Title:
On group equations

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ON GROUP EQUATIONS

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Abstract. Suppose $f$ is a map from a non-empty finite set $X$ to a finite group $G$. Define the map $\zeta^f_G : G \rightarrow \mathbb{N} \cup \{0\}$ by $g \mapsto |f^{-1}(g)|$. In this article, we show that for a suitable choice of $f$, the map $\zeta^f_G$ is a character. We use our results to show that the solution function for the word equation $w(t_1, t_2, \ldots, t_n) = g (g \in G)$ is a character, where $w(t_1, t_2, \ldots, t_n)$ denotes the product of $t_1, t_2, \ldots, t_n, t_1^{-1}, t_2^{-1}, \ldots, t_n^{-1}$ in a randomly chosen order.

Keywords: Finite groups, word equations, group characters.


1. Introduction

Let $G$ be a finite group and let $X$ be a non-empty finite set. Suppose that $f : X \rightarrow G$ is a map. Define the map $\zeta^f_G : G \rightarrow \mathbb{N} \cup \{0\}$ given by $g \mapsto |f^{-1}(g)|$. We call the map $\zeta^f_G$, the solution function for the equation $f(x) = g$, with $g \in G$. In general, $\zeta^f_G$ need not be a characters of $G$. Suppose $w = w(t_1, t_2, \ldots, t_n)$ is a word in $n$ symbols $t_1, t_2, \ldots, t_n$. Then $w(t_1, t_2, \ldots, t_n)$ defines the map $w : G^n \rightarrow G$ given by $(g_1, g_2, \ldots, g_n) \mapsto w(g_1, g_2, \ldots, g_n)$. For any word $w$, the map $\zeta^w_G$ is a class function on $G$ but not necessarily a character.

In [1], it is shown that if $w(x) = x^n$, then $\zeta^w_G$ is a generalized character of $G$, i.e., $\mathbb{Z}$-linear combination of irreducible character of $G$. In [6], Tambour discussed the solution function for two types of words

\begin{enumerate}
  \item $t_1 t_2 \cdots t_n t_1^{-1} t_2^{-1} \cdots t_n^{-1}$,
  \item $\prod_{i=1}^{n} [t_i, s_i]$,
\end{enumerate}
and used his results to write a new proof of Itô’s theorem: the degree of any irreducible character of $G$ divides the index of any abelian normal subgroup of $G$. A word $w(t_1, t_2, \ldots, t_n)$ obtained from the word $u(t_1, t_2, \ldots, t_n) = t_1 t_2 \cdots t_n t_1^{-1} t_2^{-1} \cdots t_n^{-1}$, by shuffling $t_1^\pm 1, t_2^\pm 1, \ldots, t_n^\pm 1$, is referred to as an admissible word (cf. [2]). It is shown in [2] that if $w$ is an admissible word, then $\zeta_G^w$ is a character of $G$. This is a generalization of a result of Frobenius ([3]).

In [5], Strunkov proved the following:

**Theorem 1.1.** [5, Theorem 2] Let $X_1, X_2$ be two finite sets. Suppose that $f_i : X_i \rightarrow G$ ($i = 1, 2$) are maps. If $\zeta_G^{f_1}$ and $\zeta_G^{f_2}$ are characters (resp. generalized characters), then so is $\zeta_G^f$ where $f : X_1 \times X_2 \rightarrow G$ is given by $f(x_1, x_2) = f_1(x_1)f_2(x_2)$.

In Section 2, we generalize this result in various ways. Our main results are:

**Theorem 1.2.** Let $X_1, X_2$ be two finite sets. Suppose $H$ is a normal subgroup of $G$ and $f_1 : X_1 \rightarrow H$, $f_2 : X_2 \rightarrow G$ are maps. Assume that $\zeta_H^{f_2}$ is a constant map and $\zeta_H^{f_1}$ is constant on $G$-conjugacy classes of $H$. Define the map $\phi : X_1 \times X_2 \rightarrow G$ by $\phi(x, y) = f_1(x)^{-1}f_2(y)^{-1}f_1(x)f_2(y)$. If $\zeta_H^{f_1}$ is a generalized character (respectively, character), then so is $\zeta_G^\phi$.

**Theorem 1.3.** For $i = 1, \ldots, n$, let $f_i : X_i \rightarrow G$ be a map, where $X_i$ is a finite set. Define for $n \geq 2$, $u_n : \prod_{i=1}^n X_i \rightarrow G$ by $u_n(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n-1} [f_i(x_i), f_{i+1}(x_{i+1})]$ and $u_1 : X_i \rightarrow G$ by $u_1(x) = 1$. If $\zeta_G^{f_1}$ is a character and $\zeta_G^{f_i}$ are constant maps for each $i = 2, \ldots, n$, then $\zeta_G^{u_n}$ is a character.

In particular, if $X_i = G$ and $f_i(x) = x$ for $i = 1, 2, \ldots, n$, then $u_n$ reduces to an admissible word, namely, $\omega_n(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n-1} [x_i, x_{i+1}]$ and $\zeta_G^{\omega_n}$ is a character of $G$. In Section 3, the main result in [2] is deduced from our result by reducing the problem on admissible words to a problem essentially of the words $\omega_n$ for $n \in \mathbb{N}$. Indeed, we extend the result of Das and Nath ([2]) and prove the following theorem.

**Theorem 1.4.** Suppose $w$ is a word in $t_1, t_2, \ldots, t_n$ such that each of $t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}$ occurs in $w$ at most once. Then $\zeta_G^w$ is a character of $G$.

**Notation.** Throughout the article, $G$ denotes a finite group and $\text{Irr}(G)$, the set of all irreducible characters of $G$. If $a, b \in G$, then $ba = b^{-1}ab$, $[a, b] = a^{-1}b^{-1}ab$. For class functions $\chi$ and $\psi$ on $G$, the expression $\langle \chi, \psi \rangle_G$ denotes the standard inner product on the space of class functions on $G$, that is,

$$
\langle \chi, \psi \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}.
$$
For simplicity we prefer to denote the inner product simply by \((\chi, \psi)\) if there is no ambiguity. Furthermore, if \(H\) is a subgroup of \(G\) and \(\chi\) a character of \(G\), \(\chi_{|H}^G\) denotes the restriction of \(\chi\) to \(H\).

2. Equations in finite groups

We start by recalling the well known result, which are used frequently.

**Lemma 2.1.** [4, Lemma 2.25] Let \(\rho\) be an irreducible \(\mathbb{C}\)-representation of \(G\) of degree \(n\). Suppose \(A\) is an \(n \times n\) matrix over \(\mathbb{C}\) which commutes with \(\rho(g)\) for all \(g \in G\). Then \(A = \lambda I_n\) for some \(\lambda \in \mathbb{C}\).

**Theorem 2.2.** [4, Theorem 2.13] The following holds for every \(h \in G\).

\[
\frac{1}{|G|} \sum_{g \in G} \chi_i(gh)\chi_j(g^{-1}) = \delta_{ij} \frac{\chi_i(h)}{\chi_i(1)},
\]

where \(\chi_i, \chi_j \in \text{Irr}(G)\).

**Proposition 2.3.** Let \(H\) be a subgroup of \(G\) and let \(X_1, X_2\) be two finite sets. Suppose \(f_1 : X_1 \to H\) and \(f_2 : X_2 \to G\) are maps. Define the map \(\psi : X_1 \times X_2 \to G\) by \(\psi(x, y) = f_2(y)^{-1}f_1(x)f_2(y)\) and assume that \(\zeta^\psi_H\) is a constant map. If \(\zeta_H^{f_1}\) is a generalized character (resp. character), then so is \(\zeta^\psi_G\). In particular, if \(H = G\), then \(\zeta^\psi_G = |X_2| \zeta_H^{f_1}\).

**Proof.** Since \(H\) is a subgroup of \(G\), for any \(\chi \in \text{Irr}(G)\), we have

\[
\chi_{|H}^G = \sum_{\phi_i \in \text{Irr}(H)} n_i \phi_i,
\]

for some \(n_i \in \mathbb{N} \cup \{0\}\). Suppose \(\chi\) is afforded by the irreducible representation \(\rho\) of \(G\). Consider an element

\[
(2.1) \quad z = \sum_{x \in X_1, y \in X_2} f_2(y)^{-1}f_1(x)f_2(y).
\]

then, \(\rho(z) = \sum_{x \in X_1, y \in X_2} \rho(f_2(y)^{-1}f_1(x)f_2(y))\). Now take the trace on both sides,

\[
\chi(z) = |X_2| \sum_{x \in X_1} \chi_{|H}^G(f_1(x))
\]

\[
(2.2) = |H||X_2| \sum_{\phi_i \in \text{Irr}(H)} n_i (\zeta_H^{f_1}, \phi_i).
\]

By definition of \(\zeta^\psi\), we have

\[
(2.3) \quad \chi(z) = |G| \langle \zeta^\psi_G, \chi \rangle.
\]
Therefore, from (2.2) and (2.3), we get
\[ \langle \zeta_G^\phi, \chi \rangle = \frac{|H||X_2|}{|G|} \sum_{\phi_i \in \text{Irr}(H)} n_i \langle \zeta_H^f, \bar{\phi}_i \rangle. \]

Hence, the result follows from the hypothesis.

Let \( H = G \). Then (2.2) reduces to
\[ (2.4) \quad \chi(z) = |X_2||G| \langle \zeta_G^f, \chi \rangle. \]

So, by (2.3) and (2.4), \( \zeta_G^w = |X_2| \zeta_G^f \).

With the help of the above proposition, we have the following corollary.

**Corollary 2.4.** Let \( H \) be a subgroup of a group \( G \). Suppose \( w:H \times G \rightarrow G \) is defined by \( w(x_1, x_2) = x_2^{-1}x_1x_2 \). Then \( \zeta_G^w \) is a character of \( G \).

**Proof of Theorem 1.2.** Let \( \chi_1, \chi_2, \ldots, \chi_r \) be the irreducible characters of \( G \). Suppose \( \chi_i \) is afforded by the irreducible representation \( \rho_i \) of \( G \). Consider an element \( z = \sum_{x \in X_1} \sum_{y \in X_2} f_1(x)^{-1}f_2(y)^{-1}f_1(x)f_2(y). \)

Then
\[ (2.5) \quad \rho_i(z) = \sum_{x \in X_1} \rho_i(f_1(x)^{-1})A_i(x) \]

where,
\[ (2.6) \quad A_i(x) = \sum_{y \in X_2} \rho_i(f_2(y)^{-1}f_1(x)f_2(y)). \]

As \( \zeta_G^f \) is a constant function, \( A_i(x) \) commutes with \( \rho_i(g), \forall g \in G \). By Lemma 2.1,
\[ (2.7) \quad A_i(x) = \lambda_x I \]

for some \( \lambda_x \in \mathbb{C} \). Now take the trace on the both sides of (2.7) and use (2.6) to obtain
\[ (2.8) \quad \lambda_x = \frac{|X_2|}{\chi_i(1)} \chi_i(f_1(x)). \]

Again take the trace on both side of (2.5) and use (2.7),(2.8) to get
\[ \chi_i(z) = \frac{|X_2|}{\chi_i(1)} \sum_{h \in H} \zeta_H^f(h) \chi_i(h) \chi_i(h^{-1}) \]
\[ = \frac{|X_2|}{\chi_i(1)} |H| \langle \zeta_H^f, \chi_i \rangle_H. \]

By definition of \( \zeta_G^\phi \), we have
\[ (2.10) \quad \chi_i(z) = |G| \langle \zeta_G^\phi, \chi_i \rangle. \]
Therefore, by (2.9) and (2.10), we get

\[(2.11) \quad \zeta_G^\phi = \frac{|X_2|}{|G|} \sum_{i=1}^{r} \frac{|H|}{\chi_i(1)} \langle \zeta_H^f[X_i, X_i] \rangle H \chi_i.\]

As \(\zeta_G^f_{|G|}\) is a constant map, \(|G|\) divides \(|X_2|\). Hence, to make the assertion it is enough to prove that \(\frac{|H|}{\chi_i(1)} \langle \zeta_H^f[X_i, X_i] \rangle H \) is an integer. Since \(H\) is normal in \(G\), \(H = \bigcup_{1 \leq j \leq k} Cl_G(x_j)\) for some \(x_j \in H\). For any irreducible character \(\chi_i\) of \(G\), we have

\[|H| \langle \zeta_H^f[X_i, X_i] \rangle H = \sum_{h \in H} \zeta_H^f(h) \chi_i(h) \chi_i(h^{-1}) = \sum_{1 \leq j \leq k} |Cl_G(x_j)| \langle \zeta_H^f(x_j) \chi_i(x_j) \chi_i(x_j) \rangle\]

Suppose \(K_j = \sum_{x \in Cl_G(x_j)} x\), for \(1 \leq j \leq k\). Then by [4, Theorem 3.7], \(w_{\chi_i}(K_j) = \chi_i(x_j)|Cl_G(x_j)|^{\chi_i(1)}\) is an algebraic integer for any irreducible character. Hence, we have

\[|H| \langle \zeta_H^f[X_i, X_i] \rangle H = \sum_{1 \leq j \leq k} \zeta_H^f(x_j) \chi_i(1) w_{\chi_i}(K_j) \chi_i(x_j)^{-1}.\]

Therefore,

\[\frac{|H|}{\chi_i(1)} \langle \zeta_H^f[X_i, X_i] \rangle H = \sum_{1 \leq j \leq k} \zeta_H^f(x_j) w_{\chi_i}(K_j) \chi_i(x_j)^{-1}.\]

The right-hand side of the latter equality is an algebraic integer. Therefore the result follows by the hypothesis. \(\square\)

In Theorem 1.2, if we take \(H = G\), then we get the following result.

**Corollary 2.5.** Let \(X_1, X_2\) be two finite sets. Suppose \(f_i : X_i \to G\) is a map for \(i = 1, 2\). Set \(\phi(x) := f_1(x)^{-1} f_2(y)^{-1} f_1(x) f_2(y)\). If \(\zeta^f_{|G|}\) is a constant map and \(\zeta_G^f\) is a generalized character (resp. character), then \(\phi_G^\phi\) is a generalized character (resp. character).

**Proof.** From Theorem 1.2, we have

\[\zeta_G^\phi = \sum_{i=1}^{r} \frac{|X_2|}{\chi_i(1)} \langle \zeta_H^f[X_i, X_i] \rangle H \chi_i.\]

As \(\zeta_G^f_{|G|}\) is a constant map, \(\chi_i(1)\) divides \(|X_2|\). Hence, the assertion follows. \(\square\)

**Corollary 2.6.** Let \(H\) be a normal subgroup of a group \(G\). Suppose \(w : H \times G \to G\) is defined by \(w(x, y) = x^{-1} y^{-1} xy\). Then \(\zeta_G^w\) is a character of \(G\).
Proof. In view of Theorem 1.2, \( \zeta^n_G \) is a character.

In Theorem 1.3, we discuss the solution function \( \zeta^n_G \), where \( f \) is a product of commutators of functions.

Proof of Theorem 1.3. Suppose for \( n \geq 1 \), the map \( v_n : \prod_{i=1}^n X_i \rightarrow G \) is defined by

\[
v_n := f_n(x_n)u_n(x_1, x_2, \ldots, x_n)f_n(x_n)^{-1}.
\]

By induction, we show that for any finite group \( G \), \( \zeta^n_G \) and \( \zeta^n_G^\circ \) are characters of \( G \). For \( n = 2 \), in view of Corollary 2.5, \( \zeta^{n^2}_G \) and \( \zeta^{n^2}_G^\circ \) are characters. Let \( n \geq 3 \). Now we assume that \( \zeta^{n-i}_G \) and \( \zeta^{n-i}_G^\circ \) are characters of \( G \) for \( 3 \leq i \leq (n-1) \). Suppose \( \rho_1, \rho_2, \ldots, \rho_r \) are irreducible representations of \( G \). Since \( \zeta^f_G \) is a constant map for \( i \geq 2 \), \( \zeta^f_G(g) = c_i \) for some \( c_i \in \mathbb{N} \). Now consider an element

\[
\omega = \sum_{x_i \in X_i} u_n(x_1, x_2, \ldots, x_n).
\]

Then,

\[
(2.13) \quad \rho_i(\omega) = \sum_{x_i \in X_i} \rho_i(B_{n-1}) \cdot A(x_{n-1}),
\]

where

\[
(2.14) \quad A(x_{n-1}) = \sum_{x_n \in X_n} \rho_i(f_n(x_n)^{-1}f_{n-1}(x_{n-1})f_n(x_n))
\]

and \( B_{n-1} = u_{n-1}(x_1, x_2, \ldots, x_{n-1})f_{n-1}(x_{n-1})^{-1} \). Since \( \zeta^f_n \) is a constant function, \( A(x_{n-1}) \) commutes with \( \rho_i(g) \) \( \forall g \in G \). Therefore, by Lemma 2.1,

\[
(2.15) \quad A(x_{n-1}) = \lambda(x_{n-1})I
\]

for a scalar \( \lambda(x_{n-1}) \in \mathbb{C} \). Take the trace on both sides of (2.14) and use (2.15) to get

\[
(2.16) \quad \lambda(x_{n-1}) = \frac{|X_n|}{\chi_i(1)} \cdot \chi_i(f_{n-1}(x_{n-1})).
\]

Now take the trace on both sides of (2.13) and use (2.15) and (2.16) to get

\[
\chi_i(\omega) = \frac{|X_n|}{\chi_i(1)} \sum_{x_i \in X_i} \chi_i(B_{n-1}) \cdot \chi_i(f_{n-1}(x_{n-1})).
\]
Set $C_{n-1} = f_{n-1}(x_{n-1})^{-1}u_{n-2}(x_1, x_2, \ldots, x_{n-2})$. It is easy to see that $B_{n-1}$ and $C_{n-1}$ are conjugates, for any $f_1(x_1), f_2(x_2), \ldots, f_{n-1}(x_{n-1}) \in G$. Therefore, we have

$$
\chi_i(\omega) = \frac{|X_n|}{\chi_i(1)} \sum_{x_i \in X_i} \chi_i(C_{n-1}) \cdot \chi_i(f_{n-1}(x_{n-1}))
$$

$$
= \frac{|X_n|}{\chi_i(1)} \sum_{h \in G} \zeta_{G}^{u_{n-2}}(h) \sum_{g \in G} \zeta_{G}^{f_{n-1}}(g^{-1}h) \chi_i(g) \quad (\text{set } \nu_{n-2} = h)
$$

$$
= \frac{|X_n||G||C_{n-1}|}{\chi_i(1)^2} \sum_{h \in G} \zeta_{G}^{\nu_{n-2}}(h) \cdot \chi_i(h) \quad (\text{use Theorem 2.2})
$$

$$
(2.17) \quad = \frac{|X_n||G||C_{n-1}|}{\chi_i(1)^2} \langle \zeta_{G}^{\nu_{n-2}}, \bar{\chi}_i \rangle \quad (\text{since } \zeta_{G}^{\nu_{n-2}} \text{ is a character of } G)
$$

Now use the definition of $\zeta_{G}^{u_n}$ to write

$$
(2.18) \quad \chi_i(\omega) = |G| \langle \zeta_{G}^{u_n}, \bar{\chi}_i \rangle.
$$

From (2.17) and (2.18), we have

$$
\zeta_{G}^{u_n} = \sum_{i=1}^{r} \frac{|X_n||G||C_{n-1}|}{\chi_i(1)^2} \langle \zeta_{G}^{\nu_{n-2}}, \bar{\chi}_i \rangle \chi_i
$$

so that $\zeta_{G}^{u_n}$ is a character of $G$. Next to show that $\zeta_{G}^{v_n}$ is a character. In $\mathbb{Z}[G]$, set

$$
z := \sum_{x_i \in X_i, i=1,\ldots,n} f_n(x_n)u_n(x_1, x_2, \ldots, x_n)f_n(x_n)^{-1}.
$$

Then,

$$
\chi_i(z) = \sum_{x_i \in X_i, i=1,\ldots,n} \chi_i(u_n(x_1, x_2, \ldots, x_n))
$$

$$
= \sum_{h \in G} \zeta_{G}^{u_n}(h) \chi_i(h)
$$

$$
(2.19) \quad = |G| \langle \zeta_{G}^{u_n}, \bar{\chi}_i \rangle.
$$

On the other hand, use the definition of $\zeta_{G}^{v_n}$ to write

$$
(2.20) \quad \chi_i(z) = |G| \langle \zeta_{G}^{v_n}, \bar{\chi}_i \rangle.
$$

Finally, by (2.19) and (2.20), $\zeta_{G}^{v_n} = \zeta_{G}^{u_n}$. Hence $\zeta_{G}^{v_n}$ is a character. This completes the proof. \hfill \Box

In particular, if $X_i = G$ and $f_i(x) = x$ for $i = 1, 2, \ldots, n$, then we have the following corollary.

**Corollary 2.7.** Suppose $w(x_1, \ldots, x_n) := \prod_{i=1}^{n-1} [x_i, x_{i+1}]$, where $x_1, \ldots, x_n$ are $n$ distinct letters. Then $\zeta_{G}^{w}$ is a character of $G$. 
Corollary 2.8. Let \( w : \prod_{i=1}^{2n} G \rightarrow G \) be the map defined by
\[
w(x_1, \ldots, x_n, y_1, \ldots, y_n) := \left( [ \ldots [ [y_1 x_1, y_2 x_2], y_3 x_3], \ldots | y_n x_n] \right).
\]
Then \( \zeta_G^w \) is a character of \( G \).

Observe that if \( \langle |G|, k \rangle = 1 \), then the map \( g \mapsto g^k \) is a bijection on \( G \). Hence, in view of Corollary 2.7, we have the following.

Corollary 2.9. Let \( w(x_1, \ldots, x_n) := [x_1^{k_1}, x_2^{k_2}, x_3^{k_3}, \ldots, x_n^{k_n}] \), where \( k_i \in \mathbb{N} \) with \( \langle |G|, k_i \rangle = 1 \) for each \( i = 1, \ldots, n \). Then \( \zeta_G^w \) is a character of \( G \).

3. Application

In this section we prove Theorem 1.4. In the Theorem 1.4, we extend the main result of Das and Nath ([2]) a little bit. We begin to prove the following lemmas.

Lemma 3.1. Let \( w_1 \) and \( w_2 \) be words in \( t_1, t_2, \ldots, t_n \). If \( w := w_1 t_{n+1} w_2 \), then \( \zeta_G^w \) is a constant function on \( G \).

Proof. Observe that \( (a_1, a_2, \ldots, a_n, a_{n+1}) \in G^{n+1} \) is a solution of \( w = g \) if and only if
\[a_{n+1} = w_1(a_1, a_2, \ldots, a_n)^{-1} g w_2(a_1, a_2, \ldots, a_n)^{-1}.
\]
Therefore, \( \zeta^w(g) = |G|^n \). \( \square \)

Lemma 3.2. Suppose \( w_1 \) is a word in \( t_1, \ldots, t_k \) and \( w_2 \), in \( t_{k+1}, \ldots, t_n \) such that both \( \zeta_G^{w_1} \) and \( \zeta_G^{w_2} \) are characters of \( G \). If \( w = w_1 t_k^{-1} w_2 t_k \), then \( \zeta_G^w \) is a character of \( G \).

Proof. Observe that \( w(x_1, \ldots, x_n) = g \) if and only if \( w_2(x_{k+1}, \ldots, x_n) = x_k w_1(x_1, \ldots, x_k)^{-1} g x_k^{-1} \). Therefore,
\[
\zeta_G^w(g) = \sum_{x_1, \ldots, x_k \in G} \zeta_G^{w_2}(w_1 x_1, \ldots, x_k)^{-1} g x_k^{-1}
\]
\[
= \sum_{x_1, \ldots, x_k \in G} \zeta_G^{w_2}(w_1 x_1, \ldots, x_k)^{-1} g (\text{since } \zeta_G^{w_2} \text{ is a character})
\]
\[
= \sum_{h \in G} \zeta_G^{w_2}(h g) \zeta_G^{w_1}(h^{-1}) \text{ (use the equation: } w_1 = h^{-1}).
\]
Since \( \zeta_G^{w_1} \) and \( \zeta_G^{w_2} \) are characters, write \( \zeta_G^{w_1} = \sum_{i=1}^{r} n_i \chi_i ; \zeta_G^{w_2} = \sum_{i=1}^{r} m_i \chi_i \), where \( m_i, n_i \) are non-negative integers for \( 1 \leq i \leq r \) and \( \chi_1, \chi_2, \ldots, \chi_r \) are the distinct
irreducible characters of $G$. Then,

$$
\zeta^w_G(g) = \sum_{h \in G} \sum_{1 \leq i, j \leq r} m_i n_j \chi_i(hg) \chi_j(h^{-1}) = \sum_{1 \leq i, j \leq r} m_i n_j \sum_{h \in G} \chi_i(hg) \chi_j(h^{-1}) = \sum_{1 \leq i, j \leq r} m_i n_j \frac{|G|}{\chi_i(1)} \delta_{i,j} \chi_i(g) \quad \text{(use the Theorem 2.2)}
$$

Since the coefficients of $\chi_i$ are non-negative integers, $\zeta^w_G$ is a character of $G$. □

Now we are ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** By Lemma 3.1, it suffices to show that $\zeta^w_G$ is a character of $G$ if $w$ is an admissible word. The proof is by induction on $n$. The case $n = 1$ is trivial. The case $n = 2$ is essentially the case $w = \omega_2$. Now assume that $\zeta^w_G$ is a character for any admissible word $w$ in $t_1, t_2, \ldots, t_k$ for $k \leq (n - 1)$. Suppose $w$ is a word in $t_1, t_2, \ldots, t_n$, without loss of generality, we may assume

$$w(t_1, t_2, \ldots, t_n) = t_1^{-1}w_1t_1w_2$$

where $w_1, w_2$ are words in the letters $t_2, t_3, \ldots, t_n$. Observe that if $w_1(t_2, t_3, \ldots, t_n)$ is an admissible word in $t_{i_1}, t_{i_2}, \ldots, t_{i_k}$ for some $1 \leq k \leq (n - 1)$, then $t_{i_1}, t_{i_2}, \ldots, t_{i_k}$ do not occur in $w_2(t_2, t_3, \ldots, t_n)$ and $w_2(t_2, t_3, \ldots, t_n)$ is an admissible word in the rest of the letters. Therefore, in this case, use the induction hypothesis and Theorem 1.1 to conclude that $\zeta^w_G$ is a character of $G$. Thus, we assume that $w_1(t_2, t_3, \ldots, t_n)$ and hence, $w_2(t_2, t_3, \ldots, t_n)$ is not admissible. Again, without loss of generality, assume that $t_2^{-1}$ occurs in $w_1(t_2, t_3, \ldots, t_n)$ and $t_2$ does not. Then,

$$w_1(t_2, t_3, \ldots, t_n) = w_3t_2^{-1}w_4,$$

$$w_2(t_2, t_3, \ldots, t_n) = w_5t_2w_6,$$

where $w_3, w_4, w_5, w_6$ are words in $t_3, \ldots, t_n$ such that each of $t_3^{\pm 1}, \ldots, t_n^{\pm 1}$ occurs at most once. Apply the automorphism $\sigma$ to $F(\{t_1, t_2, \ldots, t_n\})$ given by

$$\sigma(t_i) = \begin{cases} t_i & \text{if } i \in \{1, 2, \ldots, n\} - \{2\}, \\ w_4(t_3, t_4, \ldots, t_n)t_2w_3(t_3, t_4, \ldots, t_n) & \text{if } i = 2. \end{cases}$$

Then

$$\sigma(w)(t_1, t_2, \ldots, t_n) = t_1^{-1}t_2^{-1}t_1w_3w_4t_2w_3w_6.$$

Since $\zeta^w_G = \zeta^w_G$, we may assume without loss of generality that

$$w(t_1, t_2, \ldots, t_n) = t_1^{-1}t_2^{-1}t_1w_7t_2w_8$$

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where $w_7, w_8$ are words in letters $t_3, \ldots, t_n$. Observe that $t_3, \ldots, t_n$ and their inverses occur at most once in $w_7$ and $w_8$, and that $w_7 w_8$ is admissible. Thus we have following three cases. Case 1: $w_7 = 1$, case 2: $w_8 = 1$ and case 3: neither $w_7$ nor $w_8$ is the empty word. In case 3, if $w_7$ admissible then so is $w_8$ and therefore, the result follows from Theorem 1.1. Otherwise, we will split $w_7$ and $w_8$, in the same way that we have split $w_1, w_2$ in (3.1) and (3.2) and continue the process. Thus, we have the following three possibilities:

(a) $w = \prod_{i=1}^{k-1} [t_i, t_{i+1}] w_9(t_{k+1}, t_{k+2}, \ldots, t_n)$,

(b) $w = \prod_{i=1}^{k-1} [t_i, t_{i+1}] t_k^{-1} w_{10}(t_{k+1}, t_{k+2}, \ldots, t_n) t_k$,

(c) $w = \prod_{i=1}^{n-1} [t_i, t_{i+1}]$,

where $3 \leq k \leq n$, and $w_9, w_{10}$ are admissible words in $t_{k+1}, t_{k+2}, \ldots, t_n$. Use Theorem 1.1 and the induction hypothesis for (a), Lemma 3.2, the induction hypothesis and Corollary 2.7 for (b), and Corollary 2.7 for (c) to complete the proof of the theorem.

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