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ON GROUP EQUATIONS

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Abstract. Suppose f is a map from a non-empty finite set X to a finite group G. Define the map $\zeta_G^f : G \longrightarrow \mathbb{N} \cup \{0\}$ by $g \mapsto |f^{-1}(g)|$. In this article, we show that for a suitable choice of f, the map ζ_G^f is a character. We use our results to show that the solution function for the word equation $w(t_1, t_2, \ldots, t_n) = g$ $(g \in G)$ is a character, where $w(t_1, t_2, \ldots, t_n)$ denotes the product of $t_1, t_2, \ldots, t_n, t_1^{-1}, t_2^{-1}, \ldots, t_n^{-1}$ in a randomly chosen order.

Keywords: Finite groups, word equations, group characters.
MSC(2010): Primary: 20C15.

1. Introduction

Let G be a finite group and let X be a non-empty finite set. Suppose that $f : X \longrightarrow G$ is a map. Define the map $\zeta_G^f : G \longrightarrow \mathbb{N} \cup \{0\}$ given by $g \mapsto |f^{-1}(g)|$. We call the map ζ_G^f, the solution function for the equation $f(x) = g$, with $g \in G$. In general, ζ_G^f need not be a characters of G. Suppose $w = w(t_1, t_2, \ldots, t_n)$ is a word in n symbols $t_1, t_2, \ldots, t_n$. Then $w(t_1, t_2, \ldots, t_n)$ defines the map $w : G^n \longrightarrow G$ given by $(g_1, g_2, \ldots, g_n) \mapsto w(g_1, g_2, \ldots, g_n)$. For any word w, the map ζ_G^w is a class function on G but not necessarily a character.

In [1], it is shown that if $w(x) = x^n$, then ζ_G^w is a generalized character of G, i.e., $\mathbb{Z}$-linear combination of irreducible character of G. In [6], Tambour discussed the solution function for two types of words

(1) $t_1 t_2 \cdots t_n t_1^{-1} t_2^{-1} \cdots t_n^{-1}$,
(2) $\prod_{i=1}^{n} [t_i, s_i]$,

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and used his results to write a new proof of Itô's theorem: the degree of any irreducible character of G divides the index of any abelian normal subgroup of G. A word $w(t_1, t_2, \ldots, t_n)$ obtained from the word $u(t_1, t_2, \ldots, t_n) = t_1 t_2 \cdots t_n t_1^{-1} t_2^{-1} \cdots t_n^{-1}$, by shuffling $t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}$, is referred to as an *admissible* word (cf. [2]). It is shown in [2] that if w is an admissible word, then ζ_G^w is a character of G. This is a generalization of a result of Frobenious ([3]).

In [5], Strunkov proved the following:

Theorem 1.1. [5, Theorem 2] *Let X_1, X_2 be two finite sets. Suppose that $f_i : X_i \longrightarrow G$ $(i = 1, 2)$ are maps. If $\zeta_G^{f_1}$ and $\zeta_G^{f_2}$ are characters (resp. generalized characters), then so is ζ_G^f where $f : X_1 \times X_2 \longrightarrow G$ is given by $f(x_1, x_2) = f_1(x_1) f_2(x_2)$.*

In Section 2, we generalize this result in various ways. Our main results are:

Theorem 1.2. *Let X_1, X_2 be two finite sets. Suppose H is a normal subgroup of G and $f_1 : X_1 \to H$, $f_2 : X_2 \to G$ are maps. Assume that $\zeta_G^{f_2}$ is a constant map and $\zeta_H^{f_1}$ is constant on G-conjugacy classes of H. Define the map $\phi : X_1 \times X_2 \longrightarrow G$ by $\phi(x, y) = f_1(x)^{-1} f_2(y)^{-1} f_1(x) f_2(y)$. If $\zeta_H^{f_1}$ is a generalized character (respectively, character), then so is ζ_G^{ϕ}.*

Theorem 1.3. *For $i = 1, \ldots, n$, let $f_i : X_i \to G$ be a map, where X_i is a finite set. Define for $n \geq 2$, $u_n : \prod_{i=1}^{n} X_i \to G$ by $u_n(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n-1} [f_i(x_i), f_{i+1}(x_{i+1})]$ and $u_1 : X_1 \to G$ by $u_1(x) = 1$. If $\zeta_G^{f_1}$ is a character and $\zeta_G^{f_i}$ are constant maps for each $i = 2, \ldots, n$, then $\zeta_G^{u_n}$ is a character.*

In particular, if $X_i = G$ and $f_i(x) = x$ for $i = 1, 2, \ldots, n$, then u_n reduces to an admissible word, namely, $\omega_n(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n-1} [x_i, x_{i+1}]$ and $\zeta_G^{w_n}$ is a character of G. In Section 3, the main result in [2] is deduced from our result by reducing the problem on admissible words to a problem essentially of the words ω_n for $n \in \mathbb{N}$. Indeed, we extend the result of Das and Nath ([2]) and prove the following theorem.

Theorem 1.4. *Suppose w is a word in $t_1, t_2, \ldots,$ t_n such that each of $t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}$ occurs in w at most once. Then ζ_G^w is a character of G.*

Notation. Throughout the article, G denotes a finite group and $\mathrm{Irr}(G)$, the set of all irreducible characters of G. If $a, b \in G$, then ${}^b a = b^{-1} a b$, $[a, b] = a^{-1} b^{-1} a b$. For class functions χ and ψ on G, the expression $\langle \chi, \psi \rangle_G$ denotes the standard inner product on the space of class functions on G, that is,

$$\langle \chi, \psi \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}.$$

For simplicity we prefer to denote the inner product simply by $\langle \chi, \psi \rangle$ if there is no ambiguity. Furthermore, if H is a subgroup of G and χ a character of G, $\chi\downarrow_H^G$ denotes the restriction of χ to H.

2. **Equations in finite groups**

We start by recalling the well know result, which are used frequently.

Lemma 2.1. [4, Lemma 2.25] *Let ρ be an irreducible $\mathbb{C}$-representation of G of degree n. Suppose A is an $n \times n$ matrix over $\mathbb{C}$ which commutes with $\rho(g)$ for all $g \in G$. Then $A = \lambda I_n$ for some $\lambda \in \mathbb{C}$.*

Theorem 2.2. [4, Theorem 2.13] *The following holds for every $h \in G$.*

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(gh)\chi_j(g^{-1}) = \delta_{ij} \frac{\chi_i(h)}{\chi_i(1)},$$

where $\chi_i, \chi_j \in \mathrm{Irr}(G)$.

Proposition 2.3. *Let H be a subgroup of G and let X_1, X_2 be two finite sets. Suppose $f_1 : X_1 \to H$ and $f_2 : X_2 \to G$ are maps. Define the map $\psi : X_1 \times X_2 \longrightarrow G$ by $\psi(x, y) = f_2(y)^{-1} f_1(x) f_2(y)$ and assume that $\zeta_G^{f_2}$ is a constant map. If $\zeta_H^{f_1}$ is a generalized character (resp. character), then so is ζ_G^{ψ}. In particular, if $H = G$, then $\zeta_G^{\psi} = |X_2|\ \zeta_G^{f_1}$.*

Proof. Since H is a subgroup of G, for any $\chi \in \mathrm{Irr}(G)$, we have

$$\chi\downarrow_H^G = \sum_{\phi_i \in \mathrm{Irr}(H)} n_i \phi_i,$$

for some $n_i \in \mathbb{N} \cup \{0\}$. Suppose χ is afforded by the irreducible representation ρ of G. Consider an element

$$z = \sum_{x \in X_1, y \in X_2} f_2(y)^{-1} f_1(x) f_2(y). \tag{2.1}$$

then, $\rho(z) = \sum_{x \in X_1, y \in X_2} \rho(f_2(y)^{-1} f_1(x) f_2(y))$. Now take the trace on both sides,

$$\begin{aligned} \chi(z) &= |X_2| \sum_{x \in X_1} \chi\downarrow_H^G (f_1(x)) \\ &= |H||X_2| \sum_{\phi_i \in \mathrm{Irr}(H)} n_i \langle \zeta_H^{f_1}, \overline{\phi_i} \rangle. \end{aligned} \tag{2.2}$$

By definition of ζ_G^{ψ}, we have

$$\chi(z) = |G|\ \langle \zeta_G^{\psi}, \overline{\chi} \rangle. \tag{2.3}$$

Therefore, from (2.2) and (2.3), we get

$$\langle \zeta_G^{\psi}, \chi \rangle = \frac{|H||X_2|}{|G|} \sum_{\phi_i \in \mathrm{Irr}(H)} n_i \langle \zeta_H^{f_1}, \overline{\phi_i} \rangle.$$

Hence, the result follows from the hypothesis.

Let $H = G$. Then (2.2) reduces to

$$\chi(z) \quad = \quad |X_2||G| \, \langle \zeta_G^{f_1}, \overline{\chi} \rangle. \tag{2.4}$$

So, by (2.3) and (2.4), $\zeta_G^{\psi} = |X_2| \, \zeta_G^{f_1}$. □

With the help of the above proposition, we have the following corollary.

Corollary 2.4. *Let H be a subgroup of a group G. Suppose $w : H \times G \longrightarrow G$ is defined by $w(x_1, x_2) = x_2^{-1} x_1 x_2$. Then ζ_G^w is a character of G.*

Proof of Theorem 1.2. Let $\chi_1, \chi_2, ..., \chi_r$ be the irreducible characters of G. Suppose χ_i is afforded by the irreducible representation ρ_i of G. Consider an element $z = \sum_{\substack{x \in X_1 \\ y \in X_2}} f_1(x)^{-1} f_2(y)^{-1} f_1(x) f_2(y)$.

Then

$$\rho_i(z) \quad = \quad \sum_{x \in X_1} \rho_i(f_1(x)^{-1}) A_i(x) \tag{2.5}$$

where,

$$A_i(x) = \sum_{y \in X_2} \rho_i(f_2(y)^{-1} f_1(x) f_2(y)). \tag{2.6}$$

As $\zeta_G^{f_2}$ is a constant function, $A_i(x)$ commutes with $\rho_i(g)$, $\forall \, g \in G$. By Lemma 2.1,

$$A_i(x) = \lambda_x I \tag{2.7}$$

for some $\lambda_x \in \mathbb{C}$. Now take the trace on the both sides of (2.7) and use (2.6) to obtain

$$\lambda_x \quad = \quad \frac{|X_2|}{\chi_i(1)} \chi_i(f_1(x)). \tag{2.8}$$

Again take the trace on both side of (2.5) and use (2.7),(2.8) to get

$$\begin{aligned} \chi_i(z) \quad &= \quad \frac{|X_2|}{\chi_i(1)} \sum_{h \in H} \zeta_H^{f_1}(h) \chi_i(h) \chi_i(h^{-1}) \\ &= \quad \frac{|X_2|}{\chi_i(1)} |H| \langle \zeta_H^{f_1} \overline{\chi_i}, \overline{\chi_i} \rangle_H. \end{aligned} \tag{2.9}$$

By definition of ζ_G^{ϕ}, we have

$$\chi_i(z) \quad = \quad |G| \langle \zeta_G^{\phi}, \overline{\chi_i} \rangle. \tag{2.10}$$

Therefore, by (2.9) and (2.10), we get

$$\zeta_G^{\phi} = \frac{|X_2|}{|G|} \sum_{i=1}^{r} \frac{|H|}{\chi_i(1)} \langle \zeta_H^{f_1} \overline{\chi_i}, \overline{\chi_i} \rangle_H \ \chi_i. \tag{2.11}$$

As $\zeta_G^{f_2}$ is a constant map, $|G|$ divides $|X_2|$. Hence, to make the assertion it is enough to prove that $\frac{|H|}{\chi_i(1)} \langle \zeta_H^{f_1} \overline{\chi_i}, \overline{\chi_i} \rangle_H$ is an integer. Since H is normal in G, $H = \bigcup\limits_{1 \leq j \leq k} Cl_G(x_j)$ for some $x_j \in H$. For any irreducible character χ_i of G, we have

$$\begin{aligned} |H| \ \langle \zeta_H^{f_1} \overline{\chi_i}, \overline{\chi_i} \rangle_H &= \sum_{h \in H} \zeta_H^{f_1}(h) \chi_i(h) \chi_i(h^{-1}) \\ &= \sum_{1 \leq j \leq k} |Cl_G(x_j)| \ \zeta_H^{f_1}(x_j) \ \chi_i(x_j) \ \overline{\chi_i(x_j)} \end{aligned}$$

Suppose $K_j = \sum\limits_{x \in Cl_G(x_j)} x$, for $1 \leq j \leq k$. Then by [4, Theorem 3.7], $w_{\chi_i}(K_j) = \frac{\chi_i(x_j)|Cl_G(x_j)|}{\chi_i(1)}$ is an algebraic integer for any irreducible character. Hence, we have

$$|H| \ \langle \zeta_H^{f_1} \overline{\chi_i}, \overline{\chi_i} \rangle_H = \sum_{1 \leq j \leq k} \zeta_H^{f_1}(x_j) \chi_i(1) w_{\chi_i}(K_j) \chi_i(x_j^{-1}).$$

Therefore,

$$\frac{|\ H\ |}{\chi_i(1)} \langle \zeta_H^{f_1} \overline{\chi_i}, \overline{\chi_i} \rangle_H = \sum_{1 \leq j \leq k} \zeta_H^{f_1}(x_j) w_{\chi_i}(K_j) \chi_i(x_j^{-1}).$$

The right-hand side of the latter equality is an algebraic integer. Therefore the result follows by the hypothesis. □

In Theorem 1.2, if we take $H = G$, then we get the following result.

Corollary 2.5. *Let X_1, X_2 be two finite sets. Suppose $f_i : X_i \to G$ is a map for $i = 1, 2$. Set $\phi(x, y) := f_1(x)^{-1} f_2(y)^{-1} f_1(x) f_2(y)$. If $\zeta_G^{f_2}$ is a constant map and $\zeta_G^{f_1}$ is a generalized character (resp. character), then ζ_G^{ϕ} is a generalized character (resp. character).*

Proof. From Theorem 1.2, we have

$$\zeta_G^{\phi} = \sum_{i=1}^{r} \frac{|\ X_2\ |}{\chi_i(1)} \langle \zeta_G^{f_1} \overline{\chi_i}, \overline{\chi_i} \rangle \chi_i.$$

As $\zeta_G^{f_2}$ is a constant map, $\chi_i(1)$ divides $|X_2|$. Hence, the assertion follows. □

Corollary 2.6. *Let H be a normal subgroup of a group G. Suppose $w : H \times G \longrightarrow G$ is defined by $w(x, y) = x^{-1} y^{-1} x y$. Then ζ_G^{w} is a character of G.*

Proof. In view of Theorem 1.2, ζ_G^w is a character. □

In Theorem 1.3, we discuss the solution function ζ_G^f, where f is a product of commutators of functions.

Proof of Theorem 1.3. Suppose for $n \geq 1$, the map $v_n : \prod_{i=1}^{n} X_i \longrightarrow G$ is defined by

$$v_n := f_n(x_n)u_n(x_1, x_2, \ldots, x_n)f_n(x_n)^{-1}. \tag{2.12}$$

By induction, we show that for any finite group G, $\zeta_G^{u_n}$ and $\zeta_G^{v_n}$ are characters of G. For $n = 2$, in view of Corollary 2.5, $\zeta_G^{u_2}$ and $\zeta_G^{v_2}$ are characters. Let $n \geq 3$. Now we assume that $\zeta_G^{u_i}$ and $\zeta_G^{v_i}$ are characters of G for $3 \leq i \leq (n-1)$. Suppose $\rho_1, \rho_2, \ldots, \rho_r$ are irreducible representations of G. Since $\zeta_G^{f_i}$ is a constant map for $i \geq 2$, $\zeta_G^{f_i}(g) = c_i$ for some $c_i \in \mathbb{N}$. Now consider an element

$$\omega = \sum_{\substack{x_i \in X_i \\ i=1,\ldots,n}} u_n(x_1, x_2, \ldots, x_n).$$

Then,

$$\rho_i(\omega) = \sum_{\substack{x_i \in X_i \\ i=1,\ldots,n-1}} \rho_i(B_{n-1}) \cdot A(x_{n-1}), \tag{2.13}$$

where

$$A(x_{n-1}) = \sum_{x_n \in X_n} \rho_i(f_n(x_n)^{-1} f_{n-1}(x_{n-1}) f_n(x_n)) \tag{2.14}$$

and $B_{n-1} = u_{n-1}(x_1, x_2, \ldots, x_{n-1})f_{n-1}(x_{n-1})^{-1}$. Since ζ^{f_n} is a constant function, $A(x_{n-1})$ commutes with $\rho_i(g)$ $\forall g \in G$. Therefore, by Lemma 2.1,

$$A(x_{n-1}) = \lambda(x_{n-1})I \tag{2.15}$$

for a scalar $\lambda(x_{n-1}) \in \mathbb{C}$. Take the trace on both sides of (2.14) and use (2.15) to get

$$\lambda(x_{n-1}) = \frac{|X_n|}{\chi_i(1)} \cdot \chi_i(f_{n-1}(x_{n-1})). \tag{2.16}$$

Now take the trace on both sides of (2.13) and use (2.15) and (2.16) to get

$$\chi_i(\omega) \quad = \quad \frac{|X_n|}{\chi_i(1)} \sum_{\substack{x_i \in X_i \\ i=1,\ldots,n-1}} \chi_i(B_{n-1}) \cdot \chi_i(f_{n-1}(x_{n-1})).$$

Set $C_{n-1} = f_{n-1}(x_{n-1})^{-1} v_{n-2}(x_1, x_2, \ldots, x_{n-2})$. It is easy to see that B_{n-1} and C_{n-1} are conjugates, for any $f_1(x_1), f_2(x_2), \ldots, f_{n-1}(x_{n-1}) \in G$. Therefore, we have

$$\begin{aligned}
\chi_i(\omega) &= \frac{|X_n|}{\chi_i(1)} \sum_{\substack{x_i \in X_i \\ i=1,\ldots,n-1}} \chi_i(C_{n-1}) \cdot \chi_i(f_{n-1}(x_{n-1})) \\
&= \frac{|X_n|}{\chi_i(1)} \sum_{h \in G} \zeta_G^{v_{n-2}}(h) \sum_{g \in G} \zeta_G^{f_{n-1}}(g^{-1}) \chi_i(g^{-1}h) \chi_i(g) \text{ (set } v_{n-2} = h) \\
&= \frac{|X_n||G|c_{n-1}}{\chi_i(1)^2} \sum_{h \in G} \zeta_G^{v_{n-2}}(h) \cdot \chi_i(h) \quad \text{(use Theorem 2.2)} \\
&= \frac{|X_n||G|^2 c_{n-1}}{\chi_i(1)^2} \langle \zeta_G^{v_{n-2}}, \bar{\chi}_i \rangle \text{ (since } \zeta_G^{v_{n-2}} \text{ is a character of } G) \qquad (2.17)
\end{aligned}$$

Now use the definition of $\zeta_G^{u_n}$ to write

$$\chi_i(\omega) = |G| \langle \zeta_G^{u_n}, \bar{\chi}_i \rangle. \qquad (2.18)$$

From (2.17) and (2.18), we have

$$\zeta_G^{u_n} = \sum_{i=1}^{r} \frac{|X_n||G|c_{n-1}}{\chi_i(1)^2} \langle \zeta_G^{v_{n-2}}, \bar{\chi}_i \rangle \chi_i$$

so that $\zeta_G^{u_n}$ is a character of G. Next to show that $\zeta_G^{v_n}$ is a character. In $\mathbb{Z}[G]$, set

$$z := \sum_{x_i \in X_i, i=1,\ldots,n} f_n(x_n) u_n(x_1, x_2, \ldots, x_n) f_n(x_n)^{-1}.$$

Then,

$$\begin{aligned}
\chi_i(z) &= \sum_{x_i \in X_i, i=1,\ldots,n} \chi_i(u_n(x_1, x_2, \ldots, x_n)) \\
&= \sum_{h \in G} \zeta_G^{u_n}(h) \chi_i(h) \\
&= |G| \ \langle \zeta_G^{u_n}, \bar{\chi}_i \rangle. \qquad (2.19)
\end{aligned}$$

On the other hand, use the definition of $\zeta_G^{v_n}$ to write

$$\chi_i(z) = |G| \langle \zeta_G^{v_n}, \bar{\chi}_i \rangle. \qquad (2.20)$$

Finally, by (2.19) and (2.20), $\zeta_G^{v_n} = \zeta_G^{u_n}$. Hence $\zeta_G^{v_n}$ is a character. This completes the proof. $\square$

In particular, if $X_i = G$ and $f_i(x) = x$ for $i = 1, 2, \ldots, n$, then we have the following corollary.

Corollary 2.7. *Suppose $w(x_1, \ldots, x_n) := \prod_{i=1}^{n-1} [x_i, x_{i+1}]$, where $x_1, \ldots, x_n$ are n distinct letters. Then ζ_G^w is a character of G.*

Corollary 2.8. *Let* $w : \prod_{i=1}^{2n} G \longrightarrow G$ *be the map defined by*

$$w(x_1, \ldots, x_n, y_1, \ldots, y_n) := [\ [\ldots [\ [^{y_1} x_1, ^{y_2} x_2], ^{y_3} x_3], \ldots]^{y_n} x_n] .$$

Then ζ_G^w *is a character of* G.

Observe that if $(|G|, k) = 1$, then the map $g \mapsto g^k$ is a bijection on G. Hence, in view of Corollary 2.7, we have the following.

Corollary 2.9. *Let* $w(x_1, \ldots, x_n) := [x_1^{k_1}, x_2^{k_2}][x_3^{k_3}, x_4^{k_4}] \ldots [x_{n-1}^{k_{n-1}}, x_n^{k_n}]$, *where* $k_i \in \mathbb{N}$ *with* $(|G|, k_i) = 1$ *for each* $i = 1, \ldots, n$. *Then* ζ_G^w *is a character of* G.

3. Application

In this section we prove Theorem 1.4. In the Theorem 1.4, we extend the main result of Das and Nath ([2]) a little bit. We begin to prove the following lemmas.

Lemma 3.1. *Let* w_1 *and* w_2 *be words in* $t_1, t_2, \ldots, t_n$. *If* $w := w_1\, t_{n+1}\, w_2$, *then* ζ_G^w *is a constant function on* G.

Proof. Observe that $(a_1, a_2, \ldots, a_n, a_{n+1}) \in G^{n+1}$ is a solution of $w = g$ if and only if

$$a_{n+1} = w_1(a_1, a_2, \ldots, a_n)^{-1}\, g\, w_2(a_1, a_2, \ldots, a_n)^{-1}.$$

Therefore, $\zeta^w(g) = |G|^n$. $\square$

Lemma 3.2. *Suppose* w_1 *is a word in* $t_1, \ldots, t_k$ *and* w_2, *in* $t_{k+1}, \ldots, t_n$ *such that both* $\zeta_G^{w_1}$ *and* $\zeta_G^{w_2}$ *are characters of* G. *If* $w = w_1\, t_k^{-1}\, w_2\, t_k$, *then* ζ_G^w *is a character of* G.

Proof. Observe that $w(x_1, \ldots, x_n) = g$ if and only if $w_2(x_{k+1}, \ldots, x_n) = x_k w_1(x_1, \ldots, x_k)^{-1} g x_k^{-1}$. Therefore,

$$\begin{aligned} \zeta_G^w(g) &= \sum_{x_1, \ldots, x_k \in G} \zeta_G^{w_2}(x_k w_1(x_1, \ldots, x_k)^{-1} g x_k^{-1}) \\ &= \sum_{x_1, \ldots, x_k \in G} \zeta_G^{w_2}(w_1(x_1, \ldots, x_k)^{-1} g) \text{ (since } \zeta_G^{w_2} \text{ is a character)} \\ &= \sum_{h \in G} \zeta_G^{w_2}(hg) \zeta_G^{w_1}(h^{-1}) \text{ (use the equation: } w_1 = h^{-1}). \end{aligned}$$

Since $\zeta_G^{w_1}$ and $\zeta_G^{w_2}$ are characters, write $\zeta_G^{w_1} = \sum_{i=1}^{r} n_i \chi_i$, $\zeta_G^{w_2} = \sum_{i=1}^{r} m_i \chi_i$, where m_i, n_i are non-negative integers for $1 \leq i \leq r$ and $\chi_1, \chi_2, \ldots, \chi_r$ are the distinct irreducible characters of G. Then,

$$\begin{aligned}\zeta_G^w(g) &= \sum_{h\in G}\sum_{1\leq i,j\leq r} m_i n_j \chi_i(hg)\chi_j(h^{-1}) \\ &= \sum_{1\leq i,j\leq r} m_i n_j \sum_{h\in G}\chi_i(hg)\chi_j(h^{-1}) \\ &= \sum_{1\leq i,j\leq r} m_i n_j \frac{|G|}{\chi_i(1)}\ \delta_{i,j}\ \chi_i(g) \text{ (use the Theorem 2.2)} \\ &= \sum_{1\leq i\leq r} m_i n_i \frac{|G|}{\chi_i(1)}\chi_i(g).\end{aligned}$$

Since the coefficients of χ_i are non-negative integers, ζ_G^w is a character of G. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. By Lemma 3.1, it suffices to show that ζ_G^w is a character of G if w is an admissible word. The proof is by induction on n. The case $n=1$ is trivial. The case $n=2$ is essentially the case $w=\omega_2$. Now assume that ζ_G^w is a character for any admissible word w in $t_1,t_2,\ldots,t_k$ for $k\leq(n-1)$. Suppose w is a word in $t_1,t_2,\ldots,t_n$, without loss of generality, we may assume

$$w(t_1,t_2,\ldots,t_n)=t_1^{-1}w_1t_1w_2$$

where w_1,w_2 are words in the letters $t_2,t_3,\ldots,t_n$. Observe that if $w_1(t_2,t_3,\ldots,t_n)$ is an admissible word in $t_{i_1},t_{i_2},\ldots,t_{i_k}$ for some $1\leq k\leq(n-1)$, then $t_{i_1},t_{i_2},\ldots,t_{i_k}$ do not occur in $w_2(t_2,t_3,\ldots,t_n)$ and $w_2(t_2,t_3,\ldots,t_n)$ is an admissible word in the rest of the letters. Therefore, in this case, use the induction hypothesis and Theorem 1.1 to conclude that ζ_G^w is a character of G. Thus, we assume that $w_1(t_2,t_3,\ldots,t_n)$ and hence, $w_2(t_2,t_3,\ldots,t_n)$ is not admissible. Again, without loss of generality, assume that t_2^{-1} occurs in $w_1(t_2,t_3,\ldots,t_n)$ and t_2 does not. Then,

$$w_1(t_2,t_3,\ldots,t_n) = w_3t_2^{-1}w_4, \tag{3.1}$$

$$w_2(t_2,t_3,\ldots,t_n) = w_5t_2w_6, \tag{3.2}$$

where w_3,w_4,w_5,w_6 are words in $t_3,\ldots,t_n$ such that each of $t_3^{\pm1},\ldots,t_n^{\pm1}$ occurs at most once. Apply the automorphism σ to $F(\{t_1,t_2,\ldots,t_n\})$ given by

$$\sigma(t_i)=\begin{cases} t_i & \text{if } i\in\{1,2,\ldots,n\}-\{2\}, \\ w_4(t_3,t_4,\ldots,t_n)t_2w_3(t_3,t_4,\ldots,t_n) & \text{if } i=2.\end{cases}$$

Then

$$\sigma(w)(t_1,t_2,\ldots,t_n)=t_1^{-1}t_2^{-1}t_1w_5w_4t_2w_3w_6.$$

Since $\zeta_G^w=\zeta_G^{\sigma(w)}$, we may assume without loss of generality that

$$w(t_1,t_2,\ldots,t_n)=t_1^{-1}t_2^{-1}t_1w_7t_2w_8$$

where w_7, w_8 are words in letters $t_3, \ldots, t_n$. Observe that $t_3, \ldots, t_n$ and their inverses occur at most once in w_7 and w_8, and that w_7w_8 is admissible. Thus we have following three cases. Case 1: $w_7 = 1$, case 2: $w_8 = 1$ and case 3: neither w_7 nor w_8 is the empty word. In case 3, if w_7 admissible then so is w_8 and therefore, the result follows from Theorem 1.1. Otherwise, we will split w_7 and w_8, in the same way that we have split w_1, w_2 in (3.1) and (3.2) and continue the process. Thus, we have the following three possibilities:

(a) $w = \prod_{i=1}^{k-1} [t_i, t_{i+1}]\, w_9(t_{k+1}, t_{k+2}, \ldots, t_n)$,

(b) $w = \prod_{i=1}^{k-1} [t_i, t_{i+1}]\, t_k^{-1}\, w_{10}(t_{k+1}, t_{k+2}, \ldots, t_n)\, t_k$,

(c) $w = \prod_{i=1}^{n-1} [t_i, t_{i+1}]$,

where $3 \leq k \leq n$, and w_9, w_{10} are admissible words in $t_{k+1}, t_{k+2}, \ldots, t_n$. Use Theorem 1.1 and the induction hypothesis for (a), Lemma 3.2, the induction hypothesis and Corollary 2.7 for (b), and Corollary 2.7 for (c) to complete the proof of the theorem. □

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