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STOCHASTIC DIFFERENTIAL INCLUSIONS OF SEMIMONOTONE TYPE IN HILBERT SPACES

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Abstract. In this paper, we study the existence of generalized solutions for the infinite dimensional nonlinear stochastic differential inclusions $dx(t) \in F(t, x(t))dt + G(t, x(t))dW_t$ in which the multifunction F is semimonotone and hemicontinuous and the operator-valued multifunction G satisfies a Lipschitz condition. We define the Itô stochastic integral of operator set-valued stochastic processes with respect to the cylindrical Brownian motion on separable Hilbert spaces. Then, we generalize the existence results for differential inclusions in [H. Abedi and R. Jahanipur, Nonlinear differential inclusions of semimonotone and condensing type in Hilbert spaces, *Bull. Korean Math. Soc.*, 52 (2015), no. 2, 421–438.] to the corresponding stochastic differential inclusions using the methods discussed in [R. Jahanipur, Nonlinear functional differential equations of monotone-type in Hilbert spaces, *Nonlinear Analysis* **72** (2010), no. 3-4, 1393–1408, R. Jahanipur, Stability of stochastic delay evolution equations with monotone nonlinearity, *Stoch. Anal. Appl.*, **21** (2003), 161–181, and R. Jahanipur, Stochastic functional evolution equations with monotone nonlinearity: existence and stability of the mild solutions, *J. Differential Equations* **248** (2010), no. 5, 1230–1255.]
Keywords: Stochastic differential inclusions, Stochastic set-valued integrals, Generalized solutions, Semimonotone and hemicontinuous set-valued process.
MSC(2010): 34K09, 39A50, 34A60, 49J21.

1. **Introduction**

In recent years, the theory of stochastic differential equations has become an active area of investigation due to its applications in physics, biology, economics and mechanical and electrical engineering problems ([6–8, 18, 19] and the references therein). In this paper, we are to concerned with the existence of generalized solution (in the strong sense) of the nonlinear stochastic differential

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inclusion

$$dx_t \in F(t, x_t)dt + G(t, x_t)dW_t,$$

where F and G are set-valued stochastic processes and $\{W_t\}_{t\geq 0}$ is the cylindrical Brownian motion on a separable Hilbert space H. The part $F(t, x_t)dt$ is related to the integral of a set-valued process with respect to time t and the part $G(t, x_t)dW_t$ is related to the Itô integral of a set-valued process with respect to W_t. We impose some regularity and geometric conditions on set-valued processes F and G and extend the definition of stochastic integral of set-valued stochastic processes with respect to Brownian motion introduced by Kisielewicz [15] to the infinite dimensional separable Hilbert spaces.

The set-valued integrals and stochastic set-valued integrals have been studied by many authors; see e.g. [2–5, 10–16]. Inspired by Aumann's definition [3], Hiai and Umegaki [4] and Kisielewicz [11, 12] defined the set-valued stochastic integral as a subset of the space $L^2(\Omega, \mathcal{F}, P; H)$. But this is not a decomposable set and so it is not a set-valued random variable. Then, Jung and Kim [10] introduced the set-valued stochastic integral as a closed decomposable hull of the one defiend by Kisielewicz in [12], although some of their results and proofs were not correct. Recently, Kisielewicz [15] correctly redefined the set-valued stochastic integral with respect to standard Brownian motion on finite dimensional spaces and studied its properties.

In [6, 8], Jahanipur investigated the existence and uniqueness and asymptotic properties of the mild solutions of deterministic and stochastic semilinear functional evolution equations in which the nonlinear part is monotone. Using the methods introduced in [6] and invoking some selection and fixed point theorems, we established in [1] the existence and uniqueness of the generalized solutions for deterministic nonlinear differential inclusions in Hilbert spaces in which the multifunction on the right-hand side is hemicontinuous and satisfies the semimonotone condition or is condensing. But, in some cases, deterministic models often fluctuate due to the random noise. This motivates us to switch from deterministic problems to stochastic ones and to generalize the existence results to stochastic differential inclusions of semimonotone type.

The paper is organized as follows. In Section 2, we recall some preliminaries. In sections 3 we review the definition of the Itô stochastic integral of the operator set-valued stochastic processes and discuss some of its properties. Finally, in Section 4, we prove our main theorems and results in the case that the set-valued process F is semimonotone hemicontinuous and the set-valued process G is Lipschitz continuous.

2. **Preliminaries**

Let H and K be two real separable Hilbert spaces with a norm and an inner product denoted by symbols $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Assume that $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a complete stochastic basis with a right continuous filtration.

Denote by $\mathcal{B}(H)$ the σ-field of Borel subsets of H. The σ-field generated by all sets of the form $(s,t] \times F$, $F \in \mathcal{F}_s$ and $\{0\} \times F$, $F \in \mathcal{F}_0$ is called the *predictable* σ-field and it is denoted by $\mathcal{P}_T$. We define the σ-field Σ on $[0,T] \times \Omega$ as $\Sigma = \{A \in \mathcal{P}_T : A^t = \{\omega \in \Omega : (t,\omega) \in A\} \in \mathcal{F}_t, \quad \forall t \in [0,T]\}$.

The stochastic process $\{W_t : t \geq 0\}$ is called a *cylindrical Brownian motion* on H if it satisfies the following conditions:

(i): $W_0 = 0$ and $W_t(x)$ is $\mathcal{F}_t$-adapted for every $x \in H$;

(ii): For every $x \in H$ such that $x \neq 0$, $\{W_t(x)/\|x\| : t \geq 0\}$ is a one-dimensional Brownian motion.

For the properties of cylindrical Brownian motion see [18].

Definition 2.1. *Let $f : [0,T] \times \Omega \to H$ be an H-valued predictable process such that $E\left(\int_0^T \|f(s)\|^2 \, ds\right) < \infty$. The stochastic integral of f with respect to cylindrical Brownian motion $\{W_t\}_{t \geq 0}$ on H is a real-valued continuous martingale given by*

$$\int_0^t \langle f(s), dW_s \rangle = \sum_{n=1}^{\infty} \int_0^t \langle f(s), e_n \rangle \, dW_s(e_n), \quad \forall t \in [0,T],$$

where $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal basis for H.

Suppose that $L_2(H,K)$ is the space of all Hilbert-Schmidt operators from H to K with the Hilbert-Schmidt norm $\|\cdot\|_{L_2}$.

Definition 2.2. *Let $g : [0,T] \times \Omega \to L_2(H,K)$ be an $L_2(H,K)$-valued predictable process such that $E\left(\int_0^T \|g(s)\|_{L_2}^2 \, ds\right) < \infty$. The stochastic integral of g is a K-valued continuous martingale such that*

$$\left\langle k, \int_0^t g(s) dW_s \right\rangle = \int_0^t \langle g^*(s)k, dW_s \rangle, \quad t \in [0,T], \quad k \in K,$$

where $g^*(s)$ is the adjoint operator of $g(s)$.

We assume that $P(H)$ is the family of all non-empty subsets of H and $P_c(H)$ (resp. $P_{c,v}(H)$, $P_{k,v}(H)$) is the family of all non-empty closed (resp. closed convex, compact convex) subsets of H. A set-valued map $F : H \to P(K)$ is called *upper semicontinuous* ($u.s.c$) if for every open subset U of K, the set $\{x \in H : F(x) \subseteq U\}$ is open in H. By Proposition 1.2.5 of [5], upper semicontinuity is equivalent to the following: for each sequence $\{x_n\}_{n=1}^{\infty}$ in H such that $x_n \to x$ as $n \to \infty$ and for every $\varepsilon > 0$, there exists a positive integer N such that

$$F(x_n) \subseteq F(x) + \varepsilon B, \qquad \forall n \geq N,$$

where B is an open unit ball of K. The map F is called *lower semicontinuous* ($l.s.c$) if for every open subset U of K, the set $\{x \in X : F(x) \cap U \neq \emptyset\}$ is open in H. By Proposition 1.2.6 of [5], lower semicontinuity is equivalent to the following: for each sequence $\{x_n\}_{n=1}^{\infty}$ in H which converges to x and for each $y \in F(x)$, there exists a sequence $\{y_n\}_{n=1}^{\infty}$ in K such that $y_n \in F(x_n)$ for all n and $y_n \to y$ as $n \to \infty$. Moreover, the set-valued map F is called *closed* whenever its graph $\{(x, y) : y \in F(x)\}$ is a closed subset of the product space $H \times K$.

By a *set-valued random variable* on the Hilbert space H we mean a closed valued $\mathcal{F}$-measurable multifunction, i.e., a set-valued map $X : \Omega \to P_c(H)$ such that $X^{-1}(C) = \{\omega \in \Omega : X(\omega) \cap C \neq \emptyset\} \in \mathcal{F}$ for every $C \in P_c(H)$. If H is separable, then by the Kuratowski-Ryll Nardzewski selection theorem ([5]), there is a sequence of random variables $x_n : \Omega \to H$, $n \geq 1$, such that $X(\omega) = cl\{x_n(\omega)\}$ for $\omega \in \Omega$, where cl denotes the closure in the space H. Assume that $p \geq 1$, we denote by $S^p(X)$ the family of all Bochner L^p-integrable selections of X. If $S^1(X) \neq \emptyset$, then X is said to be Aumann integrable. $S^p(X)$ is a closed and *decomposable* subset of $L^p(\Omega, \mathcal{F}, P; H)$, i.e., if $x_1, x_2 \in S^p(X)$, then $1_A x_1 + 1_{\Omega - A} x_2 \in S^p(X)$ for all $A \in \mathcal{F}$; see [5]. A set-valued random variable X is called *integrably bounded* if there exists $m \in L^1(\Omega, \mathcal{F}, P; \mathbb{R}^+)$ such that $\|X(\omega)\| = \sup\limits_{x \in X(\omega)} \|x\| \leq m(\omega)$ for a. e. $\omega \in \Omega$. If $m \in L^2(\Omega, \mathcal{F}, P; \mathbb{R}^+)$, then X is said to be *square integrably bounded*. It is easy to see that X is integrably bounded if and only if $S^1(X)$ is a non-empty bounded subset of $L^1(\Omega, \mathcal{F}, P; H)$; see [5]. The decomposable hull of the set $C \subseteq L^p(\Omega, \mathcal{F}, P; H)$ is the smallest decomposable subset of $L^p(\Omega, \mathcal{F}, P; H)$ containing C and is denoted by $dec(C)$. It turns out that (see [5]) $\overline{dec}(C)$ is the set of those $f \in L^p(\Omega, \mathcal{F}, P; H)$ such that given $\varepsilon > 0$, there exist a finite $\mathcal{F}$-measurable partition $\{A_k\}_{k=1}^{N}$ of Ω and $\{u_k\}_{k=1}^{N} \subseteq C$ for which $E\left(\left\|f - \sum_{k=1}^{N} 1_{A_k} u_k\right\|^p\right) < \varepsilon$. Moreover, we have $\overline{dec}(S^p(X)) = S^p(X)$ for all $p \geq 1$. If C is convex, then $dec(C)$ is convex and $\overline{dec}(C)$ is weakly closed convex. Also, if the set-valued random variable X is integrably bounded with convex values, then $S^1(X)$ is a weakly compact, convex, decomposable subset of $L^1(\Omega, \mathcal{F}, P; H)$, see [11, 12, 15].

The set-valued map $X : [0, T] \times \Omega \to P_c(H)$ is called a *set-valued stochastic process* if $X(t, \cdot)$ is a set-valued random variable for every $t \in [0, T]$. Moreover, we say that X is measurable if it is $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable. The set-valued stochastic process X is called *square integrable* if $E(\int_0^T \|X_t\|^2 \, dt) < \infty$, where

$$\|X_t(\omega)\| = \sup\{\|x\| : x \in X(t, \omega), \quad (t, \omega) \in [0, T] \times \Omega\}.$$

We denote by $S^2(X)$ (resp. $S^2_{\Sigma}(X)$) the set of all square integrable selections of the set-valued stochastic process X which are $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable (resp. Σ-measurable). Let the set-valued stochastic process X be square integrable (integrably bounded). If $S^2(X) \neq \emptyset$, we say that X is a Aumann square integrable (integrably bounded). Similarly, X is a Σ-Aumann square integrable (integrably bounded) if $S^2_{\Sigma}(X) \neq \emptyset$. Note that in these cases, $S^2(X)$ and $S^2_{\Sigma}(X)$ are closed (bounded), decomposable subsets of $L^2([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{F}, dt \times$

$P;\ H)$ and $L^2([0,T]\times\Omega,\Sigma,dt\times P;\ H)$, respectively and as before, if X has convex values, $S^2(X)$ and $S^2_\Sigma(X)$ are convex and weakly compact subsets of the above spaces, see [11, 12, 15].

The following inequality constitutes some part of our main tools in this paper to obtain the existence results.

Theorem 2.3. (Ito-type inequality)([9]) *Let ξ be a K-valued, $\mathcal{F}_0$-measurable random variable. Suppose that $p\geq 2$, $f\in L^p([0,T]\times\Omega,\Sigma,P;K)$ and $g\in L^p([0,T]\times\Omega,\Sigma,P;L_2(H,K))$. If $X(t)=\xi+\int_0^t f(s)ds+\int_0^t g(s)dW_s$, then*

$$\|X(t)\|^p\leq\|\xi\|^p+p\int_0^t\|X(s)\|^{p-2}\langle X(s),f(s)\rangle\,ds+p\int_0^t\|X(s)\|^{p-2}\langle X(s),g(s)dW_s\rangle + \frac{p(p-1)}{2}\int_0^t\|X(s)\|^{p-2}\|g(s)\|^2_{L_2}\,ds.$$

Let Y be a metric space. By the *Caratheodory conditions* on a two-variable multifunction $F:\Omega\times Y\to P(Y)$ we mean that (i) for every $x\in Y$, the function $\omega\to F(\omega,x)$ is measurable; (ii) for almost all $\omega\in\Omega$, the function $x\mapsto F(\omega,x)$ is *u.s.c.* The (measurable) function $x:\Omega\to Y$ such that $x(\omega)\in F(\omega,x(\omega))$ for each $\omega\in\Omega$ is called a (random) fixed point of F. The following random fixed point theorem that is a direct consequence of the well-known selection theorem of Kuratowski-Ryll Nardzewski is needed in the later sections (Theorem 7.1 of [17]).

Theorem 2.4. *Let $(\Omega,\mathcal{F})$ be a measurable space and Z be a non-empty complete separable subset of the metric space Y. Suppose that the Caratheodory set-valued map $F:\Omega\times Z\to P_c(Y)$ satisfies the following condition*

$$\left\{\begin{array}{l}\text{For every } \omega\in\Omega \text{ and sequence } \{x_n\}_{n=1}^{\infty} \text{ in } Z \text{ and } D\in P_c(Z),\\ \text{if } d(x_n,D)\to 0 \text{ and } d(x_n,F(x_n))\to 0, \text{ when } n\to\infty,\\ \text{then } F \text{ has a fixed point in } D.\end{array}\right.$$

Then, F has a random fixed point if and only if it has a fixed point.

3. Set-valued stochastic integrals

In this section, we study the Itô stochastic integrals of set-valued stochastic processes with respect to cylindrical Brownian motion on infinite dimensional separable Hilbert spaces. For properties of stochastic integral for single-valued processes with respect to the cylindrical Brownian motion, see [18] and for set-valued stochastic integrals on finite dimensional spaces, see [11–15].

Let X be a square integrable set-valued stochastic process on the Hilbert space H. Aumann defined the set-valued integral of X as, [3]

$$L_X(t)=\left\{\int_0^t\varphi(s)ds:\varphi\in S^2(X)\right\},$$

for every $t\in[0,T]$. One can see that $L_X(t)$ is a decomposable subset of $L^2(\Omega,\mathcal{F},P;H)$, so by Hiai-Umegaki's theorem (Theorem 3.1 of [4]), there is a unique set-valued stochastic process $Y : [0,T]\times\Omega \to P_c(H)$ such that $S^2(Y_t) = L_X(t)$ for all $t \in [0,T]$. The process Y_t is called the Aumann-Lebesgue integral of X_t and is denoted by $\int_0^t X_s ds$. In the same way, if $S^2_\Sigma(X) \neq \emptyset$, we define

$$L^\Sigma_X(t) = \left\{ \int_0^t \varphi(s)ds : \varphi \in S^2_\Sigma(X) \right\}.$$

Note that in this case, $L^\Sigma_X(t)$ may not be decomposable. But, by Hiai-Umegaki's theorem, there exists a unique set-valued stochastic process $\int_0^t X_s ds$ such that $S^2(\int_0^t X_s ds) = \overline{dec}(L^\Sigma_X(t))$ for each $t \in [0,T]$. To study more about these processes, we refer to [4, 11, 15].

Now, we are ready to define the Itô set-valued stochastic integral for an operator set-valued stochastic process with respect to the cylindrical Brownian motion $\{W_t : t \geq 0\}$. Assume that the set-valued stochastic process $F : [0,T]\times\Omega \to P_c(L_2(H,K))$ is such that $E\left(\int_0^T \|F(s)\|^2_{L_2} ds\right) < \infty$. We define the set-valued map $J_F : [0,T] \to P(L^2(\Omega,\mathcal{F}_T,P;K))$ as

$$J_F(t) = \left\{ \int_0^t \varphi(s)dW_s : \varphi : [0,T]\times\Omega \to L_2(H,K), \quad \varphi \in S^2_\Sigma(F) \right\}, \forall t \in [0,T].$$

An immediate consequence of this definition is that $J_F(t) \subseteq L^2(\Omega,\mathcal{F}_t,P;K)$. Moreover, $E(J_F(t)) = \{0\}$ for each $t \in [0,T]$. The following proposition presents a few properties of the values of the set-valued map J_F and can be proved similar to Theorem 2.1 of [15] using the Itô-type inequality (Theorem 2.3) for $p = 2$.

Proposition 3.1. *Let F be the set-valued stochastic process defined above. Then,*

- **i):** *for every $t \in [0,T]$, $J_F(t)$ is a closed subset of $L^2(\Omega,\mathcal{F}_t,P;K)$. In addition, if F is square integrably bounded, then $J_F(t)$ is a bounded subset of $L^2(\Omega,\mathcal{F}_t,P;K)$;*
- **ii):** *$J_F(t)$ is a decomposable subset of $L^2(\Omega,\mathcal{F}_t,P;K)$ if and only if $J_F(t)$ is singleton. Moreover, if $\mathrm{Int}\left(\overline{dec}(J_F(t))\right) \neq \emptyset$, then $\overline{dec}(J_F(t)) = L^2(\Omega,\mathcal{F}_t,P;K)$;*
- **iii):** *if F is a convex-valued, then $J_F(t)$ and $\overline{dec}(J_F(t))$ are convex and weakly closed subsets of $L^2(\Omega,\mathcal{F}_t,P;K)$. If, in addition, F is square integrably bounded, then $J_F(t)$ is convex and weakly compact subset of $L^2(\Omega,\mathcal{F}_t,P;K)$ and there exists a sequence $\{\varphi_n\}_{n=1}^\infty$ in $S^2_\Sigma(F)$ such that*

  $$J_F(t) = cl_w \left\{ \int_0^t \varphi_n(s)dW_s : n \geq 1 \right\}$$

  and

  $$\overline{dec}(J_F(t)) = cl_w \left\{ dec(\int_0^t \varphi_n(s)dW_s) : n \geq 1 \right\},$$

where cl_w stands for the weak closure in the space $L^2(\Omega, \mathcal{F}_t, P; K)$.

Again by Theorem of Hiai-Umegaki, we can find the set-valued random variable $Y_t : \Omega \to P_C(K)$ such that $S^2(Y_t) = \overline{dec}(J_F(t))$ for every $t \in [0, T]$. We call the process $\{Y_t\}_{0 \leq t \leq T}$, the Ito stochastic integral of set-valued stochastic process F with respect to the cylindrical Brownian motion W and denote it by $\int_0^t F(s)dW_s$. In addition, if F has convex values, then by the previous definition and Proposition 3.1, we have

$$\begin{aligned} S^2\Big(\int_0^t F(s)dW_s\Big) &= \overline{dec}(J_F(t)) \\ &= \overline{co}(\overline{dec}(J_F(t))) = \overline{co}\Big(S^2\Big(\int_0^t F(s)dW_s\Big)\Big) \\ &= S^2\Big(\overline{co}\Big(\int_0^t F(s)dW_s\Big)\Big). \end{aligned}$$

This implies that $\int_0^t F(s)dW_s$ is a closed convex subset of $L^2(\Omega, \mathcal{F}_t, P; K)$.

We have the following Ito-type inequality for the set-valued stochastic integrals.

Theorem 3.2. *Let $p \geq 2$ and $F : [0, T] \times \Omega \to P_c(L_2(H, K))$ be a stochastic process such that $E\left(\int_0^T \|F(s)\|_{L_2}^p \, ds\right) < \infty$. Then,*

$$E(\left\|\int_0^t F(s)dW_s\right\|^p) \leq C_{p,T} \int_0^t E\left(\|F(s)\|_{L_2}^p\right) ds,$$

for every $t \in [0, T]$, where $C_{p,t} = e^{(p-1)(p-2)t}(\frac{p-1}{2} + \frac{9p^2}{2} t^{\frac{p-2}{p}})$. Moreover,

$$E\left(\sup_{0 \leq t \leq T} \left\|\int_0^t F(s)dW_s\right\|^p\right) \leq K_p \int_0^T E\left(\|F(s)\|_{L_2}^p\right) ds,$$

where $K_p = \left(\frac{p}{p-1}\right)^p C_{p,T}$.

Proof. Let $t \in [0, T]$, by definition of the Ito set-valued stochastic integral and Theorem 2.2 of [4], we get

$$\begin{aligned} E\left(\left\|\int_0^t F(s)dW_s\right\|^p\right) &= E\left[\sup\left\{\|z\|^p : z \in \int_0^t F(s)dW_s\right\}\right] \\ &= \sup\left\{E\left[\|u\|^p\right] \, : \, u \in S^p\left(\int_0^t F(s)dW_s\right)\right\} \\ &= \sup\left\{E\left[\|u\|^p\right] \, : \, u \in \overline{dec}(J_F(t))\right\} \\ &= \sup\left\{E\left[\|u\|^p\right] \, : \, u \in dec(J_F(t))\right\}. \end{aligned}$$

For $u \in dec(J_F(t))$, there exist a finite $\mathcal{F}_t$-measurable partition $\{A_k\}_{k=1}^N$ of Ω and $\{\varphi_k\}_{k=1}^N \subseteq S^p_\Sigma(F)$ such that $u = \sum_{k=1}^{N} 1_{A_k} \int_0^t \varphi_k(s)dW_s$. Thus, we have

$$E\left(\left\|\int_0^t F(s)dW_s\right\|^p\right) \leq \sup_{\{\varphi_k\}_{k=1}^N \subseteq S^p_\Sigma(F)} E\left(\max_{1\leq k\leq N}\left\|\int_0^t \varphi_k(s)dW_s\right\|^p\right).$$

Note that, the Burkholder-Davis inequality (see [18]) with the Ito-type inequality, implies that

$$E\left(\sup_{0\leq s\leq t}\left\|\int_0^t \varphi_k(s)dW(s)\right\|^p\right) \leq C_{p,t}\int_0^t E\left(\|\varphi_k(s)\|^p_{L_2}\right)ds, \tag{3.1}$$

where $C_{p,t} = e^{(p-1)(p-2)t}(\frac{p-1}{2} + \frac{9p^2}{2}t^{\frac{p-2}{p}})$. Now, with a similar argument as in the proof of Inequality (3.1), we conclude that

$$\begin{aligned} E\left(\left\|\int_0^t F(s)dW_s\right\|^p\right) &< C_{p,t} \sup_{\{\varphi k\}_{k=1}^N \subseteq S^p_\Sigma(F)} E\left(\max_{1\leq N}\int_0^t \|\varphi k(s)\|^p_{L_2} ds\right) \\ &\leq C_{p,t}\int_0^t E\left(\|F(s)\|^p_{L_2} ds.\right) \end{aligned}$$

We set $\phi(t) = \sup\left\{\|Z\|^p : Z \in \int_0^t F(s)dW_s\right\}$, then, $\{\phi(t)\}_{t\in[0,T]}$ is a real-valued $\mathcal{F}_t$-submartingle ([4]). So, by the maximal inequality for real-valued submartingles, we obtain

$$E\left(\sup_{0\leq t\leq T}\left\|\int_0^t F(s)dW_s\right\|^p\right) = E\Big(\sup_{0\leq t\leq T}\phi(t)\Big) \leq \left(\frac{p}{p-1}\right)^p E\left(\phi(T)\right).$$

□

4. Existence results

In this section, we study the existence of generalized solution for nonlinear stochastic differential inclusion

$$\begin{cases} dx(t) \in F(t, x(t))dt + G(t, x(t))dW_t, & t \in [0, T], \\ x(0) = \xi, \end{cases} \tag{4.1}$$

in which W is the cylindrical Brownian motion on the real separable Hilbert space H. The multifunctions F and G on the right-hand side are defined on $[0,T] \times \Omega \times K$ with values in spaces $P_c(K)$ and $P_c(L_2(H,K))$, respectively. First, we give the definition of a generalized solution for the problem (4.1).

Definition 4.1. *The mean square continuous Σ-measurable stochastic process $\{x_t\}_{t\in[0,T]}$ is said to be a generalized solution of* (4.1) *if $x(0) = \xi$ and there* *are stochastic processes* $f:[0,T]\times\Omega\to K$ *and* $g:[0,T]\times\Omega\to L_2(H,K)$ *such that the following integral equation holds:*

$$x_t-x_s=\int_s^t f(\tau)d\tau+\int_s^t g(\tau)dW_\tau,\qquad \forall s,t\in[0,T],$$

where $\int_s^t f(\tau)d\tau\in S^2\left(\int_s^t F(\tau,x_\tau)d\tau\right)$ *and* $\int_s^t g(\tau)dW_\tau\in S^2\left(\int_s^t G(\tau,x_\tau)dW_\tau\right)$ *for every* $s,t\in[0,T]$.

Along with the stochastic differential inclusion (4.1), we consider the stochastic integral inclusion

$$\begin{cases} x_t-x_s\in\int_s^t F(\tau,x_\tau)d\tau+\int_s^t G(\tau,x_\tau)dW_\tau, \quad s,\ t\in[0,T] \quad \text{and } a.e.\ \omega\in\Omega,\\ x(0)=\xi,\end{cases}$$

and the stochastic functional inclusion

$$\begin{cases} x_t-x_s\in\ \mathrm{cl}_{L^2}\left(L^{\Sigma}_{1_{[s,t]}\tilde F}(t)+J_{1_{[s,t]}\tilde G}(t)\right),\quad t,s\in[0,T],\\ x(0)=\xi,\end{cases}\tag{4.2}$$

where $\tilde F(t,\omega)=F(t,\omega,x(t,\omega))$ and $\tilde G(t,\omega)=G(t,\omega,x(t,\omega))$ for every $t\in[0,T]$ and $\omega\in\Omega$. Note that if $f\in S^2_\Sigma(\tilde F)$ and $g\in S^2_\Sigma(\tilde G)$ in the above definition, then $\int_0^t f(s)ds\in L^\Sigma_{\tilde F}(t)$ and $\int_0^t g(s)dW_s\in J_{\tilde G}(t)$.

The following theorem tells us how to connect differential inclusion (4.1) to the corresponding stochastic functional inclusion (4.2); see Theorem 4.2 of [13].

Theorem 4.2. *Let* $\{x_t\}_{0\le t\le T}$ *be a solution of the stochastic functional inclusion* (4.2) *such that* $\tilde F$ *and* $\tilde G$ *are* $\mathcal{F}_t$*-adapted and moreover*

$$E\left(\int_0^T\|\tilde F_t\|^2dt\right)<\infty\quad \text{and}\quad E\left(\int_0^T\|\tilde G_t\|^2_{L_2}dt\right)<\infty.$$

Then $\{x_t\}_{0\le t\le T}$ *is a generalized solution of problem* (4.1).

In order to prove the existence result for problem (4.1), we assume that F and G are $\Sigma\otimes\mathcal{B}(K)$-measurable set-valued processes such that for every Σ-measurable stochastic process $\{x_t\}_{0\le t\le T}$, the multifunctions $\{F(t,x_t)\}_{0\le t\le T}$ and $\{G(t,x_t)\}_{0\le t\le T}$ are set-valued stochastic processes. Furthermore, we impose the following hypotheses on F and G:

- **H1):** For each $t\in[0,T]$ and $\omega\in\Omega$, the set-valued map $x\mapsto F(t,\omega,x)$ is hemicontinuous on K; i.e., for each sequence $\{x_n\}_{n=1}^\infty$ in K which is convergent to x, if $y\in F(t,\omega,x)$, then there exists a sequence $\{y_n\}_{n=1}^\infty$ in K such that for all $n\ge 1$, $y_n\in F(t,\omega,x_n)$ and $y_n\xrightarrow{w}y$ in K;
- **H2):** There is a constant $C>0$ such that $\|F(t,\omega,x)\|\le C\,(1+\|x\|)$ for all $t\in[0,T]$, $x\in K$ and $\omega\in\Omega$ *a.e.*;

H3): F is semimonotone with parameter M. In other words, for every $x_1, x_2 \in K$, $t \in [0,T]$, $\omega \in \Omega$, $y_1 \in F(t,\omega,x_1)$ and $y_2 \in F(t,\omega,x_2)$, we have

$$\langle x_1 - x_2, y_1 - y_2 \rangle \leq M \|x_1 - x_2\|^2 ;$$

H4): $G : [0,T] \times \Omega \times K \to P_c(L_2(H,K))$ is a Σ-measurable set-valued process on K such that $\|G(t,\omega,x) - G(t,\omega,y)\|_{L_2} \leq C \|x-y\|$ for all $t \in [0,T]$, $\omega \in \Omega$ and $x, y \in K$.

First, we study the stochastic differential inclusion

$$dx(t) \in F(t,x(t)) + G(t)dW_t, \quad t \in [0,T], \tag{4.3}$$

where $G : [0,T] \times \Omega \to P_c(L_2(H,K))$ is a Σ-measurable set-valued process. In the case that K is finite dimensional, by virtue of Theorem 2.4, we obtain the following

Theorem 4.3. *Suppose that the set-valued map $F : [0,T] \times \Omega \times \mathbb{R}^n \to P_{c,v}(\mathbb{R}^n)$ is u.s.c with respect to the third variable, the map $(t,\omega) \mapsto F(t,\omega,x)$ is Σ-measurable and there exists a continuous function $h : [0,T] \times [0,\infty) \to [0,\infty)$ such that for each $t \in [0,T]$, the map $u \to h(t,u)$ is monotone increasing and $\|F(t,\omega,x)\| \leq h(t,\|x\|)$ for all $t \in [0,T]$, a.e. $\omega \in \Omega$ and all $x \in \mathbb{R}^n$. Let the set-valued map $G : [0,T] \times \Omega \to P_c(L_2(H,\mathbb{R}^n))$ be Σ-measurable and $\int_0^T \|G(s)\|_{L_2}^2 ds < \infty$. If the differential equation $u' = h(t,u)$ with initial condition $u(0) = 0$ has a solution $u(t)$ existing on $[0,T]$, then the stochastic differential inclusion* (4.3) *with the initial random variable $\xi = 0$, has a generalized solution.*

Proof. By the Kuratowski-Ryll Nardzewski selection theorem, there is a Σ-measurable function $\varphi : [0,T] \times \Omega \to L_2(H,\mathbb{R}^n)$ such that $\varphi(t,\omega) \in G(t,\omega)$ for every $(t,\omega) \in [0,T] \times \Omega$. We define the stochastic process y as $y(t,\omega) = x(t,\omega) - \int_0^t \varphi(s)dW_s$ and consider the following stochastic functional inclusion

$$\begin{cases} y(t) \in L^{\Sigma}_{\tilde{F}_x}(t), & t \in [0,T], \\ y(0) = \xi, \end{cases}$$

where $\tilde{F}_x(t,\omega) = \tilde{F}(t,\omega,x) = F(t,\omega,x + \int_0^t \varphi(s)dW_s)$ for $t \in [0,T]$, $\omega \in \Omega$ and $x \in \mathbb{R}^n$. By Inequality (3.1), we have

$$\left\|\tilde{F}(t,\omega,x)\right\| \leq h(t, \|x\| + \left\|\int_0^t \varphi(s)dW_s\right\|) \leq \tilde{h}(t,\|x\|),$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$, and *a.e.* $\omega \in \Omega$, where

$$\tilde{h}(t,x) = h\left(t, x + \left(K_{p,T}\int_0^T \|G(s)\|_{L_2}^2 ds\right)^{\frac{1}{2}}\right).$$

Now, as in the proof of Theorem 3.1 of [1], we define the set K of all continuous Σ-measurable stochastic processes $\{x_t\}_{0\leq t\leq T}$ such that $x(0)=\xi$ and $\|x(t,\omega)\|\leq u(t)$ for each $t\in[0,T]$ and $\omega\in\Omega$. Define the set-valued map Λ as

$$\Lambda(x)=L^{\Sigma}_{\tilde{F}_x}=\left\{ y:[0,T]\times\Omega\to\mathbb{R}^n \;\middle|\; \begin{array}{l} \text{for every } t\in[0,T], y(t)=\int_0^t v(s)ds,\\ \text{where } v\in L^2\left([0,T]\times\Omega,\ \Sigma,\ dt\times P;\ \mathbb{R}^n\right)\\ \text{and } v(s,\omega)\in\tilde{F}(s,\omega,x(s,\omega)), dt\times P-a.e. \end{array}\right\}.$$

With a similar argument as in Theorem 3.1 of [1], we conclude that Λ is a compact set-valued map with convex and compact values. Moreover, the set-valued map $\Lambda(\omega,\cdot)$ is $u.s.c.$ for every $\omega\in\Omega$ and map $\Lambda(\cdot,x)$ is measurable for every $x\in K$. Then, Kakutani's fixed point theorem guarantees that Λ satisfies the conditions of Theorem 2.4. Therefore, it has a random fixed point which is a generalized solution of stochastic differential inclusion (4.3). $\square$

If we assume in the previous theorem that the set-valued map F is $l.s.c$ on K, the result can be proved in a similar manner; see [1]. In the case that K is an infinite dimensional separable Hilbert space and the set-valued map F satisfies hypotheses **H1** $-$ **H3** and $\int_0^T \|G(s)\|^2\,ds<\infty$, we obtain a sequence of approximate solutions in finite dimensional spaces which is weakly convergent to the solution of stochastic differential inclusion (4.3). Take an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for K and let K_n be the subspace of K generated by $\{e_1,\ldots,e_n\}$. Then, $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence of finite dimensional subspaces of K with the property $\overline{\cup_{n=1}^{\infty}K_n}=K$. Suppose that P_n is the orthogonal projection of K onto K_n and define the set-valued map $P_nF:[0,T]\times\Omega\times K\to P_{c,v}(K_n)$ by $P_nF(t,\omega,x)=\{P_n(y)\,|\,y\in F(t,\omega,x)\}$. Now, as in Theorem 3.1 of [1], we can see that the stochastic differential inclusion

$$\begin{cases} dx(t)\in P_nF(t,x(t))dt+G(t)dW_t, & t\in[0,T],\\ x(0)=0, \end{cases}$$

has a Σ-measurable solution $x_n:[0,T]\times\Omega\to C_n$ such that

$$\sup_{0\leq t\leq T}\|x_n(t,\omega)\|\leq e^{(2M+1)T}\int_0^T \|\tilde{F}(s,0)\|^2 ds,$$

for every $\omega\in\Omega$. Thus, for each $\omega\in\Omega$ there is a subsequence $\{x_{n_k}(\cdot,\omega)\}_{k\geq1}$ such that

$$\int_0^T e^{-2Mt}\left\langle x_{n_k}(t,\omega),x_{n_l}(t,\omega)\right\rangle dt\to0,\quad k,l\to\infty.$$

We assume that $x(\cdot,\omega)$ is the weak limit of the above subsequence that is a generalized solution of stochastic differential inclusion (4.3) ([1]). Moreover, the hypothesis **H3** implies that this solution is unique ([7]). It remains to show the measurablity of the solution. For arbitrary $u\in K$ and $k\geq1$, the function

$\int_0^t \langle x_{n_k}(s,\omega), u\rangle\, ds$ is Σ-measurable and

$$\int_0^t \langle x_{n_k}(s,\omega), u\rangle\, ds \underset{k\to\infty}{\to} \int_0^t \langle x(s,\omega), u\rangle\, ds$$

So, the function $\int_0^t \langle x(s,\omega), u\rangle\, ds$ is also Σ-measurable with continuous integrand and we have $\langle x(t,\omega), u\rangle = \frac{d}{dt}\int_0^t \langle x(s,\omega), u\rangle\, ds$. Since K is a separable Hilbert space, $x(t,\omega)$ is Σ-measurable.

Theorem 4.4. *Let K be a separable Hilbert space and the set-valued map $F : [0,T]\times\Omega\times K \to P_{c,v}(K)$ be u.s.c with respect to the third variable, let the map $(t,\omega)\mapsto F(t,\omega,x)$ be Σ-measurable and satisfy hypotheses* **H1** $-$ **H3**. *Suppose that the set-valued map $G : [0,T]\times\Omega \to P_c(L_2(H,K))$ is Σ-measurable and $\int_0^T \|G(s)\|_{L_2}^2\, ds < \infty$. Then, the stochastic differential inclusion* (4.3) *with the initial random variable $\xi = 0$ has a unique generalized solution.*

Now, we assume that F, G are two set-valued processes defined on $[0,T]\times K$ with values in $P_{c,v}(K)$ and $P_c(L_2(H,K))$, respectively. Suppose that hypotheses **H1** $-$ **H4** hold and $\int_0^T \|G(s,0)\|_{L_2}^p\, ds < \infty$ for some $p \geq 2$. Consider the following stochastic differential inclusion

$$dx(t) \in F(t,x(t))dt + G(t,x(t))dW_t, \quad t\in[0,T], \tag{4.4}$$

with the given initial data $x(0) = \xi \in L^2(\Omega,\mathcal{F},P;K)$. We may take the initial data $\xi = 0$, since if x is a generalized solution of (4.4), then x is a continuous solution of the stochastic integral inclusion $x(t) \in L^{\Sigma}_{F_x}(t) + J_{G_x}(t)$ with $x(0) = \xi$, where $F_x(t,\omega) = F(t,\omega,x(t,\omega))$ and $G_x(t,\omega) = G(t,\omega,x(t,\omega))$ for every $(t,\omega)\in[0,T]\times\Omega$. Set $\bar{x} = x - \xi$. Then $\bar{x}$ is a continuous solution of

$$\begin{cases} \bar{x}(t) \in L^{\Sigma}_{\bar{F}_{\bar{x}}}(t) + J_{\bar{G}_{\bar{x}}}(t), \quad t\in[0,T], a.s.\ \omega\in\Omega, \\ \bar{x}(0) = 0, \end{cases}$$

where $\bar{F}_{\bar{x}}(t,\omega) = F(t,\omega,\bar{x}(t,\omega)+\xi(\omega))$ and $\bar{G}_{\bar{x}}(t,\omega) = G(t,\omega,\bar{x}(t,\omega)+\xi(\omega))$. One can easily check that the set-valued processes $\bar{F}_{\bar{x}}$ and $\bar{G}_{\bar{x}}$ satisfy the same hypotheses as for F_x and G_x. Finally, at the end of this section we prove the following existence theorem for the stochastic differential inclusion (4.4).

Theorem 4.5. *Let $p \geq 2$ and $\int_0^T \|F(s,0)\|^p\, ds < \infty$, $\int_0^T \|G(s,0)\|_{L_2}^p\, ds < \infty$. Suppose that hypotheses* **H1** $-$ **H4** *hold. Then, the stochastic differential inclusion* (4.4) *has a unique generalized solution x on $[0,T]$ such that $E\left(\sup_{0\leq s\leq t}\|x(s)\|^p\right) < \infty, \quad \forall t\in[0,T]$.*

Proof. Without lose of generality, we can assume that $\xi = 0$. Theorem 4.4 implies that the differential inclusion

$$dx(t) \in F(t,x(t))dt + G(t,0)dW_t, \quad \forall t\in[0,T],$$

has a unique generalized solution x_1 such that $x_1(0) = 0$. By Definition 4.1, there exist stochastic processes f and g on $[0,T] \times \Omega$ into spaces K and $L_2(H,K)$, respectively, such that $x_1(t) = \int_0^t f(s)ds + \int_0^t g(s)dW_s$ with $\int_0^t f(s)ds \in \overline{dec}\left(L^{\Sigma}_{F_{x_1}}(t)\right)$ and $\int_0^t g(s)dW_s \in \overline{dec}\left(J_{G_{x_1}}(t)\right)$ for every $t \in [0,T]$. Therefore, there are some finite $\mathcal{F}$-measurable partitions $\{A_k\}_{k=1}^N$ and $\{B_k\}_{k=1}^N$ of Ω and stochastic processes $\{\varphi_k\}_{k=1}^N \subseteq S^p_\Sigma(F_{x_1})$ and $\{\psi_k\}_{k=1}^N \subseteq S^p_\Sigma(G_{x_1})$ such that

$$E\left(\left\|\int_0^t f(s)ds - \sum_{k=1}^N 1_{A_k}\int_0^t \varphi_k(s)ds\right\|^p + \left\|\int_0^t g(s)dW_s - \sum_{k=1}^N 1_{A_k}\int_0^t \psi_k(s)dW_s\right\|^p\right) < 1.$$

By Inequality (3.1), we have

$$\begin{aligned}
E\left(\sup_{0\le t\le T}\|x_1(t)\|^p\right) \le 2^{2p-2}[E&\left(\sup_{0\le t\le T}\left\|\int_0^t f(s)ds - \sum_{k=1}^N 1_{A_k}\int_0^t \varphi_k(s)ds\right\|^p\right)\\
&+ E\left(\sup_{0\le t\le T}\left\|\sum_{k=1}^N 1_{A_k}\int_0^t \varphi_k(s)ds\right\|^p\right)\\
&+ E\left(\sup_{0\le t\le T}\left\|\int_0^t g(s)dW_s - \sum_{k=1}^N 1_{B_k}\int_0^t \psi_k(s)dW_s\right\|^p\right)\\
&+ E\left(\sup_{0\le t\le T}\left\|\sum_{k=1}^N 1_{B_k}\int_0^t \psi_k(s)dW_s\right\|^p\right)]
\end{aligned}$$

$$\begin{aligned}
&\le 2^{2p-2}[1 + E\left(\sup_{0\le t\le T}\max_{1\le k\le N}\left\|\int_0^t \varphi_k(s)ds\right\|^p\right)\\
&\quad + E\left(\sup_{0\le t\le T}\max_{1\le k\le N}\left\|\int_0^t \psi_k(s)dWs\right\|^p\right)]\\
&\le 2^{2p-2}[1 + E\left(\sup_{0\le t\le T}\left\|\int_0^t F(s,x_1(s))ds\right\|^p\right)\\
&\quad + E\left(\sup_{0\le t\le T}\left\|\int_0^t G(s,0)dW_s\right\|^p\right)].
\end{aligned}$$

Now, hypothesis **H3** implies that for each $t \in [0,T]$,

$$\left\|\int_0^t F(s,x_1(s))ds\right\|^2 = \sup_{f\in S_\Sigma(F_{x_1})}\left\|\int_0^t f(s)ds\right\|^2 = 2\sup_{f\in S_\Sigma(F_{x_1})}\int_0^t \langle f(s),x_1(s)\rangle\, ds$$
$$\leq 2M\int_0^t \|x_1(s)\|^2\, ds + 2\int_0^t \|F(s,0)\|\,\|x_1(s)\|\, ds$$
$$\leq (2M+1)\int_0^t \|x_1(s)\|^2\, ds + \int_0^t \|F(s,0)\|^2\, ds,$$

and by the Gronwall inequality we get

$$\left\|\int_0^t F(s,x_1(s))ds\right\|^2 \leq e^{(2M+1)t}\int_0^t \|F(s,0)\|^2\, ds.$$

Thus,

$$E\left(\sup_{0\leq t\leq T}\left\|\int_0^t F(s,x_1(s))ds\right\|^p\right) \leq e^{p(2M+1)T}T^{\frac{p}{2}-1}\int_0^t E(\|F(s,0)\|^p)ds.$$

Consequently, Theorem 3.2 yields that

$$E\left(\sup_{0\leq t\leq T}\|x_1(t)\|^p\right) \leq 2^{2p-2}\left[1+e^{p(2M+1)T}T^{\frac{p}{2}-1}\int_0^t E\left(\|F(s,0)\|^p\right)ds\right]$$
$$+\, 2^{2p-2}K_p\int_0^T E\left(\|G(s,0)\|_{L_2}^p\right)ds.$$

Also, we conclude that $E(\sup_{0\leq s\leq t}\|x_1(s)\|^p) < \infty$ for all $t\in[0,T]$. Now, we proceed by induction and assuming x_n has been defined such that $E(\sup_{0\leq s\leq t}\|x_n(s)\|^p) < \infty$, we consider the following stochastic differential inclusion:

$$dx_{n+1}(t) \in F(t,x_{n+1}(t))dt + G(t,x_n(t))dW_t. \tag{4.5}$$

Hypothesis **H4** implies that,

$$\|G(s,x_n(s))\|_{L_2} \leq C\sup_{0\leq\tau\leq s}\|x_n(\tau)\| + \|G(s,0)\|_{L_2},$$

which yields

$$E\left(\|G(s,x_n(s))\|_{L_2}^p\right) \leq 2^pC^pE\left(\sup_{0\leq\tau\leq s}\|x_n(\tau)\|^p\right) + 2^pE\left(\|G(s,0)\|_{L_2}^p\right).$$

By Theorem 3.2, we obtain

$$E\left(\sup_{0\leq s\leq t}\left\|\int_0^s G(\tau,x_n(\tau))dW_\tau\right\|^p\right) \leq \left(\frac{p}{p-1}\right)^p C_{p,T}\int_0^t E\left(\|G(s,x_n(s))\|_{L_2}^p\right)ds.$$

Hence, by Theorem 4.4, we can find a unique generalized solution x_{n+1} for (4.6) such that

$$E\left(\sup_{0\le s\le t}\|x_{n+1}\|^p\right)\le 2^{2p-2}\left[1+e^{p(2M+1)T}T^{\frac{p}{2}-1}\int_0^t E\left(\|F(s,0)\|^p\right)ds\right] + 2^{2p-2}\left(\frac{p}{p-1}\right)^p C_{p,T}\int_0^t E\left(\|G(s,x_n(s))\|_{L_2}^2\right)ds.$$

Therefore, $E(\sup_{0\le s\le t}\|x_{n+1}\|^p)<\infty$, for every $t\in[0,T]$. In order to show the convergence of the approximating solutions $\{x_n\}_{n=1}^\infty$, consider the difference of two consecutive terms of the sequence. We can assume that

$$x_{n+1}(t)-x_n(t)\in L^{\Sigma}_{F_{x_{n+1}}-F_{x_n}}(t)-J_{G_{x_n}-G_{x_{n-1}}}(t),$$

hence there are $f_n\in S^2_\Sigma(F_{x_{n+1}}-F_{x_n})$ and $g_{n-1}\in S^2_\Sigma(G_{x_n}-G_{x_{n-1}})$ such that

$$x_{n+1}(t)-x_n(t)=\int_0^t f_n(s)ds+\int_0^t g_{n-1}(s)dW_s.$$

Then, the Ito-type inequality implies that

$$\begin{aligned}\|x_{n+1}(t)-x_n(t)\|^p \le{}& p\int_0^t \|x_{n+1}(s)-x_n(s)\|^{p-2}\langle x_{n+1}(s)-x_n(s),f_n(s)\rangle\,ds\\ &+p\int_0^t\|x_{n+1}(s)-x_n(s)\|^{p-2}\langle x_{n+1}(s)-x_n(s),g_{n-1}(s)dW_s\rangle\\ &+\frac{p(p-1)}{2}\int_0^t\|x_{n+1}(s)-x_n(s)\|^{p-2}\|g_{n-1}(s)\|_{L_2}^2\,ds.\end{aligned}$$

Therefore, a similar argument as in the proof of Theorem 4.2 of [8] shows that

$$E\left(\sup_{0\le s\le t}\|x_{n+1}(t)-x_n(t)\|^p\right)\le\frac{\left(C_1e^{C_2T}\right)^{n-1}}{(n-1)!}E\left(\sup_{0\le s\le t}\|x_2(t)-x_1(t)\|^p\right),$$

where C_1 and C_2 are constants dependent on C, p and M. Thus, $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in the space $L^p(\Omega,\mathcal{F},P;K)$ and there exists a continuous $\mathcal{F}_t$-adapted process $x(t)$ such that $E(\sup_{0\le s\le t}\|x(s)\|^p)<\infty$ and $E(\sup_{0\le s\le t}\|x_n(t)-x(t)\|^p)$ $\to 0$, as $n\to\infty$ for every $t\in[0,T]$. To prove the uniqueness of solution, let x and y be two solutions of (4.4) corresponding to the initial random variable $\xi=0$ such that $x(t)\in L^\Sigma_{F_x}(t)+J_{G_x}(t)$ and $y(t)\in L^\Sigma_{F_y}(t)+J_{G_y}(t)$. We can conclude that

$$E\left(\sup_{0\le s\le t}\|x(s)-y(s)\|^2\right)\le 2M_1\int_0^t\sup_{0\le\tau\le s}\|x(\tau)-y(\tau)\|^2\,ds,\quad\forall t\in[0,T],$$

where M_1 is a positive constant. Then, by Gronwall's inequality, we have

$$E\left(\sup_{0\le s\le t}\|x(s)-y(s)\|^2\right)=0,$$

for all $t \in [0, T]$. □

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