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PERIODIC SOLUTIONS OF FOURTH-ORDER DELAY DIFFERENTIAL EQUATION

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Abstract. In this paper the periodic solutions of fourth order delay differential equation of the form

$$\ddot{\ddot{x}}(t) + a\dddot{x}(t) + f(\ddot{x}(t-\tau(t))) + g(\dot{x}(t-\tau(t))) + h(x(t-\tau(t))) = p(t)$$

is investigated. Some new positive periodic criteria are given.
Keywords: Fourth-order delay differential equation, periodic solutions.
MSC(2010): Primary: 34C25; Secondary: 34K09.

1. Introduction

The periodic solutions of delay differential equation systems has attracted attention. In the literature, Banach space technique, Fredholm operator, or L-compact solution approach have been utilized to derive sufficient conditions for periodic solutions. In [3, 13, 14], the periodic solution of delay differential equations was considered. Also, boundedness of solutions was investigated in [13]. Afterward, many books and papers dealt with the delay differential equations and given many results, see for example, [1, 2, 9–12].

On the other hand, the periodic solutions of fourth-order delay differential equations with deviating arguments has been discussed only by a few researchers. In recent years, the periodic solutions for some types of second and third-order delay differential equation with deviating argument have been investigated; see [4–6, 8]. Therefore the consequent problem for fourth-order delay differential equation with a deviating argument was discussed far less often. In [12], Sadek obtained stability and boundedness of a kind of third-order delay differential equation system. By using the continuation theorem of Mawhin's coincidence degree theory [1], we obtain some new results which extend the previous results in this content see [4–8].

A priori bounds are applied for periodic solutions of a Rayleigh equation. An existence theorem for periodic solutions can be extended by means of Mawhin's

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continuation theorems for the differential equation

$$\ddddot{x}(t) + a\dddot{x}(t) + f(\ddot{x}(t)) + g(\dot{x}(t)) + h(t, x(t)) = 0$$

where $a > 0$ and f, g, are real continuous function defined on $\mathbb{R}$; and $h(t, x)$ are periodic with common period ω. It is observed that once approximate a priori bounds for the periodic solutions of the equation

$$\ddddot{x}(t) + a\dddot{x}(t) + \lambda f(\ddot{x}(t)) + \lambda g(\dot{x}(t)) + h(t, x(t)) = 0$$

are known for each $\lambda \in (0, 1)$, then these theorems can be applied to imply the existence of periodic solutions to the equation. we will be considered with an additional delay

$$\ddddot{x}(t) + a\dddot{x}(t) + \lambda f(\ddot{x}(t - \tau(t))) + \lambda g(\dot{x}(t - \tau(t))) + h(x(t - \tau(t))) = \lambda p(t)$$

where τ and $p(t)$ are periodic with period ω. In this proof we consider $\int_0^\omega p(t)dt \neq 0$. For now, we define $|p|_0 = \max_{t\in[0,\omega]} |p(t)|$, $|x|_i := \big(\int_0^{2\pi} |x(s)|^i ds\big)^{1/i}$, $i \geq 1$. Now we consider $\mu(t) = t - \tau(t)$, then $\mu(t)$ has inverse function ν. Let $b(t) = (1 - \dot{\tau}(\nu(t)))^{-1}$, $\dot{\tau} < 1$. In this paper, we first study the continuation theorem of coincidence degree theory and Banach space lemma techniques, we establish criteria for the existence of positive periodic solutions to the following fourth-order delay differential equation. The simplified model takes the form

$$\ddddot{x}(t) + a\dddot{x}(t) + f(\ddot{x}(t - \tau(t))) + g(\dot{x}(t - \tau(t))) + h(x(t - \tau(t))) = p(t). \tag{1.1}$$

2. **Positive periodic solutions**

The main purpose of this paper is to establish the existence of positive periodic solutions to (1.1). An example to compute the main result is given.

We establish the lemma 2.1 of existence of periodic solution based on the following three conditions.

Lemma 2.1. *Let X and Z be two Banach spaces. Consider a Fredholm operator equation*

$$Lx = \lambda N(x, \lambda), \tag{2.1}$$

where $L : \mathrm{Dom}\, L \cap X \to Z$ is an operator of index zero, $\lambda \in (0, 1)$ is a parameter. Let P and Q denote two projectors such that

$$P : X \to \ker L, \quad \textit{and} \quad Q : Z \to Z/ImL.$$

Assume that $N : \bar{\Omega} \times (0, 1) \to Z$ is L-compact on $\bar{\Omega} \times (0, 1)$, where Ω is an open bounded subset in X. In addition, suppose that

(a) *For each $\lambda \in (0, 1)$ and $x \in \partial\Omega \cap \mathrm{Dom}\, L$, $Lx \neq \lambda N(x, \lambda)$*
(b) *For each $x \in \partial\Omega \cap \ker L$, $QNx \neq 0$,*
(c) $\deg\{QN, \Omega \cap \ker L, 0\} \neq 0$.

Then $Lx = N(x, 1)$ has at least one solution in $\bar{\Omega}$.

An estimate of the periodic solution of the fourth order delay differential equation (1.1) will be given in the following theorem.

Theorem 2.2. *Suppose that there exist positive constants* $\delta_1, \delta_2, \delta_3 \geqslant 0$*,* $K > 0$ *and* $D > 0$*, such that*

(H1) $|f(x)| \leqslant K + \delta_1|x|$ *for* $x \in \mathbb{R}$
(H2) $xg(x) > 0$ *and* $|g(x)| > K + |p|_0 + \delta_1|x|$ *for* $|x| \geqslant D$
(H3) $x^2h(x) > 0$ *and* $|h(x)| > K + |p|_0 + \delta_2|x|$ *for* $|x| \geqslant D$
(H4) $\lim_{x\to-\infty} \frac{h(x)}{x^2} \leqslant \delta_3$.

Then (1.1) *has at least one* ω*-periodic solution for* $a\omega + 2\delta_1|b|_2\omega^{\frac{3}{2}} + 2\delta_2|b|_2\omega^{\frac{5}{2}} + 2\omega^2(1+\omega)\delta_3 < 1$.

Proof. To use Lemma 2.1 for (1.1), we take $X = \{x \in C^3(\mathbb{R}, \mathbb{R}) : x(t+\omega) = x(t)$ for all $t \in \mathbb{R}\}$ and $Z = \{z \in C(\mathbb{R}, \mathbb{R}) : z(t+\omega) = z(t)$ for all $t \in \mathbb{R}\}$ and denote $|x|_0 = \max_{t\in[0,\omega]} |x(t)|$ and
$\|x\| = \max\{|x|_0, |\dot{x}|_0, |\ddot{x}|_0 |\dddot{x}|_0\}$. Then X and Z are Banach spaces, with the norms $\|\cdot\|$ and $|\cdot|_0$, respectively. Let

$$\begin{aligned} Lx(t) &= \ddddot{x}, \quad x \in X,\, t \in \mathbb{R}; \\ N(x(t),\lambda) &= -a\dddot{x}(t) - \lambda f(\ddot{x}(t-\tau(t))) - \lambda g(\dot{x}(t-\tau(t))) - h(x(t-\tau(t))) \\ &\quad + \lambda p(t), \quad x \in X,\, t \in \mathbb{R}; \\ Px(t) &= \frac{1}{\omega}\int_0^\omega x(t)dt, \quad Qz(t) = \frac{1}{\omega}\int_0^\omega z(t)dt, \quad x \in X,\, t \in \mathbb{R}; \end{aligned}$$

where $x \in X$, $z \in Z$, $t \in \mathbb{R}$, $\lambda \in (0,1)$.

We prove that L is a Fredholm mapping of index 0, that $P : X \to \ker L$ and $Q \to Z/\operatorname{Im} L$ are projectors, and that N is L-compact on $\bar{\Omega}$ for any given open and bounded subset Ω in X.

The equivalent differential equation for the operator $Lx = \lambda N(x,\lambda)$, $\lambda \in (0,1)$, takes the form
(2.2)
$$\ddddot{x}(t) + \lambda a\dddot{x}(t) + \lambda^2 f(\ddot{x}(t-\tau(t))) + \lambda^2 g(\dot{x}(t-\tau(t))) + \lambda h(x(t-\tau(t))) = \lambda^2 p(t).$$

Let $x \in X$ be a solution of (2.2) for a certain $\lambda \in (0,1)$. Integrating (2.2) over $[0,\omega]$, we obtain
(2.3)
$$\int_0^\omega \left[\lambda^2 f(\ddot{x}(t-\tau(t))) + \lambda^2 g(\dot{x}(t-\tau(t))) + \lambda h(x(t-\tau(t))) - \lambda^2 p(t)\right] dt = 0.$$

Thus, there is a point $\xi \in [0,\omega]$, such that

$$\lambda^2 f(\ddot{x}(\xi-\tau(\xi))) + \lambda^2 g(\dot{x}(\xi-\tau(\xi))) + \lambda h(x(\xi-\tau(\xi))) - \lambda^2 p(\xi) = 0$$

and by using the condition (H1), we get

$$\begin{aligned}|h(x(\xi-\tau(\xi)))| &\leqslant |f(\ddot{x}(\xi-\tau(\xi)))|+|g(\dot{x}(\xi-\tau(\xi)))|+|p(\xi)| \\ &\leqslant K+\delta_1|\ddot{x}(\xi-\tau(\xi))|+\delta_2|\dot{x}(\xi-\tau(\xi))|+|p|_0 \\ &\leqslant K+|p|_0+\delta_2|\ddot{x}|_0+\delta_1|\dot{x}|_0\,.\end{aligned} \tag{2.4}$$

We will prove that there is a point $t_0 \in [0,\omega]$ such that

$$|x(t_0) < |\ddot{x}|_0+|\dot{x}|_0+D. \tag{2.5}$$

Case 1: $\delta_1, \delta_2 = 0$**.** If $|x(\xi-\tau(\xi))| > D$, (H1), (H2),(H3) and (2.4) yield $K+|p|_0 < |h(x(\xi-\tau(\xi)))| \leqslant K+|p|_0$, which is a contradiction. So

$$|x(\xi-\tau(\xi))| \leqslant D. \tag{2.6}$$

Case 2: $\delta_1, \delta_2 > 0$**.** If $|x(\xi-\tau(\xi))| > D$, then $K+|p|_0+\delta_1|\dot{x}(\xi-\tau(\xi))|+\delta_2|x(\xi-\tau(\xi))| < |h(x(\xi-\tau(\xi)))| \leqslant K+|p|_0+\delta_1|\ddot{x}|_0+\delta_2|\dot{x}|_0$. So that

$$|x(\xi-\tau(\xi))| \leqslant |\ddot{x}|_0. \tag{2.7}$$

Hence from (2.6) and (2.7), we see in either case 1 or 2 that

$$|x(\xi-\tau(\xi))| \leqslant |\ddot{x}|_0+D.$$

Let $\xi-\tau(\xi) = 2k\pi+t_0$, where k is an integer and $t_0 \in [0,\omega]$. Then

$$|x(t_0)| = |x(\xi-\tau(\xi))| < |\ddot{x}|_0+D.$$

So (2.5) holds, and

$$|x|_0 \leqslant |\dot{x}(t_0)|+\int_0^\omega |\ddot{x}(s)|ds < (\omega+1)|\ddot{x}|_0+D. \tag{2.8}$$

Let $G(\theta) = a\omega+2\delta_1|b|_2\omega^{\frac{3}{2}}+2\delta_2|b|_2\omega^{\frac{5}{2}}+2\omega^2(1+\omega)(\delta_3+\theta), \theta \in [0,\infty)$. From the assumption $G(0) = a\omega+2\delta_1|b|_2\omega^{\frac{3}{2}}+2\delta_2|b|_2\omega^{\frac{5}{2}}+2\omega^2(1+\omega)\delta_3 < 1$ and $G(\theta)$ is continuous on $[0,\infty)$, we know that there must be a small constant $\theta_0 > 0$ such that $G(\theta) = a\omega+2\delta_1|b|_2\omega^{\frac{3}{2}}+2\delta_2|b|_2\omega^{\frac{5}{2}}+2\omega^2(1+\omega)(\delta_3+\theta) < 1,\ \theta \in (0,\theta_0]$. Let $\varepsilon = \theta_0/2$, once we can obtain that $a\omega+2\delta_1|b|_2\omega^{\frac{3}{2}}+2\delta_2|b|_2\omega^{\frac{5}{2}}+2\omega^2(1+\omega)(\delta_3+\varepsilon) < 1$ For such a small $\varepsilon > 0$, in view of assumption (H_4), we find that there must be a constant $\rho > D$, which is independent of λ and x, such that

$$\frac{h(x)}{x^2} < (\delta_3+\varepsilon), \quad \text{for } x < -\rho. \tag{2.9}$$

Thus putting $\Delta_1 = \{t : t \in [0,\omega], x(t-\tau(t)) > \rho\}$, $\Delta_2 = \{t : t \in [0,\omega], x(t-\tau(t)) < -\rho\}$, $\Delta_3 = \{t : t \in [0,\omega], |x(t-\tau(t))| \leqslant \rho\}$, $\Delta_4 = \{t : t \in [0,\omega], |x(t-\tau(t))| \geq \rho\}$ and $h_\rho = \sup_{|x|\leqslant\rho} h(x)$, we have

$$\int_{\Delta_1} |h(t-\tau(t))|dt < \omega(\delta_1+\varepsilon)|x|_0, \int_{\Delta_2} |h(t-\tau(t))|dt < \omega(\delta_2+\varepsilon)|x|_0,$$

$$\int_{\Delta_3} |h(t-\tau(t))|dt < \omega(\delta_3+\varepsilon)|x|_0, \int_{\Delta_4} |h(t-\tau(t))|dt \leqslant \omega h_\rho.$$

From (2.3), we get

$$\begin{aligned}\int_0^\omega h(x(t-\tau(t)))dt &= \Big(\int_{E_1}+\int_{E_2}+\int_{E_3}+\int_{E_4}\Big)h(x(t-\tau(t)))dt\\ &\leqslant \int_0^\omega |f(\ddot{x}(t-\tau(t)))|dt\\ &+\int_0^\omega |g(\dot{x}(t-\tau(t)))|dt+\int_0^\omega |h(x(t-\tau(t)))|dt+\int_0^\omega |p(t)|dt.\end{aligned}\tag{2.10}$$

That is

$$\begin{aligned}\int_{E_1}|h(x(t-\tau(t)))|dt &\leqslant \int_{E_2}|h(x(t-\tau(t)))|dt+\int_{E_3}|h(x(t-\tau(t)))|dt\\ &+\int_{E_4}|h(x(t-\tau(t)))|dt\\ +\int_0^\omega |f(\ddot{x}(t-\tau(t)))|dt &+\int_0^\omega |g(\dot{x}(t-\tau(t)))|dt+\int_0^\omega |h(x(t-\tau(t)))|dt+\omega|p|_0.\end{aligned}\tag{2.11}$$

Using condition (H1), we have

$$\begin{aligned}\int_0^\omega |f(\ddot{x}(t-\tau(t)))|dt &= \int_{-\tau(0)}^{\omega-\tau(\omega)} \frac{1}{1-\ddot{\tau}(\nu(s))}|f(\ddot{x}(s))|ds\\ &= \int_0^\omega \frac{1}{1-\ddot{\tau}(\nu(s))}|f(\ddot{x}(s)|ds\\ &\leqslant \int_0^\omega \frac{\delta_1}{1-\ddot{\tau}(\nu(s))}|\ddot{x}(s)|ds+\int_0^\omega \frac{K}{1-\ddot{\tau}(\nu(s))}ds\\ &\leqslant \delta_1|b|_2\Big(\int_0^\omega |\ddot{x}(s)|ds\Big)^{1/2}+|b|_2K\sqrt{\omega}.\end{aligned}\tag{2.12}$$

Thus, by (2.11) and (2.12), we have

$$\begin{aligned}\int_0^\omega |\ddddot{x}(s)|ds &\leqslant a\int_0^\omega |\dddot{x}(s)|ds+\int_0^\omega |f(\ddot{x}(t-\tau(t)))|dt+\int_0^\omega |g(\dot{x}(t-\tau(t)))|dt\\ &+\int_0^\omega |h(x(t-\tau(t)))|dt+\omega|p|_0\\ &= a\int_0^\omega |\dddot{x}(s)|ds+\int_0^\omega |f(\ddot{x}(t-\tau(t)))|dt+\int_0^\omega |g(\dot{x}(t-\tau(t)))|dt\\ &+\Big(\int_{\Delta_1}+\int_{\Delta_2}+\int_{\Delta_3}+\int_{\Delta_4}\Big)|h(x(t-\tau(t)))|dt+\omega|p|_0\\ \leqslant a\sqrt{\omega}\Big(\int_0^\omega |\dddot{x}(s)|^2ds\Big)^{1/2} &+2\delta_1|b|_2\Big(\int_0^\omega |\ddot{x}(s)|^2ds\Big)^{1/2}+2\delta_2|b|_2\Big(\int_0^\omega |\dot{x}(s)|^2ds\Big)^{1/2}\\ &+2\omega(\delta_3+\varepsilon)|x|_0+2K\sqrt{\omega}|b|_2+2\omega f_\rho+2|p|_0.\end{aligned}\tag{2.13}$$

Since $x(0) = x(\omega)$, there exists $t_1 \in [0,\omega]$, such that $\ddot{x}(t_1) = 0$, Hence for $t \in [0,\omega]$,

$$|\ddot{x}|_0 \leqslant \int_0^\omega |\dddot{x}(t)|dt \leqslant \sqrt{\omega}\Big(\int_0^\omega |\dddot{x}(s)|^2 ds\Big)^{1/2}, \tag{2.14}$$

$$\Big(\int_0^\omega |\ddot{x}(s)|^2 ds\Big)^{1/2} \leqslant \sqrt{\omega} \max_{t\in[0,\omega]} |\ddot{x}(t)| \leqslant \omega\Big(\int_0^\omega |\dddot{x}(s)|^2 ds\Big)^{1/2}. \tag{2.15}$$

Since $x(t)$ is periodic function, for $t \in [0,\omega]$, we have

$$|\dddot{x}(t)| \leqslant \int_0^\omega |\ddddot{x}(t)|dt, \tag{2.16}$$

$$\Big(\int_0^\omega |\ddot{x}(s)|^2 ds\Big)^{1/2} \leqslant \sqrt{\omega} \max_{t\in[0,\omega]} |\dddot{x}(t)| \leqslant \sqrt{\omega}\int_0^\omega |\ddddot{x}(t)|dt. \tag{2.17}$$

Substituting (2.17) in (2.14), we have

$$|\ddot{x}|_0 \leqslant \omega \int_0^\omega |\ddddot{x}(t)|dt. \tag{2.18}$$

Substituting (2.18) in (2.8), we get

$$|x|_0 \leqslant D + \omega(1+\omega)\int_0^\omega |\ddddot{x}(t)|dt. \tag{2.19}$$

Substituting (2.15),(2.17) and (2.19) in (2.13), and using inequality (2.16), it yields

$$|\dddot{x}|_0 \leqslant \int_0^\omega |\ddddot{x}(t)|dt \leqslant \frac{2K\sqrt{\omega}|b|_2 + 2\omega h_\rho + 2\omega|p|_0 + 2\omega(\delta_2+\varepsilon)D}{1 - a\omega - 2\delta_1|b|_2\omega^{\frac{3}{2}} - 2\delta_2|b|_2\omega^{\frac{5}{2}} - 2\omega^2(1+\omega)(\delta_3+\varepsilon)} \equiv A_3. \tag{2.20}$$

Substituting (2.20) in (2.18) and (2.19), we have

$$|x|_0 \leqslant D + \omega(1+\omega)A_3 \equiv A_1, \quad |\dot{x}|_0 \leqslant \omega A_3 \equiv A_2. \tag{2.21}$$

Let $A_0 = \max\{A_1, A_2, A_3, A_4\}$ and take $\Omega = \{x \in X : \|x\| \leqslant A_0\}$. The above bounds show that condition (a) of Lemma 2.1 is satisfied. If $x \in \partial\Omega \cap \ker L = \partial\Omega \cap \mathbb{R}$, then x is a constant with $x(t) = A_0$ or $x(t) = -A_0$. Then

$$\begin{aligned} QN(x,0) &= \frac{1}{\omega}\int_0^\omega \big[-a\dddot{x}(t) - h(x(t-\tau(t))\big]dt \\ &= \frac{1}{\omega}\int_0^\omega -f(x)dt = \frac{1}{\omega}\int_0^\omega -fA_0 dt \neq 0 \end{aligned}$$

Finally, consider the homotopy mapping

$$H(x,\mu) = \mu x + \frac{1-\mu}{\omega}\int_0^\omega h(x)dt, \quad \mu \in [0,1].$$

Since for every $\mu \in [0,1]$ and x in the intersection of $\ker L$ and $\partial\Omega$, we have

$$xH(x,\mu) = \mu x^2 + \frac{1-\mu}{\omega}\int_0^\omega xh(x)dt > 0,$$

This continues that

$$\begin{aligned}\deg\{QN(x,0),\Omega\cap\ker L,0\} &= \deg\{-h(x),\Omega\cap\ker L,0\}\\ &= \deg\{-x,\Omega\cap\ker L,0\}\\ &= \deg\{-x,\Omega\cap R,0\}\neq 0.\end{aligned}$$

All conditions in Lemma 2.1 are satisfied; therefore, (1.1) has at least one solution in Ω. The proof is complete. Now we give the examples. □

Example 1. Consider the equation

$$\begin{aligned}\ddddot{x}(t)+\frac{1}{2\pi}\dddot{x}(t)+\frac{7}{3\pi^2}\ddot{x}(t-\sin 2t))+\frac{7}{2\pi^2}\dot{x}(t-\sin 2t))\\ +\frac{3}{2}e^{-(\dot{x}(t-\sin 2t))^2}+h(x(t-\sin 2t))=\frac{1+\cos 2t}{4}\end{aligned}$$

where $p(t)=(1+\cos 2t)/4$, $\tau(t)=\sin 2t$, $f(u)=\frac{7}{3\pi^2}u+\frac{3}{2}e^{-u^2}, g(u)=\frac{7}{2\pi^2}u+\frac{3}{2}e^{-u^2}$ and

$$h(u)=\begin{cases}\frac{7}{3\pi^2}u+\frac{3}{2}+cot^{-1}u, & \text{for } u>D,\\ \left(\frac{7}{2\pi^2}+\frac{3}{2}+\frac{\pi}{2}\right), & \text{for } |u|\leqslant D,\\ \frac{7}{3\pi^2}u-\frac{3}{2}+cot^{-1}u, & \text{for } u<-D.\end{cases}$$

So we can choose $\delta_1=\delta_2=\delta_3=7/(3\pi^2)$, $D=1$, $K=1$, $|p|_0=1/2$, $|b|_2<\sqrt{\omega}$, $\omega=\pi/4$. Therefore, the fourth order delay differential equation has at least one periodic solution.

Example 2. Consider the equation

$$\begin{aligned}\ddddot{x}(t)+\frac{1}{2\pi}\dddot{x}(t)+\frac{7}{3\pi^2}\ddot{x}(t-\cos 4t))+\frac{7}{2\pi^2}\dot{x}(t-\cos 4t))\\ +\frac{3}{2}e^{-(\dot{x}(t-\cos 4t))^2}+h(x(t-\cos 4t))\\ =\frac{1+\sin 4t}{4}\end{aligned}$$

where $p(t)=(1+\sin 4t)/4$, $\tau(t)=\cos 4t$, $f(u)=\frac{7}{3\pi^2}u+\frac{3}{2}e^{-u^2}, g(u)=\frac{7}{2\pi^2}u+\frac{3}{2}e^{-u^2}$ and

$$h(u)=\begin{cases}\frac{7}{3\pi^2}u+\frac{3}{2}+tan^{-1}u, & \text{for } u>D,\\ \left(\frac{7}{2\pi^2}+\frac{3}{2}+\frac{\pi}{2}\right), & \text{for } |u|\leqslant D,\\ \frac{7}{3\pi^2}u-\frac{3}{2}+tan^{-1}u, & \text{for } u<-D.\end{cases}$$

So we can choose $\delta_1=\delta_2=\delta_3=7/(3\pi^2)$, $D=1$, $K=1$, $|p|_0=1/2$, $|b|_2<\sqrt{\omega}$, $\omega=\pi/2$. It is easy to verify that all the assumptions in Theorem 2.2 are satisfied. This equation has a periodic solution with period $\pi/2$.

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