f(z)$ be a transcendental entire function with finite order, m and n be two positive integers such that $m \geq n+2$ (or $n \geq m+2$), $Q(z)$ be a meromorphic function satisfying $\overline{N}(r, \frac{1}{Q}) = S(r, f)$ and $P(z)$ be a non-zero meromorphic function satisfying that $T(r, P) = S(r, f)$. Then $f(z)$ is not a solution of equation (1.1).*

Remark 1.4. *Corollary 1.3 is not true, if the assumption $m \geq n+2$ (or $n \geq m+2$) is omitted. The following equation*

$$f(z)^2 + f(z+c) = e^{4z}$$

can admit a transcendental entire solution $f(z) = e^{2z} - 1$, and $e^{2c} = 2$. And a transcendental entire function e^z is a solution of

$$f(z) + f(z + 2n\pi i) = 2e^z.$$

The two examples above show that when $m = n+1$ or $n = m$, equation (1.1) has transcendental entire solutions with finite order in Corollary 1.3.

Theorem 1.5. *Consider the difference equation (1.1), where $P(z)$ is a polynomial and $Q(z)$ is an entire function with finite order, m and n are positive integers such that $m \neq n$. Suppose an entire function $f(z)$ satisfies that $\sigma(Q) < \sigma(f)$ and $\mu_2(f) < 1$, then $f(z)$ is not a solution of equation (1.1).*

Remark 1.6. *If the assumption $m \neq n$ is omitted, then Theorem 1.5 cannot be valid. For example, $f(z) = sinz$ is a solution of*

$$f(z)^2 + f(z + \frac{\pi}{2})^2 = 1.$$

In addition, $f(z) = e^{e^z} + e^{\frac{z}{2}}$ is an entire function with infinite order and satisfies $f(z) - f(z + 2\pi i) = 2e^{\frac{z}{2}}$. Moreover, from the first example in Remark 1.4, we know the assumption that $\sigma(Q) < \sigma(f)$ is essential. In particular, c can be equal to zero in Theorem 1.5.

Corollary 1.7. *There exists no entire infinite order $f(z)$ with $\mu_2(f) < 1$ that satisfies the difference equation of the type*

$$f(z)^m + P(z)f(z+1) = c \sin bz \tag{1.2}$$

where $P(z)$ is a polynomial, and b, $c \in \mathbb{C}$ are non-zero constants and $m \geq 2$ is an integer.

Remark 1.8. *1 In fact, Corollary 1.7 gives a partial answer to a conjecture raised by Yang and Laine in [15]. Peng and Chen [11] have obtained Corollary 1.7. For the convenience of the reader, we give the conjecture:*

Conjecture. *There exists no entire infinite order $f(z)$ that satisfies the difference equation of the type*

$$f(z)^m + P(z)f(z+1) = c \sin bz \tag{1.3}$$

where $P(z)$ is a non-constant polynomial, and $b,\ c \in \mathbb{C}$ are non-zero constants and $m \geq 2$ is an integer.

2. Some lemmas

Lemma 2.1. *[3, Lemma 5.1] Let $f(z)$ be a finite order meromorphic function and $\varepsilon > 0$, then*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}) + O(\log r)$$

and

$$\sigma(f(z+c)) = \sigma(f(z)).$$

Thus, if $f(z)$ is a transcendental meromorphic function with finite order, then we know

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 2.2. *[6, Theorem 2.1] Let $f(z)$ be a meromorphic function with finite order, and let $c \in \mathbb{C}$ and $\delta \in (0,1)$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = o\left(\frac{T(r,f)}{r^{\delta}}\right) = S(r,f).$$

Lemma 2.3. *[8, Theorem 6 & 7] Let $f(z)$ be a non-constant meromorphic function with finite order, $c \in \mathbb{C}$. Then*

$$\overline{N}(r, \frac{1}{f(z+c)}) \leq \overline{N}(r, \frac{1}{f(z)}) + S(r, f),$$

outside of a possible exceptional set E with finite logarithmic measure.

Lemma 2.4. *[12, Lemma 2.3] Let $f(z)$ and $g(z)$ be entire functions such that $\sigma(g) < \sigma(f)$. Then, there exists a set E with $\overline{\log \operatorname{dens}}\,(E) > 0$ such that*

$$\frac{|g(z)|}{M(r,f)} = o(1)$$

for all z such that $|z| = r \in E$ is sufficiently large.

3. Proof of Theorem 1.1

Suppose by contradiction that that $f(z)$ is a transcendental meromorphic function with finite order satisfying equation (1.1). Then, we will discuss the following two cases.

Case 1. $m \geq n+4$. If $T(r,Q) = S(r,f)$, then applying Lemma 2.1 to equation (1.1), we have

$$mT(r,f) = nT(r,f) + S(r,f),$$

which contradicts the assumption that $m \geq n+4$.

If $T(r,Q) \neq S(r,f)$. Then from equation (1.1), we get

$$f^m(z) = \frac{\frac{Q'}{Q}Pf^n(z+c) - (Pf^n(z+c))'}{\frac{(f^m(z))'}{f^m(z)} - \frac{Q'}{Q}}. \tag{3.1}$$

First observe that $\frac{(f^m(z))'}{f^m(z)} - \frac{Q'}{Q}$ cannot vanish identically. Indeed, if $\frac{(f^m(z))'}{f^m(z)} - \frac{Q'}{Q} \equiv 0$, then we get

$$Q(z) = \alpha f^m(z),$$

where α is a non-zero constant. Substituting the above equality to equation (1.1), we have

$$Pf^n(z+c) = (\alpha - 1)f^m(z).$$

From Lemma 2.1 and the above equation, we immediately see that

$$mT(r,f) = nT(r,f) + S(r,f),$$

or $f(z) \equiv 0$, which is impossible.

From equation (3.1), we know

(3.2)

$$\begin{aligned} T(r,f^m) = mT(r,f) &\leq m(r,Pf^n(z+c)) + m\left(r, \frac{Q'}{Q} - \frac{(Pf^n(z+c))'}{Pf^n(z+c)}\right) \\ &+ N\left(r, \frac{Q'}{Q}Pf^n(z+c) - (Pf^n(z+c))'\right) \\ &+ m\left(r, \frac{(f^m(z))'}{f^m(z)} - \frac{Q'}{Q}\right) + N\left(r, \frac{(f^m(z))'}{f^m(z)} - \frac{Q'}{Q}\right) + S(r,f). \end{aligned}$$

Then Lemma 2.1 together with equation (1.1), implies that

$$(m-n)T(r,f) + S(r,f) \leq T(r,Q) \leq (m+n)T(r,f) + S(r,f),$$

and so

$$S(r,Q) = S(r,f). \tag{3.3}$$

To apply Lemma 2.1, Lemma 2.2 and (3.3) to equation (3.2), we obtain that

$$\begin{aligned} mT(r,f) \leq nm(r,f) &+ N\left(r, \frac{Q'}{Q}Pf^n(z+c) - (Pf^n(z+c))'\right) \\ &+ N\left(r, \frac{(f^m(z))'}{f^m(z)} - \frac{Q'}{Q}\right) + S(r,f). \end{aligned} \tag{3.4}$$

In the following, we will estimate $N\left(r, \frac{Q'}{Q}Pf^n(z+c) - (Pf^n(z+c))'\right)$ and $N\left(r, \frac{(f^m(z))'}{f^m(z)} - \frac{Q'}{Q}\right)$. Let

$$H(z) = \frac{Q'}{Q}Pf^n(z+c) - (Pf^n(z+c))', \tag{3.5}$$

and

$$G(z) = \frac{(f^m(z))'}{f^m(z)} - \frac{Q'}{Q}. \tag{3.6}$$

First of all, we deal with $N(r, H(z))$. From (1.1) and (3.5), we know the poles of $H(z)$ are at the zeros of $Q(z)$, and at the poles of $f(z)$, $f(z+c)$ and $P(z)$. (In fact, first impression of the reader is that the poles of $H(z)$ are at the poles of $Q(z)$ as well. However, looking at the equation (1.1) one realizes that the poles of $Q(z)$ should be at the poles of $f(z)$, $f(z+c)$ and $P(z)$. Hence, it is enough to discuss the poles of $f(z)$, $f(z+c)$ and $P(z)$ here.) Based on our assumption our assumption that $T(r, P) = S(r, f)$, we will ignore the poles of $P(z)$ here. If z_0 is a zero of $Q(z)$ or z_0 is a pole of $f(z)$ but not a pole of $f(z+c)$, then z_0 is at most a simple pole of $H(z)$ by (3.5). If z_0 is a pole of $f(z+c)$ but not a pole of $f(z)$, then z_0 is at most a simple pole of $H(z)$ by (3.1). If z_0 is a pole of $f(z)$ with multiplicity s and a pole of $f(z+c)$ with multiplicity t, then z_0 is a pole of $H(z)$ with the multiplicity no more than $nt+1$ by (3.5). From the above arguments and our assumption, we conclude that

$$\begin{aligned} N(r, H) &\leq \overline{N}(r, \frac{1}{Q}) + N(r, f^n(z+c)) + \overline{N}(r, f) + S(r, f) \\ &\leq nN(r, f(z+c)) + \overline{N}(r, f) + S(r, f). \end{aligned} \tag{3.7}$$

Next, we appraise $N(r, G(z))$. We obtain that the poles of $G(z)$ are at the zeros of $Q(z)$ and $f(z)$, and at the poles of $f(z)$, $f(z+c)$ from (1.1) and (3.6). If z_0 is a zero of $Q(z)$, zero of $f(z)$, or pole of $f(z+c)$, then z_0 is at most a simple pole of $H(z)$ by (3.6). If z_0 is a pole of $f(z)$ but not a pole of $f(z+c)$, then by the Laurent expansion of $G(z)$ at z_0, we obtain that $G(z)$ is analytic at z_0. Therefore, from the discussions above and our assumption, we know

$$\begin{aligned} N(r, G) &\leq \overline{N}(r, \frac{1}{Q}) + \overline{N}(r, f(z+c)) + \overline{N}(r, \frac{1}{f}) + S(r, f) \\ &\leq \overline{N}(r, f(z+c)) + \overline{N}(r, \frac{1}{f}) + S(r, f). \end{aligned} \tag{3.8}$$

From equations (3.4), (3.7) and (3.8), we have

$$\begin{aligned} mT(r,f) &\leq nm(r,f) + nN(r,f(z+c)) + \overline{N}(r,f) \\ &\quad + \overline{N}(r,f(z+c)) + \overline{N}(r,\frac{1}{f}) + S(r,f) \\ &\leq (n+3)T(r,f) + S(r,f), \end{aligned}$$

which contradicts the assumption that $m \geq n+4$.

Case 2. If $n \geq m+4$, then set $F(z) = f(z+c)$, $F(z-c) = f(z)$ follows. We obtain

$$F(z-c)^m + P^*(z)F(z)^n = Q^*(z).$$

By Lemma 2.1 and Lemma 2.3, we know $\overline{N}(r, \frac{1}{Q(z-c)}) = \overline{N}(r, \frac{1}{Q^*}) = S(r,F)$ and $T(r,P^*) = S(r,F)$. Similarly as in Case 1, we get a conclusion as well, completing the proof of Theorem 1.1.

4. **Proof of Theorem 1.5**

Let $f(z)$ be an entire solution of equation (1.1) such that $\sigma(Q) < \sigma(f)$ and $\mu_2(f) < 1$. If $P(z) = 0$, then we get $\sigma(Q) = \sigma(f)$, which contradicts our assumption. It remains to discuss the case $P(z) \neq 0$.

Case 1. If $m > n$. From (1.1), we obtain that

$$|f(z)^m| \leq |P(z)||f(z+c)^n| + |Q(z)|. \tag{4.1}$$

We know $|P(z)| \leq r^{p+1}$ for any $r > r_1$, where $p = \deg\{P(z)\}$. By Lemma 2.4, we know

$$|Q(z)| = o(1)M(r,f),\ r \in E,\ r > r_2, \tag{4.2}$$

where E is a set with $\overline{\log\operatorname{dens}}\,(E_1) > 0$. Combining equation (4.1) with (4.2), we conclude that

$$|f(z)^m| \leq r^{p+1}|f(z+c)^n| + o(1)M(r,f),\ r \in E,\ r > r_3,$$

and so

$$|f(z)^m| - o(1)M(r,f) \leq r^{p+1}|f(z+c)^n|,\ r \in E,\ r > r_3.$$

This means

$$M(r,f)^m - o(1)M(r,f) \leq r^{p+1}M(r,f(z+c))^n,\ r \in E,\ r > r_3,$$

where $r_3 = \max\{r_1, r_2\}$. By a simple geometric observation, we get

$$M(r,f)^m(1 - o(1)) \leq r^{p+1}M(r+|c|, f(z))^n,\ r \in E,\ r > r_3.$$

Moreover,

$$m\log M(r,f)\leq n\log M(r+|c|,f(z))+(p+1)\log r,\ r\in E,\ r>r_3.$$

The assumption that $\sigma(f)>\sigma(Q)$ implies $\frac{\log r}{\log M(r+|c|,f(z))}=o(1)$ for $r>r_4$. Hence,

$$\frac{m}{n}\log M(r,f)\leq \log M(r+|c|,f(z))(1+o(1)),\ r\in E,\ r>R,$$

where $R=\max\{r_3,r_4\}$. By induction, we get

$$\log M(r+k|c|,f(z))\geq \left(\frac{m}{n}\right)^k\log M(r,f)(1+o(1)),\ r\in E,\ r>R.$$

Therefore, we have

$$\begin{aligned}\log\log M(r+k|c|,f(z))&\geq k\log\frac{m}{n}+\log\log M(r,f)\\&=k\log\frac{m}{n}\left(1+\frac{\log\log M(r,f)}{k\log\frac{m}{n}}\right),r\in E,r>R.\end{aligned}$$

From the above inequality, we know

(4.3)

$$\frac{\log\log\log M(r+k|c|,f(z))}{\log(r+k|c|)}\geq\frac{\log k+\log\log\frac{m}{n}+\log\left(1+\frac{\log\log M(r,f)}{k\log\frac{m}{n}}\right)}{\log(r+k|c|)},$$

where $r\in E,\ r>R$. It follows from the definition of $\mu_2(f)$ and (4.3) that $\mu_2(f)\geq 1$, when $k\to\infty$. This contradicts the assumption.

Case 2. If $n>m$, then set $F(z)=f(z+c)$, $F(z-c)=f(z)$ follows. We obtain $F(z-c)^m+P^*(z)F(z)^n=Q^*(z)$. By the definitions of the $\sigma(f)$ and $\mu_2(f)$, we know that $\sigma(Q^*)<\sigma(F)$ and $\mu_2(F)<1$. Similarly as in Case 1, we get the conclusion, completing the proof of Theorem 1.5.

Acknowledgments

The authors would like to thank the referee for his/her helpful suggestions and comments. This work was supported by the National Natural Science Foundation of China (No. 11301220, No. 11371225 and No. 11401387) and the Tianyuan Fund for Mathematics (No. 11226094), the NSF of Shandong Province, China (No. ZR2012AQ020) and the Fund of Doctoral Program Research of University of Jinan (XBS1211).

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