Title:

Linear Weingarten hypersurfaces in a unit sphere

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LINEAR WEINGARTEN HYPERSURFACES IN A UNIT SPHERE

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(Communicated by Jost-Hinrich Eschenburg)

Abstract. In this paper, by modifying Cheng-Yau’s technique to complete hypersurfaces in \( S^{n+1}(1) \), we prove a rigidity theorem under the hypothesis of the mean curvature and the normalized scalar curvature being linearly related which improve the result of [H. Li, Hypersurfaces with constant scalar curvature in space forms, Math. Ann. 305 (1996), 665–672].

Keywords: Linear Weingarten hypersurface, generalized maximum principle, rigidity theorem.

MSC(2010): Primary: 53C20; Secondary: 53C42.

1. Introduction

Let \( M \) be an \( n \)-dimensional hypersurfaces in the unit sphere \( S^{n+1}(1) \). The investigation of curvature structures of compact hypersurfaces of \( S^{n+1}(1) \) is important and interesting, with so much attention on it. Cheng and Yau [5] studied compact hypersurfaces with constant scalar curvature in the unit sphere \( S^{n+1}(1) \). They proved that if \( M \) is an \( n \)-dimensional hypersurface with constant scalar curvature \( n(n-1)r \), \( r \geq 1 \) and the sectional curvatures of \( M \) are nonnegative, then \( M \) is isometric to a totally umbilical hypersurface \( S^n(c) \) or to the Riemannian product \( S^k(c) \times S^{n-k}(\sqrt{1-c^2}) \), \( 1 \leq k \leq n-1 \), where \( S^k(c) \) denote the sphere of radius \( c \).

Furthermore, by using of the same method which was used in [4] and the differential operator in [5], Li [6] prove that if \( M \) is an \( n \)-dimensional compact hypersurface with constant scalar curvature \( n(n-1)r \), \( r \geq 1 \) and \( S \leq (n-1) \frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2} \), then \( M \) is isometric to either totally umbilical hypersurface or the Riemannian product \( S^1(\sqrt{1-c^2}) \times S^{n-1}(c) \) with \( c^2 = \frac{n-2}{n} \leq \frac{n-2}{n} \), where \( S \) is the squared norm of the second fundamental form of \( M \). In the proof of these results, the fact that the differential operator \( \Box \) defined by \( \Box f = \sum_{i,j}^n (nH\delta_{ij} - h_{ij})f_{ij} \) is self-adjoint and degenerate elliptic...
is indispensable. So the condition of $r \geq 1$ and the assumption of constant scalar curvature is essential. Q.M.Cheng [3] generalized the results to the case $r \geq \frac{n-2}{n-1}$ under a topological condition. More precisely, he proved that if $M$ is an $n$-dimensional compact hypersurface with infinite fundamental group in $S^{n+1}(1)$, if $r \geq \frac{n-2}{n-1}$ and $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, then $M$ is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$, where $n(n-1)r$ is the scalar curvature of $M$ and $c^2 = \frac{n-2}{n}$.

On the other hand, Li [7] studied some hypersurfaces in the unit sphere with scalar curvature proportional to the mean curvature and proved the following theorem.

**Theorem 1.1** ( [7]). Let $M$ be an $n$-dimensional compact hypersurface in the unit sphere $S^{n+1}(1)$. If

1. $M$ has nonnegative sectional curvature,
2. the normalized scalar curvature $r$ and the mean curvature $H$ of $M$ satisfy the following conditions:

$$r = aH, \quad a^2 \geq \frac{4n}{n-1},$$

where $a$ is a constant, then $M$ is either totally umbilical, or $M = S^{n-k}(c) \times S^k(\sqrt{1-c^2}), 1 \leq k \leq n-1$.

Recently, Li, Suh and Wei [9] considered linear Weingarten hypersurfaces in an sphere and obtained the following rigidity theorems:

**Theorem 1.2** ( [9]). Let $M$ be an $n$-dimensional compact hypersurface in the unit sphere $S^{n+1}(1)$. If

1. $M$ has nonnegative sectional curvature,
2. the normalized scalar curvature $r$ and the mean curvature $H$ of $M$ satisfy the following conditions:

$$r = aH + b, \quad (n-1)a^2 - 4n + 4nb \geq 0,$$

then $M$ is either totally umbilical, or $M = S^{n-k}(c) \times S^k(\sqrt{1-c^2}), 1 \leq k \leq n-1$.

In these theorems, $M$ is compact and $M$ has nonnegative sectional curvature. Without the assumption of nonnegative sectional curvature, Li, Suh and Wei [9] obtained the following result, as well.

**Theorem 1.3** ( [9]). Let $M$ be an $n$-dimensional compact hypersurface in the unit sphere $S^{n+1}(1)$. If

1. $r = aH + b, (n-1)a^2 - 4n + 4nb \geq 0$,
2. $|B|^2 \leq 2\sqrt{n-1}$,

then either $|B|^2 = 0$ and $M$ is a totally umbilical hypersurface or $|B|^2 = 2\sqrt{n-1}$ and $M = S^1(c) \times S^{n-1}(\sqrt{1-c^2})$. 
In this paper, we consider complete linear Weingarten hypersurface in $S^{n+1}(1)$ with no sectional curvature restriction. By modifying Cheng-Yau's technique for complete hypersurfaces in $S^{n+1}(1)$, we will prove a rigidity theorem under the hypothesis of the mean curvature and the normalized scalar curvature being linearly related. More precisely, we prove the following theorem.

**Theorem 1.4.** Let $M^n$ be a complete hypersurface of $S^{n+1}(1)$ with bounded mean curvature. If $r = aH + b$, $a \leq 0$, $b \geq 1$, then $M^n$ is either totally umbilical, or $M = S^1(c) \times S^{n-1}(\sqrt{1-c^2})$.

If we choose $a = 0$ and $b \geq 1$ in Theorem 1.4, we get

**Corollary 1.5.** Let $M^n$ be a complete hypersurface of $S^{n+1}(1)$ with constant normalized scalar curvature $r$ satisfying $r \geq 1$. If $M^n$ has bounded mean curvature, then $M^n$ is either totally umbilical, or $M = S^1(c) \times S^{n-1}(\sqrt{1-c^2})$.

**Remark 1.6.** Corollary 1.5 is a real improvement of the result in [6] because it has no compactness restriction on $M$ and no restriction on $|B|^2$.

2. Preliminaries

Let $M$ be an $n$-dimensional complete hypersurface in the $(n+1)$-dimensional unit sphere $S^{n+1}(1)$. For any $p \in M$, we choose a local orthonormal frame $e_1, \cdots, e_n, e_{n+1}$ on $S^{n+1}(1)$ around $p$ such that $e_1, \cdots, e_n$ are tangent to $M$. Take the corresponding dual coframe $\omega_1, \cdots, \omega_n, \omega_{n+1}$. We shall make use of the following standard convention on the range of indices:

$$1 \leq A, B, C, D \cdots \leq n + 1, \quad 1 \leq i, j, k, l \cdots \leq n.$$ 

Then the structure equations of $S^{n+1}(1)$ are given by

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \quad (2.1)$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \omega_A \wedge \omega_B, \quad (2.2)$$

Restricting those forms to $M$, we have $\omega_{n+1} = 0$ and

$$0 = d\omega_{n+1} = \sum_i^n \omega_{n+1i} \wedge \omega_i. \quad (2.3)$$

By Cartan's lemma, there exist functions $h_{ij}$ such that

$$\omega_{n+1i} = \sum_j^n h_{ij} \omega_j, \quad h_{ij} = h_{ji}. \quad (2.4)$$
The structure equations of $M$ are given by

\[(2.5) \quad d\omega_i = \sum_{j=1}^{n} \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,\]

\[(2.6) \quad d\omega_{ij} = \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l=1}^{n} R_{ijkl} \omega_k \wedge \omega_l.\]

The Gauss equations are

\[(2.7) \quad R_{ijkl} = \delta_{ik} \delta_{jl} - \delta_{jl} \delta_{ik} + h_{ik} h_{jl} - h_{ij} h_{jk},\]

\[(2.8) \quad n(n-1)r = n(n-1) + n^2 H^2 - |B|^2,\]

where $R_{ijkl}$ denotes the components of the Riemannian curvature tensor of $M$, $r$ is the normalized scalar curvature of $M$ and $|B|^2 = \sum_{i,j} h_{ij}^2$ is the norm square of the second fundamental form of $M$.

By taking the exterior differentiation of (2.4), we obtain the Codazzi equations

\[(2.9) \quad h_{ijk} = h_{ikj} = h_{jik},\]

where the covariant derivative of $h_{ij}$ is defined by

\[(2.10) \quad \sum_k h_{ijk} \omega_k = dh_{ij} + \sum_k h_{kj} \omega_{ik} + \sum_k h_{ik} \omega_{kj}.\]

Similarly, the components $h_{ijkl}$ of the second derivative $\nabla^2 h$ are given by

\[(2.11) \quad \sum_l h_{ijkl} \omega_l = dh_{ijk} + \sum_l h_{jkl} \omega_{il} + \sum_l h_{ilk} \omega_{lj} + \sum_l h_{ijl} \omega_{kl}.\]

By exterior differentiation of (2.10), we can get the following Ricci identities

\[(2.12) \quad h_{ijkl} - h_{ijk} = \sum_m h_{im} R_{mjk} + \sum_m h_{jm} R_{mkj}.\]

The Laplacian $\Delta h_{ij}$ of $h_{ij}$ is defined by $\Delta h_{ij} = \sum_k h_{ijkk}$, from the Codazzi equation (2.9) and Ricci identities (2.12), we have

\[(2.13) \quad \Delta h_{ij} = \sum_k h_{kkij} + \sum_{m,k} h_{km} R_{mijk} + \sum_{m,k} h_{im} R_{mkjk}.\]

Set $\phi_{ij} = h_{ij} - H \delta_{ij}$, it is easy to check that $\phi$ is traceless and $|\phi|^2 = \sum_{i,j} \phi_{ij}^2 = |B|^2 - nH^2$. Following Cheng-Yau [5], as in [1] and [2], for any $a \geq 0$, we introduce a modified operator $\square$ acting on any $C^2$ function $f$ by

\[(2.14) \quad \square(f) = \sum_{i,j} \left((nH - \frac{n-1}{2}a)\delta_{ij} - h_{ij}\right) f_{ij},\]
where \( f_{ij} \) is given by the following

\[
\sum_j f_{ij}\omega_j = df_i + f_j\omega_{ij}.
\]

**Lemma 2.1** ([9]). Let \( M^n \) be a hypersurface of \( S^{n+1}(1) \) with \( r = aH + b, a, b \in \mathbb{R} \) and \((n-1)a^2 - 4n + 4nb \geq 0\). Then we have

\[
|\nabla B|^2 \geq n^2|\nabla H|^2.
\]

**Lemma 2.2.** Let \( M^n \) be a complete hypersurface of \( S^{n+1}(1) \) with \( r = aH + b, a, b \in \mathbb{R} \). Then we have

\[
\Box(nH) \geq |\nabla B|^2 - n^2|\nabla H|^2 + |\phi|^2 \left( -|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi| + n(1 + H^2) \right).
\]

**Proof.** First, from (2.12), we have

\[
\frac{1}{2} \Delta |B|^2 = \frac{1}{2} \Delta \sum_{i,j} h_{ij}^2 = \sum_{i,j} h_{ij} \Delta h_{ij} + \sum_{i,j,k} h_{ijk}^2
\]

\[
= \sum_{i,j,k} h_{ijk}^2 + n \sum_{i,j} h_{ij} H_{ij} + n(|B|^2 - nH^2) + nHf_3 - |B|^4,
\]

where \( f_3 = \sum_{i,j,k} h_{ij}h_{jk}h_{ki} \). On the other hand, from Gauss equation and \( r = aH + b \), we have

\[
\Delta |B|^2 = \Delta(n^2H^2 - n(n-1)(r-1))
\]

\[
= \Delta(n^2H^2 - n(n-1)(aH + b - 1))
\]

\[
= \Delta(n^2H^2 - (n-1)anH) = \Delta(nH - \frac{1}{2}(n-1)a)^2.
\]
Then from (2.17) and Okumura’s inequality [10], we get
\[
\Box(nH) = \sum_{i,j} ((nH - \frac{1}{2}(n-1)a)\delta_{ij} - h_{ij})(nH)_{ij} \\
= (nH - \frac{1}{2}(n-1)a)\Delta(nH) - \sum_{i,j} \sum_{i,j} h_{ij}(nH)_{ij} \\
= (nH - \frac{1}{2}(n-1)a)\Delta(nH - \frac{1}{2}(n-1)a) - \sum_{i,j} \sum_{i,j} h_{ij}(nH)_{ij} \\
= \frac{1}{2}\Delta(nH - \frac{1}{2}(n-1)a)^2 - \|\nabla(nH - \frac{1}{2}(n-1)a)\|^2 \\
- \sum_{i,j} \sum_{i,j} h_{ij}(nH)_{ij} \\
= \frac{1}{2}\Delta|B|^2 - n^2|\nabla H|^2 - \sum_{i,j} \sum_{i,j} h_{ij}(nH)_{ij} \\
= \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 + n(|B|^2 - nH^2) + nHf_3 - |B|^4 \\
\geq |\nabla B|^2 - n^2|\nabla H|^2 \\
+ |\phi|^2\left(-|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi| + n(1 + H^2)\right).
\]
\[\text{(2.18)}\]

On the other hand, we choose \(e_1, \ldots, e_n\) such that \(h_{ij} = \lambda_i \delta_{ij}\), then we can deduce from (2.9) and (2.12) that
\[
\frac{1}{2}\Delta|B|^2 = \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}(nH)_{ij} + \frac{1}{2}R_{ijij}(\lambda_i - \lambda_j)^2 \\
\text{(2.19)}
\]
and, as (2.18),
\[
\Box(nH) = \frac{1}{2}\Delta|B|^2 - n^2|\nabla H|^2 - \sum_{i,j} \sum_{i,j} h_{ij}(nH)_{ij} \\
= \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 + \frac{1}{2}R_{ijij}(\lambda_i - \lambda_j)^2 \\
\geq |\nabla B|^2 - n^2|\nabla H|^2 + \frac{1}{2}R_{ijij}(\lambda_i - \lambda_j)^2. \\
\text{(2.20)}
\]

**Proposition 2.3.** Let \(M^n\) be a complete hypersurface of \(S^{n+1}(1)\) with bounded mean curvature. If \(r = aH + b, b \geq 1\), then there is a sequence of points
\{p_k\} \in \mathcal{M}^n \text{ such that }
\lim_{k \to \infty} nH(p_k) = n \sup H; \quad \lim_{k \to \infty} |\nabla(nH)(p_k)| = 0; \quad \limsup_{k \to \infty}(\nabla(nH)(p_k)) \leq 0.

Proof. Choose a local orthonormal frame field \(e_1, \ldots, e_n\) at a neighbourhood of \(p \in \mathcal{M}^n\) such that \(h_{ij} = \delta_{ij}\). Thus

\begin{equation}
\Box(nH) = \sum_i [(nH - \frac{1}{2}(n-1)a) - \lambda_i](nH)_{ii}.
\end{equation}

If \(H \equiv 0\), the proposition is obvious. Let us suppose that \(H\) is not identically zero. By changing the orientation of \(\mathcal{M}^n\) if necessary, we may assume \(\sup H > 0\). From Gauss equation, we have

\begin{equation}
\lambda_i^2 \leq |\mathcal{B}|^2 = n^2H^2 - n(n-1)(r-1)
= n^2H^2 - n(n-1)(aH + b - 1)
= (nH)^2 - (n-1)a(nH) - n(n-1)(b - 1)
= (nH - \frac{1}{2}(n-1)a)^2 - \frac{1}{4}(n-1)((n-1)a^2 - 4n + 4b)
\leq (nH - \frac{1}{2}(n-1)a)^2, \quad (i = 1, 2, \ldots, n).
\end{equation}

Therefore

\begin{equation}
|\lambda_i| \leq |nH - \frac{1}{2}(n-1)a|, \quad (i = 1, 2, \ldots, n).
\end{equation}

and

\begin{equation}
R_{ijij} = 1 + \lambda_i\lambda_j \geq 1 - (nH - \frac{1}{2}(n-1)a)^2.
\end{equation}

Because \(H\) is bounded, it follows from (2.24) that the sectional curvatures are bounded from below. Therefore we may apply generalized maximum principle from [11, 12] to \(nH\), obtaining a sequence of points \(\{p_k\} \in \mathcal{M}^n\) such that

\begin{equation}
\lim_{k \to \infty} nH(p_k) = n \sup H; \quad \lim_{k \to \infty} |\nabla(nH)(p_k)| = 0; \quad \limsup_{k \to \infty}(\nabla(nH)(p_k)) \leq 0.
\end{equation}

From Gauss equation, we have

\begin{equation}
|\mathcal{B}|^2 = n^2H^2 + n(n-1)(1 - aH - b),
\end{equation}

and then

\[ nH[nH - (n-1)a] = n(n-1)(b-1) + |\mathcal{B}|^2 \geq 0, \]

if \(b \geq 1\). If \(|\mathcal{B}|^2 = 0\), from \(|\mathcal{B}|^2 \geq nH^2\), we deduce that \(H = 0\) which is the trivial case. So \(b > 1\) or \(|\mathcal{B}|^2 \neq 0\) which implies \(H \neq 0\). When \(H \neq 0\), we can obtain from (2.26) that

\[-\frac{n-1}{2}a = \frac{1}{2nH}(|\mathcal{B}|^2 - n^2H^2 + n(n-1)(b-1)).\]
Therefore,
\[
nH - \frac{n-1}{2}a - \lambda_i
\]
\[
= nH - \lambda_i + \frac{1}{2nH}(|B|^2 - n^2H^2 + n(n-1)(b-1))
\]
(2.27)
\[
= \frac{n-1}{2H}(b-1) + \frac{1}{2nH}(|B|^2 + n^2H^2 - 2nH\lambda_i).
\]
Observe now that
\[
|B|^2 + n^2H^2 - 2nH\lambda_i = \sum_{j=1}^{n} \lambda_j^2 + \left(\sum_{j=1}^{n} \lambda_j\right)^2 - 2\left(\sum_{j=1}^{n} \lambda_j\right)\lambda_i
\]
(2.28)
\[
= \sum_{j=1, j \neq i}^{n} \lambda_j^2 + \left(\sum_{j=1, j \neq i}^{n} \lambda_j\right)^2 \geq 0.
\]
So (2.27), (2.28) and \(b \geq 1\) imply that
(2.29)
\[
nH - \frac{n-1}{2}a - \lambda_i \geq 0.
\]
On the other hand, from (2.23) we get
\[
nH(p_k) - \frac{1}{2}(n-1)a - \lambda_i(p_k)
\]
\[
\leq nH(p_k) - \frac{1}{2}(n-1)a + |\lambda_i(p_k)|
\]
(2.30)
\[
\leq 2nH(p_k) - (n-1)a.
\]
Using once more the fact that \(H\) is bounded, from (2.29) and (2.30), we infer that \(nH(p_k) - \frac{1}{2}(n-1)a - \lambda_i(p_k)\) is nonnegative and bounded. By applying \(\Box(nH)\) at \(p_k\), taking the limit and using (2.25), we have
\[
\lim_{k \to \infty} \sup(\Box(nH)(p_k)) \leq \sum_{i} \lim_{k \to \infty} \sup[(nH - \frac{1}{2}(n-1)a - \lambda_i(p_k)(nH)_{ii}(p_k) \leq 0.
\]
\[
\square
\]
3. Proof of the main result

**Proof of Theorem 1.4.** If \(M^n\) is minimal, i.e., if \(H \equiv 0\), then \(r = b \geq 1\).
So, from Gauss equation, we have \(|B|^2 \equiv 0\) and \(M^n\) is totally geodesic. Let us suppose that \(H\) is not identically zero. In this case, by Proposition 2.3 it is possible to obtain a sequence of points \(\{p_k\} \in M^n\) such that
(3.1)
\[
\lim_{k \to \infty} \sup(\Box(nH)(p_k)) \leq 0, \quad \lim_{k \to \infty} H(p_k) = \sup H > 0.
\]
Moreover, using the Gauss equation, we have that
\begin{equation}
|\phi|^2 = |B|^2 - nH^2 = n(n-1)(H^2 - aH - b + 1). \tag{3.2}
\end{equation}
In view of \( \lim_{k \to \infty} H(p_k) = \sup H \) and \( a \leq 0 \), (3.2) implies that \( \lim_{k \to \infty} |\phi|^2(p_k) = \sup |\phi|^2 \). Now we consider the following polynomial given by
\begin{equation}
P_{\sup H}(x) = -x^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} \sup Hx + n(1 + \sup H^2). \tag{3.3}
\end{equation}
Because the discriminant of \( P_{\sup H}(x) \) is always positive, the smallest root of \( P_{\sup H}(x) \) is negative and the biggest root \( \mu \) of \( P_{\sup H}(x) \) is positive. We claim that \( \sup |\phi|^2 = 0 \). It’s easy to check that \( \sup |\phi|^2 - \mu^2 \leq 0 \) provided \( b \geq \frac{n-2}{n-1} \). In fact,
\begin{equation}
(\sup |\phi|)^2 = \sup |\phi|^2 = n(n-1)(\sup H^2 - a \sup H - b + 1)
\leq n(n-1)(\sup H^2 - b + 1), \tag{3.4}
\end{equation}
it is straightforward to verify that
\begin{equation}
\sup |\phi|^2 - \mu^2 \leq \frac{n-2}{2(n-1)} \left( n^2 \sup H^2 - n \sup H \sqrt{n^2 \sup H^2 + 4(n-1)} \right)
+ \frac{2n(n-1)}{n-2} \left( -(n-1)b + (n-2) \right). \tag{3.5}
\end{equation}
It can be easily seen that \( \sup |\phi|^2 - \mu^2 \leq 0 \) if and only if
\begin{align*}
n^2 \sup H^2 - n \sup H \sqrt{n^2 \sup H^2 + 4(n-1)}
- \frac{2n(n-1)}{n-2} \left( (n-1)b - (n-2) \right) & \leq 0.
\end{align*}
So, if \( b > 1 \), the last inequality is true. We deduce that \( \sup |\phi| \leq \mu \) and \( P_{\sup H}(\sup |\phi|) \geq 0 \).

Using Lemma 2.1 and evaluating (2.16) at the points \( p_k \) of the sequence, taking the limit and using (3.1), we obtain that
\begin{equation*}
0 \geq \limsup_{k \to \infty}(\Box(nH)(p_k)) \geq \sup |\phi|^2 P_{\sup H}(\sup |\phi|) \geq 0,
\end{equation*}
so \( \sup |\phi|^2 P_{\sup H}(\sup |\phi|) = 0 \). Then we have either \( \sup |\phi|^2 = 0 \) which shows that \( M^n \) is totally umbilical or \( P_{\sup H}(\sup |\phi|) = 0 \). So (2.16) is also equality, then the Okumura’s formula implies that \( M^n \) has two distinct principal curvatures one of which is simple and \( M^n = S^1(c) \times S^{n-1}(\sqrt{1-c^2}) \).

\textbf{Remark 3.1.} \textit{Using our method, we can easily generalize Theorem 1.2 and Theorem 1.3 to the complete case. But we don’t state it here.}

\textbf{Acknowledgments}

I would like to express my gratitude to anonymous referees for valuable comments and suggestions.
REFERENCES


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