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ON CONVOLUTION PROPERTIES FOR SOME CLASSES OF MEROMORPHIC FUNCTIONS ASSOCIATED WITH LINEAR OPERATOR

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Abstract. In this paper, we defined two classes $S_p^*(n,\lambda,A,B)$ and $K_p(n,\lambda,A,B)$ of meromorphic $p-$valent functions associated with a new linear operator. We obtained convolution properties for functions in these classes.

Keywords: Meromorphic functions, subordination, Hadamard product, linear operator.
MSC(2010): Primary: 30C45.

1. Introduction

In this paper, we obtain conditions on meromorphic functions defined by (1.1) to be in the classes $S_p^*(n,\lambda,A,B)$ and $K_p(n,\lambda,A,B)$ which defined by (1.12) and (1.13), using convolution properties. Also we obtain conditions required for inclusion of the classes $S_p^*(n,\lambda,A,B)$ and $S_p^*(n+1,\lambda,A,B)$ and the classes $K_p(n,\lambda,A,B)$ and $K_p(n+1,\lambda,A,B)$.
Let $\sum_p$ be the class of functions of the form:

$$f(z)=z^{-p}+\sum_{k=1}^{\infty}a_k z^{k-p} \qquad (p\in\mathbb{N}=\{1,2,...\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disk $U^*=U\backslash\{0\}$, where $U=\{z:z\in\mathbb{C}\ ,\ |z|<1\}$. If f and g are analytic functions in U, we say that f is subordinate to g, denoted by $f\prec g$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0)=0$ and $|w(z)|<1$ for all $z\in U$, such that $f(z)=g(w(z)),\ z\in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence (see [13]):

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$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions f given by (1.1) and $g \in \sum_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p}, \tag{1.2}$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g * f)(z).$$

Aouf et al. [4] defined the linear operator $D^n_{\lambda,p}(f * g)(z) : \Sigma_p \longrightarrow \Sigma_p$ $(f, g \in \Sigma_p,\ \lambda \geq 0,\ p \in \mathbb{N},\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ as follows:

$$D^0_{\lambda,p}(f * g)(z) = (f * g)(z),$$

$$D^1_{\lambda,p}(f * g)(z) = D_{\lambda,p}(f * g)(z) = (1 - \lambda)(f * g)(z) + \lambda z^{-p}\ (z^{p+1}(f * g)(z))',$$

$$= z^{-p} + \sum_{k=1}^{\infty}(1 + \lambda k) a_k b_k z^{k-p}\ (\lambda \geq 0;\ p \in \mathbb{N}),$$

$$D^2_{\lambda,p}(f * g)(z) = D_{\lambda,p}(D_{\lambda,p}(f * g))(z),$$

$$= (1 - \lambda) D_{\lambda,p}(f * g)(z) + \lambda z^{-p}\ (z^{p+1} D_{\lambda,p}(f * g)(z))'$$

$$= z^{-p} + \sum_{k=1}^{\infty}(1 + \lambda k)^2 a_k b_k z^{k-p}\ (\lambda \geq 0;\ p \in \mathbb{N})$$

and (in general)

$$D^n_{\lambda,p}(f * g)(z) = D_{\lambda,p}(D^{n-1}_{\lambda,p}(f * g)(z))$$

$$= z^{-p} + \sum_{k=1}^{\infty}(1 + \lambda k)^n a_k b_k z^{k-p}\ (\lambda \geq 0;\ p \in \mathbb{N};\ n \in \mathbb{N}_0). \tag{1.3}$$

From (1.3) it is easy to verify that:

$$z(D^n_{\lambda,p}(f * g)(z))' = \frac{1}{\lambda} D^{n+1}_{\lambda,p}(f * g)(z) - (p + \frac{1}{\lambda}) D^n_{\lambda,p}(f * g)(z)\ (\lambda > 0). \tag{1.4}$$

Specializing the parameters n, λ and the function g, we obtain some operators different from those studied earlier.

(i) For $n = 0$ and $b_k = \Gamma_k(\alpha_1)$,

$$\Gamma_k(\alpha_1) = \frac{(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1)_k ... (\beta_s)_k (1)_k}\quad (\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s \in \mathbb{C}, \tag{1.5}$$

$$\beta_j \notin Z_0^- = \{0, -1, -2, ...\}; j = 1, 2, ..., s, q \leq s + 1; s, q \in \mathbb{N}_0),$$

where

$$(d)_k = \begin{cases} 1 & (k = 0; d \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}) \\ d(d+1)...(d+k-1) & (k \in \mathbb{N}; d \in \mathbb{C}). \end{cases}$$

we have

$$D^0_{\lambda,p}(f * g)(z) = H_{p,q,s}(\alpha_1)f(z) = z^{-p} + \sum_{k=1}^{\infty} \Gamma_k(\alpha_1)a_k z^{k-p}, \tag{1.6}$$

where the operator $H_{p,q,s}(\alpha_1)$ was investigated and studied by Liu and Srivastava [11] (see also [1, 10] and [12]). The operator $H_{p,q,s}(\alpha_1)$ has as special cases interesting operaors such as $L_p(a, c)(a > 0, c \neq 0, -1, -2, ..)$ (see [10]) and $D^{\nu+p-1}(\nu > -p)$ (see [2] and [10]).

(*ii*) For $n = 0$ and $b_k = \left(\frac{l+\delta k}{l}\right)^m$ $(l > 0, \delta \geq 0, m \in \mathbb{N}_0)$, we have

$$D^0_{\lambda,p}(f * g)(z) = I^m_p(\delta, l)f(z) = z^{-p} + \sum_{k=1}^{\infty} \left(\frac{l + \delta k}{l}\right)^m a_k z^{k-p}, \tag{1.7}$$

where the operator $I^m_p(\delta, l)$ was introduced and studied by El-Ashawh and Aouf [8];

(*iii*) For $n = 0$ and $b_k = \left(\frac{l}{l+\delta(k+p)}\right)^m$ $(l > 0, \delta \geq 0, m \in \mathbb{N}_0)$, we have

$$D^0_{\lambda,p}(f * g)(z) = J^m_p(\delta, l)f(z) = z^{-p} + \sum_{k=1-p}^{\infty} \left(\frac{l}{l + \delta(k + p)}\right)^m a_k z^k, \tag{1.8}$$

where the operator $J^m_p(\delta, l)$ was introduced and studied by El-Ashawh [7].

For $-1 \leq A < B \leq 1, B \geq 0$ and $z \in U^*$, Mogra [14] defined the class

$$\sum[p, A, B] = \left\{ f \in \sum\nolimits_p : -\frac{zf'(z)}{f(z)} \prec p\frac{1 + Az}{1 + Bz}, z \in U \right\} \tag{1.9}$$

and Srivastava et al. [16] defined the class

$$\sum K_p[A, B] = \left\{ f \in \sum\nolimits_p : -[1 + \frac{zf''(z)}{f'(z)}] \prec p\frac{1 + Az}{1 + Bz}, z \in U \right\}. \tag{1.10}$$

It is clear that

$$f(z) \in \sum K_p[A, B] \Leftrightarrow -\frac{zf'(z)}{p} \in \sum[p, A, B], \tag{1.11}$$

$$\sum[1, 2\alpha - 1, 1] = \sum S^*(\alpha) \text{ (see Juneja and Reddy [9])},$$

$$\sum K_1[2\alpha - 1, 1] = \sum K(\alpha) \ (0 \leq \alpha < 1) \text{ (see Srivastava et al. [16])},$$

$$\sum[p, \frac{2\alpha}{p} - 1, 1] = \sum S^*_p(\alpha) \text{ (see Aouf and Hossen [3]) } (0 \leq \alpha < p)$$

and

$$\sum K_p[\frac{2\alpha}{p}-1,1]=\sum K_p(\alpha) \text{ (see Aouf and Srivastava [5]) } (0\leq\alpha<p).$$

Using the operator $D^n_{\lambda,p}(f*g)(z), -1\leq B<A\leq 1, \lambda\geq 0, n\in\mathbb{N}_0$ and $z\in U^*$ we define the classes $S^*_p(n,\lambda,A,B)$ and $K_p(n,\lambda,A,B)$ as follows:

$$S^*_p(n,\lambda,A,B)=\left\{f\in\sum\nolimits_p : D^n_{\lambda,p}(f*g)(z)\in\sum[p,A,B], z\in U\right\}, \tag{1.12}$$

and

$$K_p(n,\lambda,A,B)=\left\{f\in\sum\nolimits_p : D^n_{\lambda,p}(f*g)(z)\in\sum K_p[A,B], z\in U\right\}. \tag{1.13}$$

We notice that

$$f(z)\in K_p(n,\lambda,A,B)\Leftrightarrow -\frac{zf'(z)}{p}\in S^*_p(n,\lambda,A,B). \tag{1.14}$$

2. **Main results**

Unless otherwise mentioned, we shall assume in this paper that $-1\leq B<A\leq 1, 0\leq A<1, \lambda>0, n\in\mathbb{N}_0, 0<\theta<2\pi, p\in\mathbb{N}, z\in U^*$ and $g(z)$ is given by (1.2).

To prove our results we need the following lemmas.

Lemma 2.1. *[15]. The function $f(z)$ defined by* (1.1) *is in the class $\sum[p,A,B]$ if and only if*

$$z^p\left[f(z)*\frac{1+(D-1)z}{z^p(1-z)^2}\right]\neq 0, \tag{2.1}$$

where

$$D=\frac{e^{-i\theta}+B}{p(A-B)}. \tag{2.2}$$

.

Lemma 2.2. *[15] The function $f(z)$ defined by* (1.1) *is in the class $\sum K_p[A,B]$ if and only if* $z^p\left\{f(z)*\left[\frac{p-\{2+p-(p-1)(D-1)\}z-(p+1)(D-1)z^2}{pz^p(1-z)^3}\right]\right\}\neq 0.$

Lemma 2.3. *[6]. Let h be convex (univalent) in U, with $Re[\beta h(z)+\gamma]>0$ for all $z\in U$. If p is analytic in U, with $p(0)=h(0)$, then*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec h(z)\Rightarrow p(z)\prec h(z). \tag{2.3}$$

Theorem 2.4. *The function $f(z)$ defined by* (1.1) *is in the class $S^*_p(n,\lambda,A,B)$ if and only if*

$$1+\sum_{k=1}^{\infty}\left[\frac{ke^{-i\theta}+pA+(k-p)B}{p(A-B)}\right](1+\lambda k)^n a_k b_k z^k\neq 0. \tag{2.4}$$

Proof. From Lemma 1, we find that $f(z) \in S_p^*(n,\lambda,A,B)$ if and only if

$$z^p\left[D_{\lambda,p}^n(f*g)(z)*\frac{1+(D-1)z}{z^p(1-z)^2}\right]\neq 0.$$

Expanding $\frac{1+(D-1)z}{z^p(1-z)^2}$, we have (2.4) which completes the proof of Theorem 1.

Theorem 2.5. *The function $f(z)$ defined by* (1.1) *is in the class $K_p(n,\lambda,A,B)$ if and only if*

$$1-\sum_{k=1}^{\infty}\left[\frac{k\left[ke^{-i\theta}+pA+(k-p)B\right]}{p^2(A-B)}\right](1+\lambda k)^n a_k b_k z^k\neq 0. \tag{2.5}$$

Proof. From Lemma 2, we find that $f(z) \in K_p(n,\lambda,A,B)$ if and only if

$$z^p\left\{D_{\lambda,p}^n(f*g)(z)*\left[\frac{p-\{2+p-(p-1)(D-1)\}z-(p+1)(D-1)z^2}{pz^p(1-z)^3}\right]\right\}\neq 0. \tag{2.6}$$

Now it can be easily shown that

$$z^{-p}(1-z)^{-3}=z^{-p}+\sum_{k=1}^{\infty}\frac{(k+1)(k+2)}{2}z^{k-p}, \tag{2.7}$$

$$z^{1-p}(1-z)^{-3}=\sum_{k=1}^{\infty}\frac{k(k+1)}{2}z^{k-p}, \tag{2.8}$$

$$z^{2-p}(1-z)^{-3}=\sum_{k=1}^{\infty}\frac{k(k-1)}{2}z^{k-p}. \tag{2.9}$$

Using (2.7)−(2.9) and (2.2) in (2.6), we have the desired result which completes the proof of Theorem 2.

Theorem 2.6. *If the function $f(z)$ defined by* (1.1) *belongs to the class $S_p^*(n,\lambda,A,B)$, then*

$$\sum_{k=1}^{\infty}\left[k+pA+(k-p)B\right](1+\lambda k)^n|a_k|\leq p(A-B). \tag{2.10}$$

Proof. Since

$$\left|\frac{ke^{-i\theta}+pA+(k-p)B}{p(A-B)}\right|=\frac{\left|ke^{-i\theta}+pA+(k-p)B\right|}{p(A-B)}\leq\frac{k+pA+(k-p)B}{p(A-B)}$$

and

$$\left|1+\sum_{k=1}^{\infty}\frac{\left[ke^{-i\theta}+pA+(k-p)B\right]}{p(A-B)}(1+\lambda k)^{n}\left|a_{k}z^{k}\right|\right|$$
$$> \quad 1-\sum_{k=1}^{\infty}\left|\frac{\left[ke^{-i\theta}+pA+(k-p)B\right]}{p(A-B)}\right|(1+\lambda k)^{n}\left|a_{k}\right|.$$

The result follows from Theorem 1.

Using the same technique, we can also prove the following theorem.

Theorem 2.7. *If the function $f(z)$ defined by* (1.1) *belongs to the class $K_p(n,\lambda,A,B)$, then*

$$\sum_{k=1}^{\infty}k\left[k+pA+(k-p)B\right](1+\lambda k)^{n}\left|a_{k}\right|\leq p^{2}(A-B). \tag{2.11}$$

Theorem 2.8. *Let the function $f(z)$ be defined by* (1.1). *If*

$$\frac{1+AB+(A+B)\cos\theta}{1+B^{2}+2B\cos\theta}\leq\frac{\lambda p+1}{\lambda^{2}} \tag{2.12}$$

and $f(z)\in S_p^*(n{+}1,\lambda,A,B)$ with $D^n_{\lambda,p}(f{*}g)(z)\neq 0$, then $f(z)\in S_p^*(n,\lambda,A,B)$.

Proof. Let $f(z)\in S_p^*(n+1,\lambda,A,B)$ and define the function

$$P(z)=-\frac{z\left(D^{n}_{\lambda,p}(f*g)(z)\right)'}{D^{n}_{\lambda,p}(f*g)(z)}, \tag{2.13}$$

we see that P is analytic in U with $P(0)=1$. Using the identity (1.4) in (2.13) we have

$$\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{D^{n}_{\lambda,p}(f*g)(z)}=-\lambda P(z)+(p+\frac{1}{\lambda}). \tag{2.14}$$

Differentiating (2.14) logarithmically and using (2.13), we have

$$-\frac{z\left(D^{n+1}_{\lambda,p}(f*g)(z)\right)'}{D^{n+1}_{\lambda,p}(f*g)(z)}=p(z)+\frac{zP'(z)}{-P(z)+\frac{1}{\lambda}(p+\frac{1}{\lambda})}\prec\frac{1+Az}{1+Bz}=h(z). \tag{2.15}$$

Simple computations show that the inequalty $\Re\{-h(z)+\frac{1}{\lambda}(p+\frac{1}{\lambda})\}>0$ can be written in the form

$$\Re\frac{1+Az}{1+Bz}-\frac{1}{\lambda}(p+\frac{1}{\lambda})<0,$$

which is equivalent to (2.12). Since the function $h(z)$ is a convex function, by applying Lemma 3, we see that the subordination (2.15) implies $P(z)\prec h(z)$. This completes the proof of Theorem 2.8.

Putting $n = 0$ and $b_k = \Gamma_k(\alpha_1)$, where $\Gamma_k(\alpha_1)$ is defined by (1.5), in Theorem 2.8, we have the follwoing corollary which improves the result obtained by Sarkar et al. [15, Theorem 2.7].

Corollary 2.9. *Let the function $f(z)$ be defined by* (1.1). *If $\alpha_1 > 0$,*

$$\frac{1 + AB + (A + B)\cos\theta}{1 + B^2 + 2\cos\theta} \leq \frac{p + \alpha_1}{p^2} \tag{2.16}$$

and $f(z) \in S^*_{p,q,s}(\alpha_1 + 1, A, B)$ with $H_{p,q,s}f(z) \neq 0$, then $f(z) \in S^*_p(\alpha_1, A, B)$.
Using (1.14) *and the fact that*

$$D^n_{\lambda,p}(-zf')(z) = -z\left(D^n_{\lambda,p}f(z)\right)',$$

Theorem 5 yields the following theorem.

Theorem 2.10. *Let the function $f(z)$ be defined by* (1.1). *If* (2.12) *holds and $f(z) \in K_p(n + 1, \lambda, A, B)$ with $D^n_{\lambda,p}(f * g)(z) \neq 0$, then $f(z) \in K_p(n, \lambda, A, B)$.*

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