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COHOMOGENEITY ONE ANTI DE SITTER SPACE $H_1^{3\dagger}$

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ABSTRACT. We study cohomogeneity one anti de Sitter space H_1^3. When the action is proper we determine the orbits up to isometry and the acting groups up to conjugacy.

1. **Introduction**

One of the central problems in pseudo Riemannian geometry is to study a pseudo Riemannian manifold M via its isometry group $Iso(M)$. The larger $Iso(M)$ is, the simpler M is. Many manifolds have isometry group large enough so that Lie theory can be applied. In non transitive cases, $Iso(M)$ is a geometric invariant of M ranking in importance with its curvature and geodesics. This is one of the reasons that non transitive actions are of so much interest to mathematicians. A (pseudo) Riemannian manifold M has cohomogeneity one if some closed Lie subgroup $G \subset Iso(M)$ acts on M with a codimension one orbit. So, the isometry group of M, $Iso(M)$, is quite large. This fact enables one to study such manifolds quite successfully, and this is why there are so many interesting papers on this subject (see [1], [2], [3], [5], [10], [11], [13] for example). Here, continuing of our study of cohomogeneity one Lorentzian manifolds, we study H_1^3 where a closed Lie subgroup

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$G \subset Iso(H_1^3)$ can act on H_1^3 with an orbit of dimension 2. We characterize the orbits up to isometry, determine the possible acting groups up to conjugacy and give the orbit space(s). Our main results are Theorems 3.1−3.4.

2. **Preliminaries**

We begin with the definition of proper action and recall some of its properties from [6].

Definition 2.1. ([6], p.53). An action of the group G on the manifold M is said to be proper if the mapping $\phi : G \times M \to M \times M$, $(g, x) \mapsto (g.x, x)$ is proper.

The orbit space M/G of a proper action of G on M is Hausdorff, the orbits are closed submanifolds in M, and the isotropy subgroups are compact. Throughout the paper, we assume that the action is effective and proper.

A result by Mostert (see [11]), for the compact case, and Berard Bergery (see [4]), for the general case, says that the orbit space M/G, equipped with the quotient topology, is homeomorphic to $\mathbb{R}$, S^1, $[0, +\infty)$ or $[0, 1]$.

Consider the projection map $M \to M/G$ to the orbit space. Given a point $x \in M$, we say that the orbit $G(x)$ is *principal* (resp. *singular*) if the corresponding image in the orbit space M/G is an internal (resp. boundary) point. A point x whose orbit is principal (resp. singular) will be called *regular* (resp. *singular*). All principal orbits are diffeomorphic to each other, and each singular orbit is of dimension less than or equal to $n - 1$, where $n = dimM$. A singular orbit of dimension $n - 1$ is called an *exceptional* orbit. Note that no exceptional orbit is simply connected, and if M is simply connected no exceptional orbit may exist.

Definition 2.2. Let M be a Lorentzian manifold and G be a closed and connected Lie subgroup of $Iso(M)$ which acts properly and isometrically on M. The orbit $G(x)$, for $x \in M$, is called degenerate if the induced metric on $G(x)$ is degenerate.

Notation: Throughout the paper, $\mathbb{R}_2^4$ denotes the 4-dimensional real vector space $\mathbb{R}^4$ with the scalar product of signature $(2, 2)$ given by

$$\langle X, Y\rangle = -\frac{1}{2}(x_1y_4 - x_2y_3 - x_3y_2 + x_4y_1),$$

and

$$H_1^3 = \{X \in \mathbb{R}_2^4 | \langle X, X \rangle = -1\}$$

is the anti de Sitter space of dimension 3. The notation $G_1 \cong G_2$ means that G_1 is isomorphic to G_2. We use $Aff_\circ(\mathbb{R})$ for the connected component of the Lie group of affine transformations of the real line (this group is simply connected with trivial center).

Let $\pi : \mathbb{R}_1^2 \longrightarrow \mathbb{R}_1^2/\mathbb{Z}^2$ be the pseudo Riemannian covering (map) space of the Lorentzian Torus T_1^2 (see [12], p.191), where $\mathbb{R}_1^2$ is the Minkowski space with the metric generated by the form $ds^2 = -dx_1^2 + dx_2^2$. Then, T_1^2 is a compact flat 2-dimensional Lorentzian space.

We need the following lemma to prove our main results.

Lemma 2.3. ([9]). *Let G be a Lie group which acts on a manifold M. Let H be a closed Lie subgroup of G such that G/H is compact. Then, G acts properly if and only if H does.*

3. Main results

We now state our results.

Theorem 3.1. *If H_1^3 is of cohomogeneity one under the proper action of a connected, closed Lie subgroup $G \subset Iso(H_1^3)$, then there is no space-like orbit.*

Theorem 3.2. *Let H_1^3 be as in Theorem 3.1. If there is a degenerate orbit, then we have:*
(1) There is no singular orbit and each principal orbit is diffeomorphic to $\mathbb{R}^2$.
(2) The orbit space H_1^3/G is homeomorphic to S^1.
(3) The action is free, and the group G is isomorphic to either the Lie group $\mathbb{R}^2$ or $Aff_\circ(\mathbb{R})$. Furthermore, if all orbits are degenerate, then G is isomorphic to $Aff_\circ(\mathbb{R})$.

Theorem 3.3. *Let H_1^3 be as in Theorem 3.1. Then, the following assertions are equivalent:*
(1) There is neither a degenerate nor a singular orbit.
(2) The orbit space H_1^3/G is homeomorphic to $\mathbb{R}$.
(3) G is isomorphic to $\mathbb{R} \times SO(2)$.

(4) Each principal orbit is isometric to $\mathbb{R} \times B$*, where* B *is anti-isometric to* S^1*.*

Theorem 3.4. *Let* H_1^3 *be as in Theorem 3.1. Then, the following statements are equivalent:*
1) There is a singular orbit B *anti-isometric to* S^1*.*
2) The orbit space H_1^3/G *is homeomorphic to* $[0, +\infty)$*.*
3) G *is conjugate to* $SO(2) \times SO(2)$*.*
4) Each principal orbit is isometric to a flat Lorentzian torus.

Proof. We first determine the acting groups up to conjugacy. Then, we find the orbits up to isometry. Finally, we characterize the orbit space H_1^3/G. As the result, we get the proof of Theorems 3.1 to 3.4.
Let $M(2,\mathbb{R})$ be the 2×2 matrices with real entries equiped with the metric defined by

$$\langle X|X\rangle = -det(X) \quad , \quad X \in M(2,\mathbb{R}),$$

using polarization. Consider the isometry

$$(\mathbb{R}_2^4, \langle,\rangle) \longrightarrow (M(2,\mathbb{R}), \langle\ |\ \rangle),$$

defined by

$$(a,b,c,d) \mapsto \left[\begin{array}{cc} a & b \\ c & d \end{array}\right],$$

which implies that $(H_1^3, \langle,\rangle)$ is isometric to $(M = SL(2,\mathbb{R}), \langle\ |\ \rangle)$. Consider the isometric action of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ on M defined by:

$$((g, g'), h)) \mapsto ghg'^{-1}.$$

Then, there exists the homomorphism,

$$\theta : SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \longrightarrow Iso(M),$$

with

$$\theta(g,g')(h) = ghg'^{-1}, \qquad \forall h \in M$$

and

$$ker(\theta) = \{(I,I), (-I,-I)\},$$

where I denotes the identity matrix. Thus, we have (see [14]):

$$Iso_\circ(H_1^3) = SO_\circ(2,2) \cong \frac{SL(2,\mathbb{R}) \times SL(2,\mathbb{R})}{\mathbb{Z}_2} = Iso_\circ(M).$$

Therefore, we may assume that $(M, \langle\ |\ \rangle)$ is of cohomogeneity one under the proper action of a connected and closed Lie subgroup,

$$G \subseteq SL(2,\mathbb{R}) \times SL(2,\mathbb{R}).$$

It is known that each connected one dimensional Lie subgroup of $SL(2,\mathbb{R})$ is conjugate to one of the groups (see [8], p.436),

$$A = \left\{ \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \mid t \in \mathbb{R} \right\},$$

$$N = \left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mid t \in \mathbb{R} \right\},$$

$$K = \left\{ \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \mid t \in \mathbb{R} \right\},$$

and each one of them is a one parameter subgroup defined by,

$$X_\circ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, Y_\circ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, Z_\circ = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

respectively. ($\{X_\circ, Y_\circ, Z_\circ\}$ is a basis for $\mathfrak{sl}(2,\mathbb{R})$ and we fix this basis throughout the discussion.)
Each two dimensional connected closed Lie subgroup of $SL(2,\mathbb{R})$ is conjugate to

$$\left\{ \begin{bmatrix} e^t & s \\ 0 & e^{-t} \end{bmatrix} \mid t, s \in \mathbb{R} \right\},$$

which is isomorphic to $Aff_\circ(\mathbb{R})$.
Let M be of cohomogeneity one under the proper action of a connected and closed Lie subgroup $G \subset SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ and let $p_i : G \to SL(2,\mathbb{R})$, $i = 1, 2$, be projections on the first and the second factor. We continue the proof by considering the following cases.

Case 1. $dim(p_1(G)) = dim(p_2(G)) = 1$.

Subcase 1.1. $p_1(G)$ and $p_2(G)$ are noncompact.
Since $dim(p_1(G)) = dim(p_2(G)) = 1$, then,

$$G = \{(g^t = \exp(tV), h^s = \exp(sW)) \mid s, t \in \mathbb{R}\},$$

where $V, W \in \mathfrak{sl}(2,\mathbb{R})$. If $V = W$, then the isotropy subgroup G_I is not compact, and so the action is not proper. We show that if V, W are conjugate in $SL(2,\mathbb{R})$, then the action is not proper either. In fact, if $V = pWp^{-1}$ for some $p \in M$, then,

$$\exp(tV) = p\exp(tW)p^{-1} \Longrightarrow g^t = ph^tp^{-1},$$

for each $t \in \mathbb{R}$. Hence, the isotropy subgroup at p is:

$$G_p = \{(g^t, h^s) \in G \mid (g^t, h^s).p = p\} = \{(g^t, h^t) \in G \mid g^t = ph^tp^{-1}\},$$

which is a noncompact subgroup of G, and so the action is not proper. Hence, we may assume:

$$V = X_\circ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad W = Y_\circ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Fix an arbitrary $p = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M$,

$$(g^t, h^s).p = g^tph^{-s} = \exp(tX_\circ)p\exp(-sY_\circ)$$

For arbitrary fixed $\alpha \in \mathbb{R}$, let $s = \alpha t$ and

$$\Psi_t(p) = (g^t, h^{\alpha t}).p\ .$$

then

$$-det(\frac{d}{dt}\Psi_t(p)|_{t=0}) = -det(X_\circ p - \alpha pY_\circ) = 1 + 2\alpha xz$$

Hence, if $xz \neq 0$ then the polynomial $1 + 2\alpha xz$ can be positive, zero and negative for various values of α, which shows that the orbit $G(p)$ is a Lorentzian orbit. If $xz = 0$, then,

$$-det(X_\circ p - \alpha pY_\circ) = 1,$$

for each $\alpha \in \mathbb{R}$. On the other hand, for

$$\Phi_s(p) = (I, h^s).p = p\exp(-sY_\circ)$$

we have,

$$-det(\frac{d}{dt}\Phi_s(p)|_{s=0}) = -det(pY_\circ) = 0,$$

which shows that the orbit $G(p)$ is a degenerate principal orbit when $xz = 0$.

One can see that $G_p = \{I\}$, for each $p \in M$, and so the action is free, which implies that there is no singular orbit, and G is isomorphic to the Lie group $\mathbb{R}^2$, and the orbit space M/G is diffeomorphic to $\mathbb{R}$ or S^1. If M/G is diffeomorphic to $\mathbb{R}$, then M (so H^3_1) must be diffeomorphic to $\mathbb{R} \times G(p) = \mathbb{R} \times \mathbb{R}^2$, which is obviously not true, since H^3_1 is diffeomorphic to $S^1 \times \mathbb{R}^2$ by Lemma 4.25 of [12]. Hence, M/G is diffeomorphic to S^1.

Subcase 1.2 $dim(p_1(G)) = dim(p_2(G)) = 1$ and $p_i(G)$ is compact for some $1 \leq i \leq 2$. Hence, by the cohomogeneity one assumption, G is conjugate to one of the following groups:

$$G_1 = \{(\exp(tX_\circ), \exp(sZ_\circ)|\ t, s \in \mathbb{R}\},$$
$$G_2 = \{(\exp(tY_\circ), \exp(sZ_\circ)|\ t, s \in \mathbb{R}\},$$
$$G_3 = \{(\exp(tZ_\circ), \exp(sZ_\circ)|\ t, s \in \mathbb{R}\}.$$

So, the action is proper by Lemma 2.3.
If $G = G_1 \cong \mathbb{R} \times SO(2)$, then for fix $p = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M$ let,

$$\Psi(t) = \exp(tX_\circ)p\exp(\alpha tZ_\circ),$$

where α is an arbitrary fixed real number. Then,

$$-det(\frac{d}{dt}\Psi_t(p)|_{t=0}) = -det(X_\circ p - \alpha pZ_\circ) = -(\alpha^2 + 2(xz + yw)\alpha - 1).$$

Since the polynomial $u^2 + 2(xz + yw)u - 1 = 0$ has two roots, then $-det(X_\circ p - \alpha pZ_\circ)$ can be negative, zero or positive, which shows that all orbits are Lorentzian. So for each $p \in M$, $G_p = \{I\}$, there is no singular orbit, and hence the orbit space M/G is homeomorphic to $\mathbb{R}$ or S^1. We claim that the orbit space M/G is homeomorphic to $\mathbb{R}$. If M/G is homeomorphic to S^1, then the projection $\pi : M \to S^1$ is a fibration with fiber G/K, where K is the isotropy subgroup of a regular point (see [3]). By Theorem 4.41 in [7, p.379] we get the following exact sequence,

$$0 \to \pi_1(G/K) \to \pi_1(M) \to \pi_1(S^1) \to 0,$$

that is, the short exact sequence,

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0,$$

which obviously can not occur. Thus M/G is homeomorphic to $\mathbb{R}$.

For $G = G_2$, the discussion is similar to that of the case $G = G_1$.
If $G = G_3 \cong SO(2) \times SO(2)$, then,

$G_I = \{\exp(tZ_\circ), \exp(tZ_\circ)|\ t \in \mathbb{R}\} \cong SO(2)$,
where I is the identity matrix. So, $G(I)$ is a singular orbit diffeomorphic to S^1. For fix $p = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M$, let

$\Phi(t) = \exp(tZ_\circ)p\exp(\alpha t Z_\circ),$

where α is an arbitrary fixed real number. Then,

$$-det(\frac{d}{dt}\Phi_t(p)|_{t=0}) = -det(Z_\circ p - \alpha p Z_\circ) = -(\alpha^2+(x^2+y^2+z^2+w^2)\alpha+1).$$

Since $p \in SL(2,\mathbb{R})$, then $x^2+y^2+z^2+w^2 \geq 2$. If $x^2+y^2+z^2+w^2 = 2$, then $G(p)$ is a time-like singular orbit anti isometric to S^1. By Theorem 3.1 in [5], the singular orbit $G(p)$ is unique, and so all such points p, where $x^2+y^2+z^2+w^2 = 2$, belong to the orbit $G(I)$. If $x^2+y^2+z^2+w^2 > 2$, then,

$$-det(\tfrac{d}{dt}\Phi_t(p)|_{t=0})$$

can be positive, zero or negative for different values of α, and so $G(p)$ is a Lorentzian principal orbit. The Lie group G is compact, and so $G(p)$ is compact for each $p \in M$. We give the results related to *Case* 1 in Table 1.

$G = \{(\exp(tV), \exp(sW)) \| t, s \in \mathbb{R}\}$	*Number of singular orbits*	*Causal character of principal orbit* $G(p)$ $p = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M$	M/G
$V = X_\circ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $W = Z_\circ = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	0	*Lorentzian*	$\mathbb{R}$
$V = X_\circ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $W = Y_\circ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	0	$xz = 0 \rightarrow$ *degenerate* $xz \neq 0 \rightarrow$ *Lorentzian*	S^1
$V = Z_\circ = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $W = Y_\circ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	0	*Lorentzian*	$\mathbb{R}$
$V = W = Z_\circ = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	1 (*time* − *like*)	*Lorentzian* (*torus*)	$[0, +\infty)$

TABLE 1. $p_1(G)$ and $p_2(G)$ are one dimensional

Case 2. $dim(p_i(G)) \geq 2$, for some $1 \leq i \leq 2$.
Without loss of generality, we may assume that $dim(p_1(G)) = 2$, and so $p_1(G) = Aff_\circ(\mathbb{R})$ up to conjugacy.

Subcase 2.1 $p_2(G)$ is noncompact. Then, $p_2(G)$ is isomorphic to either $SL(2,\mathbb{R})$ or $Aff_\circ(\mathbb{R})$ or $\mathbb{R}$. The action is of cohomogeneity one, and so the case $SL(2,\mathbb{R})$ can not occur. Hence, G is simply connected. Thus, by the properness of the action each isotropy subgroup is trivial; i.e., the action is free, and so $dim(G) = 2$. Hence, the kernel of the map $p_1 : G \longrightarrow Aff_\circ(\mathbb{R})$ is discrete, and hence p_1 is a covering map. Since $Aff_\circ(\mathbb{R})$ is simply connected, then the map p_1 is injective. Therefore, G is the group consists of the elements of the form,

$$(g, \rho(g)) \in SL(2,\mathbb{R}) \times SL(2,\mathbb{R}),$$

where $g \in Aff_\circ(\mathbb{R})$ and $\rho = p_2 o p_1^{-1}$. There are two cases.

Subsubcase 2.1.1 $p_2(G)$ is isomorphic to $\mathbb{R}$. Since $p_2(G)$ is abelian, then the commutator subgroup $[Aff_\circ(\mathbb{R}), Aff_\circ(\mathbb{R})]$ must be contained in the kernel of ρ. On the other hand, $[Aff_\circ(\mathbb{R}), Aff_\circ(\mathbb{R})]$ is the one dimensional subgroup spanned by unipotent elements $\exp(Y_\circ)$, which shows that there is only one homomorphism $\rho : Aff_\circ(\mathbb{R}) \longrightarrow \mathbb{R}$ up to conjugacy in $Aff_\circ(\mathbb{R})$. If $\rho(Aff_\circ(\mathbb{R}))$ is conjugate to $\{\exp(tX_\circ) \mid t \in \mathbb{R}\}$, then the action is not proper. So, $\rho(Aff_\circ(\mathbb{R}))$ is conjugate to $\{\exp(tY_\circ) \mid t \in \mathbb{R}\}$ in $SL(2,\mathbb{R})$. Hence, up to conjugacy,

$$\rho\left(\begin{bmatrix} e^t & s \\ 0 & e^{-t} \end{bmatrix}\right) = \begin{bmatrix} 1 & at \\ 0 & 1 \end{bmatrix},$$

where a is a fixed nonzero real number. Thus,

$$G = \left\{\left(\begin{bmatrix} e^t & s \\ 0 & e^{-t} \end{bmatrix}, \begin{bmatrix} 1 & at \\ 0 & 1 \end{bmatrix}\right) \mid t, s \in \mathbb{R}\right\}.$$

Hence, $\{(X_\circ, aY_\circ), (Y_\circ, 0)\}$ is a basis for the Lie algebra $\mathfrak{g}$.
Now, we determine the causal character of the orbits. Fix $p = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M$. Then, for $(Y_\circ, 0) \in \mathfrak{g}$ let

$\Phi_t(p) = \exp(tY_\circ)p$.
We then have,
$-det(\frac{d}{dt}\Phi_t(p)|_{t=0}) = 0$.

So, the orbit $G(p)$ can not be space-like. Now, for a fixed $\beta \in \mathbb{R}$ let $(V_t, W_t) = t(X_\circ, aY_\circ) + \beta t(Y_\circ, 0)$ and

$\Psi_t(p) = \exp(V_t)p\exp(W_t)^{-1}$.
Then
$-det(\frac{d}{dt}\Psi_t(p)|_{t=0}) = 1 + 2axz + a\beta z^2$,

which shows that if $z = 0$ then $G(p)$ is a degenerate principal orbit and if $z \neq 0$ then $G(p)$ is a Lorentzian principal orbit.

Subsubcase 2.1.2 $p_2(G)$ is isomorphic to $Aff_\circ(\mathbb{R})$. Since automorphisms of $Aff_\circ(\mathbb{R})$ are conjugacies, then for each automorphism $\rho \in Aut(Aff_\circ(\mathbb{R}))$, there is an $h \in Aff_\circ(\mathbb{R})$ such that

$$\begin{aligned} \rho : Aff_\circ(\mathbb{R}) &\longrightarrow Aff_\circ(\mathbb{R}) \\ g &\longmapsto h^{-1}gh. \end{aligned}$$

Hence, $G = \{(g, h^{-1}gh) \mid g \in Aff_\circ(\mathbb{R})\}$, which shows that $G_h = Aff_\circ(\mathbb{R})$; i.e., the action is not proper. Thus, subsubcase 2.1.2 can not occur.

Subcase 2.2 $p_1(G) = Aff_\circ(\mathbb{R})$ and $p_2(G)$ is compact. We study this subcase by considering the following subsubcases.

Subsubcase 2.2.1 $p_1(G) = Aff_\circ(\mathbb{R})$ and $p_2(G)$ is trivial. This case reduces to the left action of

$$G = \left\{ \begin{bmatrix} e^t & s \\ 0 & e^{-t} \end{bmatrix} \mid t, s \in \mathbb{R} \right\} = Aff_\circ(\mathbb{R})$$

on $M = SL(2, \mathbb{R})$, which is obviously proper and free. So, there is no singular orbit, and the orbit space M/G is diffeomorphic to S^1. Let

$$\left\{ X_\circ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, Y_\circ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

be a basis for the Lie algebra $\mathfrak{g}$. Then, for each $p \in M$ define,

$\Psi_t(p) = \exp(tX_\circ)p,$

and

$\Phi_t(p) = \exp(tY_\circ)p.$

Hence,

$\frac{d}{dt}\Psi_t(p)dt|_{t=0} = X_\circ p \Longrightarrow -det(\frac{d}{dt}\Psi_t(p)|_{t=0}) = 1,$
$\frac{d}{dt}\Phi_t(p)|_{t=0} = Y_\circ p \Longrightarrow -det(\frac{d}{dt}\Phi_t(p)|_{t=0}) = 0,$

which shows that the orbit $G(p)$ is a degenerate orbit.

Subsubcase 2.2.2 $p_1(G) = Aff_\circ(\mathbb{R})$ and $p_2(G)$ is conjugate to $SO(2)$.
Without loss of generality, we may assume:

$$p_1(\mathfrak{g}) = \left\{ \begin{bmatrix} t & s \\ 0 & -t \end{bmatrix} \mid s, t \in \mathbb{R} \right\},$$

$$p_2(\mathfrak{g}) = \left\{ \begin{bmatrix} 0 & -u \\ u & 0 \end{bmatrix} \mid u \in \mathbb{R} \right\},$$

where $p_1(\mathfrak{g})$ and $p_2(\mathfrak{g})$ are the Lie algebras of $p_1(G)$ and $p_2(G)$, respectively. If u is independend of s and t, then there will be an orbit of dimension three which is in contrast with the cohomogeneity one assumption. So, $u = u(t, s)$ is a linear function $u : \mathbb{R}^2 \to \mathbb{R}$; i.e., $u = at + bs$, for some $a, b \in \mathbb{R}$. But the relation $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$, implies $b = 0$, and without loss of generality, we can assume that $a = 1$. Thus,

$$\mathfrak{g} = \left\{ \left(\begin{bmatrix} t & s \\ 0 & -t \end{bmatrix}, \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} \right) \mid s, t \in \mathbb{R} \right\},$$

which implies:

$$G = \left\{ \left(\begin{bmatrix} e^t & v \\ 0 & e^{-t} \end{bmatrix}, \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \right) \mid v, t \in \mathbb{R} \right\}.$$

The group G is isomorphic to $Aff_\circ(\mathbb{R})$, and so the action is free (the maximal compact subgroup of G is trivial) and there is no singular orbit. Hence, the orbit space M/G is diffeomorphic to S^1, and by a

similar discussion to that of the *subcases* 1.1 and 2.1, one gets that each orbit is Lorentzian.
Since the tables 1 and 2 illustrate all connected closed Lie subgroups (up to conjugacy), which can act properly and of cohomogeneity one on M, one gets the proof of Theorems 3.1−3.4 . □

G	*Number of singular orbits*	*Causal character of the principal orbit* $G(p)$ $p=\begin{bmatrix} x & y \\ z & w \end{bmatrix}$	M/G
$Aff_{\circ}(\mathbb{R}) = (\begin{bmatrix} e^t & s \\ 0 & e^{-t} \end{bmatrix}, \{I\})$	0	*degenerate*	S^1
$(\begin{bmatrix} e^t & s \\ 0 & e^{-t} \end{bmatrix}, \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix})$	0	*Lorentzian*	S^1
$(\begin{bmatrix} e^t & s \\ 0 & e^{-t} \end{bmatrix}, \begin{bmatrix} 1 & at \\ 0 & 1 \end{bmatrix})$	0	$z \neq 0 \rightarrow$ *Lorentzian* $z = 0 \rightarrow$ *degenerate*	S^1

TABLE 2. $p_1(G)$ is conjugate to $Aff_{\circ}(\mathbb{R})$

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