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**The distributional Henstock-Kurzweil integral and measure differential equations**

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## THE DISTRIBUTIONAL HENSTOCK-KURZWEIL INTEGRAL AND MEASURE DIFFERENTIAL EQUATIONS

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**ABSTRACT.** In the present paper, measure differential equations involving the distributional Henstock-Kurzweil integral are investigated. Theorems on the existence and structure of the set of solutions are established by using Schauder's fixed point theorem and Vidossich theorem. Two examples of the main results paper are presented. The new results are generalizations of some previous results in the literatures.

**Keywords:** Distributional Henstock-Kurzweil integral, measure differential equation, Henstock-Stieltjes integral, distributional derivative.

**MSC(2010):** Primary: 26A39; Secondary: 28B05, 34A12, 46F05, 46G12.

### 1. Introduction

In this paper, we consider the following measure differential equation (MDE for short)

$$(1.1) \quad \begin{cases} Dx = f(x, t) + g(x, t)Du, \\ x(a) = x_0, \end{cases}$$

where  $t \in [a, b] \subset \mathbb{R}$ ,  $-\infty < a < b < +\infty$ ,  $x \in C([a, b])$ ,  $u : [a, b] \rightarrow \mathbb{R}$  is a right continuous function of bounded variation,  $f : \mathbb{R}^n \times [a, b] \rightarrow \mathbb{R}^n$  is distributionally Henstock-Kurzweil integrable,  $g : \mathbb{R}^n \times [a, b] \rightarrow \mathbb{R}^n$  is Henstock-Stieltjes (*HS*) integrable.

MDEs have been studied by many authors, we refer the readers to [5, 6, 11, 13, 15, 19] and the references therein. For instance, in [15], Tanwani, et al. considered the MDE (1.1) in a special case where  $f(x, t) = f(x(t))$ ,  $g(x, t) = g(x(t))$ . Schmaedeke [11] considered the equation (1.1) in a complicated case where  $g(x(t))$  is a continuous  $n \times m$  matrix. Furthermore, Das and Sharma [6] generalized the results in [11] for a functional differential equation, i.e.  $Dx = f(t, x_t) + g(t, x_t)Du$ .

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All these papers use the ordinary derivative to discuss the MDEs. In this paper, we will use the distributional derivative to study the MDE (1.1). It is well known that the notion of a distributional derivative is a general concept, including ordinary derivatives and approximate derivatives. The distributional Henstock-Kurzweil integral is defined by the distributional derivative. In this case, we present some new results which are generalizations of the previous results in [5, 19].

The paper is organized as follows. In Section 2, we introduce some fundamental concepts and basic results of the distributional Henstock-Kurzweil integral. In Section 3, we apply the Schauder's fixed point theorem to verify the existence of solutions of the MDE (1.1). We also show that the set of solutions of the MDE (1.1) is an  $R_\delta$  by using the Vidossich theorem stated in [16]. Here, an  $R_\delta$  is the intersection of a decreasing sequence of compact absolute retracts (see [2] for details). In Section 4, we give two examples to show that Theorem 3.3 in Section 3 is more extensive.

## 2. The distributional Henstock-Kurzweil integral

In this section, we present the definition and some basic properties of the distributional Henstock-Kurzweil integral.

Define the space

$$C_c^\infty = \{\phi : \mathbb{R} \rightarrow \mathbb{R} \mid \phi \in C^\infty/\mathbb{R} \text{ and has compact support in } \mathbb{R}\},$$

where *support* of  $\phi$  is denoted by  $\text{supp}(\phi)$ . A sequence  $\{\phi_n\} \subset C_c^\infty$  converges to  $\phi \in C_c^\infty$  if there is a compact set  $K$  such that all  $\phi_n$  have supports in  $K$  and the sequence of derivatives  $\phi_n^{(m)}$  converges to  $\phi^{(m)}$  uniformly on  $K$  for every  $m \in \mathbb{N} \cup \{0\}$ ,  $\phi^{(0)} = \phi$ . Denote  $C_c^\infty$  endowed with this convergence property by  $\mathcal{D}$ . Also,  $\phi$  is called a *test function* if  $\phi \in \mathcal{D}$ . The dual space to  $\mathcal{D}$  is denoted by  $\mathcal{D}'$  and its elements are called distributions. That is, if  $f \in \mathcal{D}'$  then  $f : \mathcal{D} \rightarrow \mathbb{R}$ , and we write  $\langle f, \phi \rangle \in \mathbb{R}$ , for  $\phi \in \mathcal{D}$ .

For all  $f \in \mathcal{D}'$ , we define the distributional derivative  $Df$  of  $f$  to be a distribution satisfying  $\langle Df, \phi \rangle = -\langle f, \phi' \rangle$ , where  $\phi$  is a test function and  $\phi'$  is the ordinary derivative of  $\phi$ . With this definition, all distributions have derivatives of all orders and each derivative is a distribution.

Let  $(a, b)$  be an open interval in  $\mathbb{R}$ . We define

$$\mathcal{D}((a, b)) = \{\phi : (a, b) \rightarrow \mathbb{R} \mid \phi \in C_c^\infty \text{ and has compact support in } (a, b)\}.$$

The dual space of  $\mathcal{D}((a, b))$  is denoted by  $\mathcal{D}'((a, b))$ .

Define

$$B_C = \{F \in C([a, b]) \mid F(a) = 0\}.$$

$B_C$  is a Banach space with the uniform norm  $\|F\|_\infty = \max_{[a, b]} |F|$ .

Now we are able to introduce the definition of the  $D_{HK}$ -integral.

**Definition 2.1.** A distribution  $f$  is distributionally Henstock-Kurzweil integrable or briefly  $D_{HK}$ -integrable on  $[a, b]$  if  $f$  is the distributional derivative of a continuous function  $F \in B_C$ .

The  $D_{HK}$ -integral of  $f$  on  $[a, b]$  is denoted by  $(D_{HK}) \int_a^b f = F(b)$ , where  $F$  is called the primitive of  $f$  and “ $(D_{HK}) \int$ ” denotes the  $D_{HK}$ -integral. Notice that if  $f \in D_{HK}$  then  $f$  has many primitives in  $C([a, b])$ , but  $f$  has exactly one primitive in  $B_C$ .

The space of  $D_{HK}$ -integrable distributions is defined by

$$D_{HK} = \{f \in \mathcal{D}'((a, b)) \mid f = DF \text{ for some } F \in B_C\}.$$

With this definition, if  $f \in D_{HK}$  then, for all  $\phi \in \mathcal{D}((a, b))$ ,

$$\langle f, \phi \rangle = \langle DF, \phi \rangle = -\langle F, \phi' \rangle = -\int_a^b F \phi'.$$

Now we show that  $D_{HK}$ -integral includes the  $HK$ -integral, and hence includes Lebesgue and Riemann integrals (See [8–10] for details). So our results will extend the previous ones in [3, 4].

**Remark 2.2.** In [9], Lee pointed out that if  $F$  is a continuous function and pointwise differentiable nearly everywhere on  $[a, b]$ , then  $F$  is  $ACG^*$ . So the primitive  $F$  of the  $HK$ -integrable function  $f$  is generalized absolutely continuous or briefly  $ACG^*$  (see [9, 10]). Furthermore, if  $F$  is a continuous function which is differentiable nowhere on  $[a, b]$ , then  $F$  is not  $ACG^*$ . Therefore, if  $F \in C([a, b])$  but is differentiable nowhere on  $[a, b]$ , then  $DF$  exists and is  $D_{HK}$ -integrable but not  $HK$ -integrable. Conversely, if  $F$  is  $ACG^*$  then it belongs to  $C([a, b])$ . Hence  $F'$  is not only  $HK$ -integrable but also  $D_{HK}$ -integrable.

Let us introduce some basic properties of the distributional Henstock-Kurzweil integral needed later.

**Lemma 2.3** ([14, Theorem 4], Fundamental Theorem of Calculus).

(a) Let  $f \in D_{HK}$ , and define  $F(t) = (D_{HK}) \int_a^t f$ . Then  $F \in B_C$  and  $DF = f$ .

(b) Let  $F \in C([a, b])$ . Then  $(D_{HK}) \int_a^t DF = F(t) - F(a)$  for all  $t \in [a, b]$ .

From definition of the positive measure, we impose a partial ordering on  $D_{HK}$ : for  $f, g \in D_{HK}$ , we say that  $f \succeq g$  (or  $g \preceq f$ ) if and only if  $f - g$  is a positive measure on  $[a, b]$ . By this definition, if  $f, g \in D_{HK}$  then  $(D_{HK}) \int_a^b f \geq (D_{HK}) \int_a^b g$ , whenever  $f \succeq g$ . See [1] for details.

It is shown that the following result holds.

**Lemma 2.4** ([1, Corollary 1]). If  $f_1, f_2, f_3 \in \mathcal{D}'((a, b))$ ,  $f_1 \preceq f_2 \preceq f_3$ , and if  $f_1$  and  $f_3$  are  $D_{HK}$ -integrable, then  $f_2$  is also  $D_{HK}$ -integrable.

We now define the Alexiewicz norm by  $\|f\| = \|F\|_\infty = \max_{[a, b]} |F|$  for  $f \in D_{HK}$  and  $F \in B_C$  with  $DF = f$ . Also, we say a sequence  $\{f_n\} \subset D_{HK}$  converges strongly to  $f \in D_{HK}$  if  $\|f_n - f\| \rightarrow 0$  as  $n \rightarrow \infty$ .

Then the following holds.

**Lemma 2.5** ([14, Theorem 2]). *With the Alexiewicz norm,  $D_{HK}$  is a Banach space.*

The Lebesgue integral has very important applications since the space  $L^1$  of Lebesgue integrable functions is a Banach space and there are excellent convergence theorems. We have shown that  $D_{HK}$  is a Banach space, and by [14],  $\mathcal{BV}$  is the dual space of  $D_{HK}$ . Next we will look at convergence theorems of  $D_{HK}$ -integral.

**Lemma 2.6** ([1, Corollary 5, Dominated convergence theorem for the  $D_{HK}$ -integral]). *Let  $\{f_n\}_{n=0}^\infty$  be a sequence in  $D_{HK}$  such that  $f_n \rightarrow f$  in  $\mathcal{D}'$ . Suppose there exist  $f_-, f_+ \in D_{HK}$  satisfying  $f_- \preceq f_n \preceq f_+, \forall n \in \mathbb{N}$ . Then  $f \in D_{HK}$  and  $\lim_{n \rightarrow \infty} (D_{HK}) \int_a^b f_n = (D_{HK}) \int_a^b f$ .*

If  $g : [a, b] \rightarrow \mathbb{R}$ , its variation is  $Vg = \sup \sum_n |g(t_n) - g(s_n)|$ , where the supremum is taken over every sequence  $\{(t_n, s_n)\}$  of disjoint intervals in  $[a, b]$ . If  $Vg < \infty$  then  $g$  is called a function with bounded variation. Denote the set of functions with bounded variation by  $\mathcal{BV}$ . As it is known that the dual space of  $D_{HK}$  is  $\mathcal{BV}$  (see details in [14]), the next results have been presented.

**Lemma 2.7** ([14, Definition 6, Integration by parts]). *Let  $f \in D_{HK}$ , and  $g \in \mathcal{BV}$ . Define  $fg = DH$ , where  $H(t) = F(t)g(t) - \int_a^t Fdg$ . Then  $fg \in D_{HK}$  and*

$$\int_a^b fg = F(b)g(b) - \int_a^b Fdg.$$

**Lemma 2.8** ([17, Theorem 2]). *Let  $f(t)$  be a vector-valued function such that  $f(t)$  is Henstock-Stieltjes integrable with respect to  $g(t)$  on  $[a, b]$ . If  $g(t)$  satisfies Lipschitz condition on  $[a, b]$ , then  $F(t)$  is continuous on  $[a, b]$ , where*

$$F(t) = (HS) \int_a^t f(s)dg(s).$$

**Lemma 2.9** ([18, Theorem 2]). *Let  $g(t)$  be a function of bounded variation on  $[a, b]$  and,  $\{f_n(t)\}_{n=1}^\infty$  be a sequence of vector-valued functions such that  $f_n(t)$  is Henstock-Stieltjes integrable with respect to  $g(t)$  on  $[a, b]$ . If  $f_n(t)$  converges uniformly to  $f(t)$ , then  $f(t)$  is Henstock-Stieltjes integrable with respect to  $g(t)$  on  $[a, b]$  and*

$$\lim_{n \rightarrow \infty} (HS) \int_a^b f_n(t)dg(t) = (HS) \int_a^b f(t)dg(t).$$

### 3. Main results

In this section, we shall consider the existence of solutions and investigate the topological characterization of the set of solutions of the MDE (1.1). The main results are Theorems 3.3 and 3.5.

Define

$$B = \{x \in C([a, b]) : \|x - x_0\|_\infty \leq r, r > 0\},$$

$$E = \{(x, t) : t \in [a, b], x \in B\}.$$

Now, we impose some assumptions on  $f$  and  $g$  in (1.1).

- (C<sub>1</sub>)  $f(x(\cdot), \cdot)$  is  $D_{HK}$ -integrable for each  $x \in B$ ;
- (C<sub>2</sub>)  $f(\cdot, t)$  is continuous for all  $t \in [a, b]$ ;
- (C<sub>3</sub>) There exist  $f_-, f_+ \in D_{HK}$  such that  $f_-(\cdot) \preceq f(x, \cdot) \preceq f_+(\cdot)$  for each  $x \in B$ ;
- (C<sub>4</sub>)  $g(x(\cdot), \cdot)$  is Henstock-Stieltjes integrable for each  $x \in B$ ;
- (C<sub>5</sub>)  $g(\cdot, t)$  is continuous for all  $t \in [a, b]$ ;
- (C<sub>6</sub>) There exist  $g_-, g_+$  such that

$$g_-(\cdot) \leq g(x, \cdot) \leq g_+(\cdot), \quad (x, t) \in E,$$

where  $g_-, g_+$  are Henstock-Stieltjes ( $HS$ ) integrable.

Here  $f(x(\cdot), \cdot)$  is  $D_{HK}$ -integrable if each component of  $f(x(\cdot), \cdot)$  is  $D_{HK}$ -integrable. In the space of  $D_{HK}$ ,  $f \preceq g$  if and only if  $f^i \preceq g^i$ , where  $f^i, g^i$  denote the  $i$ th component of  $f, g$ ,  $i = 1, 2, \dots, n$ .

The following two lemmas will play an important role to study the existence of solutions of MDE (1.1).

**Lemma 3.1.** *Under assumptions (C<sub>1</sub>) – (C<sub>6</sub>), the solutions of MDE (1.1) satisfy the integral equation for all  $t \in [a, b]$ ,*

$$(3.1) \quad x(t) = x_0 + (D_{HK}) \int_a^t f(x(s), s) ds + (HS) \int_a^t g(x(s), s) du(s).$$

*The converse also holds.*

*Proof.* It follows from (C<sub>3</sub>), (C<sub>6</sub>) and Lemma 2.4 that the two integrals ( $D_{HK}$ )  $\int_a^t f(x(s), s) ds$  and ( $HS$ )  $\int_a^t g(x(s), s) du(s)$  exist. Denote  $X^i = x^i(\cdot)$ , where  $x^i(\cdot)$  is the  $i$ th component of  $x(\cdot)$ .

Let  $x(\cdot)$  satisfies (3.1). Obviously,  $x(a) = x_0$ . Since  $\varphi \in C_c^\infty((a, b))$ , define

$$(3.2) \quad \langle X^i, \varphi \rangle = \int_a^b \left[ x_0^i + \int_a^t f^i(x(s), s) ds + \int_a^t g^i(x(s), s) du(s) \right] \varphi(t) dt, \quad i = 1, 2, \dots, n.$$

Therefore,

$$\begin{aligned}
 \langle DX^i, \varphi \rangle &= -\langle X^i, \varphi' \rangle \\
 &= -\int_a^b \left[ x_0^i + \int_a^t f^i(x(s), s) ds \right. \\
 &\quad \left. + \int_a^t g^i(x(s), s) du(s) \right] \varphi'(t) dt, \\
 &\quad i = 1, 2, \dots, n.
 \end{aligned}
 \tag{3.3}$$

Since  $\varphi \in C_c^\infty((a, b))$ , using Lemma 2.7, we have

$$\begin{aligned}
 -\int_a^b \left[ x_0^i + \int_a^t f^i(x(s), s) ds \right] \varphi'(t) dt &= \int_a^b \varphi(t) f^i(x(t), t) dt, \\
 &\quad i = 1, 2, \dots, n.
 \end{aligned}
 \tag{3.4}$$

Moreover,  $\int_a^t g^i(x(s), s) du(s)$  is right continuous on  $[a, b]$ ,

$$\begin{aligned}
 &\int_a^b \left[ \int_a^t g^i(x(s), s) du(s) \right] \varphi'(t) dt \\
 &= -\int_a^b \varphi(t) d \left[ \int_a^t g^i(x(s), s) du(s) \right] \\
 &= -\int_a^b g^i(x(t), t) \varphi(t) du(t), \quad i = 1, 2, \dots, n.
 \end{aligned}
 \tag{3.5}$$

It follows from (3.4), (3.5) that

$$\begin{aligned}
 \langle DX^i, \varphi \rangle &= \int_a^b \varphi(t) f^i(x(t), t) dt + \int_a^b g^i(x(t), t) \varphi(t) du(t), \\
 &\quad i = 1, 2, \dots, n.
 \end{aligned}
 \tag{3.6}$$

Thus,  $DX^i$  is identified with  $f^i(x(t), t) + g^i(x(t), t)Du$ ,  $i = 1, 2, \dots, n$ . Therefore,  $x(\cdot)$  satisfies (1.1).

On the other side, suppose that  $x(\cdot)$  satisfies (1.1). According to assumptions  $(C_1)$  and  $(C_4)$ , by integrating equation (1.1) from  $a$  to  $t$ , we obtain that the integral equation (3.1) holds.

Hence the MDE (1.1) is equivalent to the integral equation (3.1).

This completes the proof.  $\square$

**Lemma 3.2** ([12, Theorem 6.15]). *Let  $M$  be a convex, closed subset of a normed space  $X$ . Let  $T$  be a continuous map of  $M$  into a compact subset  $K$  of  $M$ . Then  $T$  has a fixed point.*

The first main result based on Lemma 3.2, is the existence of solutions of the MDE (1.1).

**Theorem 3.3.** *Assume that the functions  $f, g$  in (1.1) satisfy assumptions  $(C_1) - (C_6)$ . Then there exists at least one solution of the MDE (1.1).*

*Proof.* We first assume that

$$h_1 = \max_{t \in [a, b]} \left\{ \left| (D_{HK}) \int_a^t f_- \right|, \left| (D_{HK}) \int_a^t f_+ \right| \right\}.$$

Then for each  $t \in [a, b]$ , we have

$$(3.7) \quad -h_1 \leq (D_{HK}) \int_a^t f_-, \quad (D_{HK}) \int_a^t f_+ \leq h_1.$$

According to Lemma 2.8, let

$$h_2 = \max_{t \in [a, b]} \left\{ \left| (HS) \int_a^t g_- du \right|, \left| (HS) \int_a^t g_+ du \right| \right\}.$$

It follows from Lemma 2.4 and assumptions  $(C_3)$  and  $(C_6)$  that

$$(3.8) \quad \left| (D_{HK}) \int_a^t f(x(s), s) ds \right| \leq h_1,$$

$$(3.9) \quad \left| (HS) \int_a^t g(x(s), s) du(s) \right| \leq h_2.$$

Let  $B = \{x \in C([a, b]) : \|x - x_0\|_\infty \leq r, r = h_1 + h_2 > 0\}$ . For each  $x \in B$ , we define an operator  $T : B \rightarrow C([a, b])$ , satisfying

$$(3.10) \quad \begin{aligned} T(x)(t) = & x_0 + (D_{HK}) \int_a^t f(x(s), s) ds \\ & + (HS) \int_a^t g(x(s), s) du(s), t \in [a, b]. \end{aligned}$$

Now we prove this theorem in three steps.

Step 1:  $T : B \rightarrow B$ .

By (3.10),

$$(3.11) \quad \begin{aligned} \|T(x) - x_0\|_\infty = & \max_{t \in [a, b]} \left| (D_{HK}) \int_a^t f(x(s), s) ds \right. \\ & \left. + (HS) \int_a^t g(x(s), s) du(s) \right|. \end{aligned}$$

Furthermore, by using (3.7)-(3.11), we have

$$(3.12) \quad \begin{aligned} \|T(x) - x_0\|_\infty \leq & \max_{t \in [a, b]} \left| (D_{HK}) \int_a^t f(x(s), s) ds \right| \\ & + \max_{t \in [a, b]} \left| (HS) \int_a^t g(x(s), s) du(s) \right| \\ \leq & h_1 + h_2 = r. \end{aligned}$$



This implies that  $T(x) \in B$ . Hence,  $T(B) \subseteq B$ .

Step 2:  $T(B)$  is equi-continuous.

Let  $t_1, t_2 \in [a, b], t_1 < t_2, x \in B$ . By  $(C_3)$  and  $(C_6)$ , one has

$$\begin{aligned}
 |Tx(t_1) - Tx(t_2)| &\leq \left| (D_{HK}) \int_{t_1}^{t_2} f(x(s), s) ds \right| \\
 &\quad + \left| (HS) \int_{t_1}^{t_2} g(x(s), s) du(s) \right| \\
 (3.13) \qquad &\leq \left| (HS) \int_{t_1}^{t_2} g_-(s) du(s) \right| \\
 &\quad + \left| (HS) \int_{t_1}^{t_2} g_+(s) du(s) \right| \\
 &\quad + \left| (D_{HK}) \int_{t_1}^{t_2} f_- \right| + \left| (D_{HK}) \int_{t_1}^{t_2} f_+ \right|.
 \end{aligned}$$

Since  $g_-, g_+ \in HS$  and  $u(t)$  is a right continuous function of bounded variation on  $[a, b]$ , by Lemma 2.8, we know that  $(HS) \int_{t_1}^{t_2} g_- du(s)$  and  $(HS) \int_{t_1}^{t_2} g_+ du(s)$  are continuous on  $[a, b]$  and, hence, uniformly continuous on  $[a, b]$ . Besides,  $f_-, f_+ \in D_{HK}$ , the primitives of them are also uniformly continuous on  $[a, b]$ . Thus,  $T(B)$  is equiuniformly continuous on  $[a, b]$  for all  $x \in B$ .

In view of Ascoli-Arzelà theorem,  $T(B)$  is relatively compact.

Step 3:  $T$  is a continuous mapping.

Let  $x \in B, \{x_n\}_{n \in \mathbb{N}}$  be a sequence in  $B$  and  $x_n \rightarrow x$  as  $n \rightarrow \infty$ .

By  $(C_2)$  and  $(C_5)$ , one has

$$\begin{aligned}
 f(x_n, \cdot) &\rightarrow f(x, \cdot) \quad \text{as } n \rightarrow \infty, \\
 g(x_n, \cdot) &\rightarrow g(x, \cdot) \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

According to assumption  $(C_1)$  and Lemma 2.6, we have

$$\lim_{n \rightarrow \infty} (D_{HK}) \int_a^t f(x_n(s), s) ds = (D_{HK}) \int_a^t f(x(s), s) ds, \quad t \in [a, b].$$

It follows from Lemma 2.9 that

$$\lim_{n \rightarrow \infty} (HS) \int_a^t g(x_n(s), s) du(s) = (HS) \int_a^t g(x(s), s) du(s).$$

Then  $\lim_{n \rightarrow \infty} T(x_n) = T(x)$ , so  $T$  is continuous.

Hence  $T$  satisfies the hypotheses of Lemma 3.2, then there exists a fixed point of  $T$  which is a solution of the MDE (1.1). □

We now consider the topological characterization of the set of solutions of the MDE (1.1).

Let  $C_u(K, Y)$  be the space of all continuous mappings  $x : K \rightarrow Y$ , where  $K$  is a compact convex subset of a normed space and  $Y$  is a metric space equipped

with the topology of uniform convergence. Denote by  $B(t_0, \varepsilon)$  the closed ball with center  $t_0$  and radius  $\varepsilon$ . Denote by  $x|_A$  the restriction of the map  $x$  to  $A$ .

Now we present the well-known Vidossich theorem.

**Lemma 3.4** ([16, Corollary 1.2, Vidossich theorem]). *Let  $K$  be a compact convex subset of a normed space,  $Y$  a closed convex subset of a Banach space  $Y_0$ ,  $F$  a compact mapping  $C_u(K, Y) \rightarrow C_u(K, Y)$ . Suppose that there exist  $t_0 \in K$  and  $y_0 \in Y$  such that the following two conditions hold.*

(i)  $F(x)(t_0) = y_0 \quad (x \in C(K, Y))$ .

(ii) For every  $\varepsilon > 0$ ,

$$x|_{K_\varepsilon} = y|_{K_\varepsilon} \Rightarrow F(x)|_{K_\varepsilon} = F(y)|_{K_\varepsilon} \quad (x, y \in C(K, Y)),$$

where  $K_\varepsilon = B(t_0, \varepsilon) \cap K$ . Then the set of fixed points of  $F$  is an  $R_\delta$ .

Recall that if a set is an  $R_\delta$ , it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts. Furthermore, G. Vidossich pointed out that  $R_\delta$  is a nonempty, compact and connected set in [16].

The second main result concerned with the structure of the set of solutions of the MDE (1.1) can be stated as follow.

**Theorem 3.5.** *Under the above assumptions  $(C_1) - (C_6)$ , the solution set of MDE (1.1) on  $[a, b]$  is an  $R_\delta$ .*

*Proof.* In Theorem 3.3, we have verified that the mapping  $T : B \rightarrow B$  is compact. Hence, conditions (i) and (ii) of Lemma 3.4 hold. Therefore, the solution set of the MDE (1.1) is an  $R_\delta$ .  $\square$

#### 4. Examples

In this section, we first show a lemma which is obtained by Y. T. Xu in literature [19]. And then, we will give two examples to explain that Theorem 3.3 in Section 3 is more generalized than Theorem 6.2 in [19] and Theorem 1 in [5].

**Lemma 4.1** ([19, Theorem 6.2]). *Let  $f$  and  $g$  in (1.1) satisfy the following conditions:*

- (1)  $f(x, t)$  is measurable in  $t$  for each  $x$  and continuous in  $x$  for all  $t$ ;
- (2) there exists a Lebesgue integrable function  $f(t)$  such that

$$|f(x, t)| \leq f(t), \quad (x, t) \in E;$$

- (3)  $g(x, t)$  is continuous in  $x$  for all  $t$  and  $du$ -integrable for each  $x \in \mathcal{BV}([a, b], B)$ ;

- (4) there exists a  $dV_u$ -integrable function  $g(t)$  such that

$$|g(x, t)| \leq g(t), \quad (x, t) \in E,$$

where  $V_u$  denotes the total variation function of  $u$ .

Then there exists at least one solution of the MDE (1.1).

We can see that if  $f$  and  $g$  in (1.1) satisfy assumptions (1) – (4) in Lemma 4.1, then  $f$  and  $g$  satisfy  $(C_1)$ – $(C_6)$ . According to Theorem 3.3, the MDE (1.1) has at least one solution. However, the following example shows that Lemma 4.1 is only a special case of Theorem 3.3.

**Example 1.** Consider the following MDE

$$(4.1) \quad Dx(t) = r(t) + x(t) + \frac{2}{t+1}x(t)DH(t), \quad x(0) = 0, \quad t \in [0, 2],$$

where  $x(t)$  is continuous and  $H(t)$  is the Heaviside function, i.e.,  $H(t) = 0$  if  $t < 0$  and  $H(t) = 1$  if  $t \geq 0$ , then  $DH = \delta$  is the Dirac measure. Let  $r(t)$  be the distributional derivative of Riemann function  $R(t) = \sum_{n=1}^{\infty} \frac{\sin n^2 \pi t}{n^2}$ . Then equation (4.1) has at least one solution. Moreover, the solution set is an  $R_\delta$ .

We can assume that  $f(x, t) = r(t) + x(t)$ ,  $g(x, t) = \frac{2}{t+1}x(t)$  and  $u(t) = H(t)$ , then (4.1) can be regarded as the MDE (1.1).

It is easy to see that  $f(x, t)$  is continuous in  $x$  and  $D_{HK}$ -integrable on  $[0, 2]$ . Since  $x(t)$  is continuous, there exists  $M_0 > 0$  such that  $-M_0 \leq x(t) \leq M_0$ . Note that  $r(t)$  is  $D_{HK}$ -integrable, we have  $r(t) \pm M_0 \in D_{HK}$ . Moreover,

$$r(t) - M_0 \preceq f(x, t) \preceq r(t) + M_0, \quad t \in [0, 2].$$

Therefore,  $f(x, t)$  satisfies  $(C_1)$  –  $(C_3)$ .

Obviously,  $\frac{2}{t+1}x(t)$  is continuous in  $x$ . Because  $H(t)$  is of bounded variation and right continuous on  $[a, b]$ , so  $(HS) \int_a^t \frac{2}{s+1}x(s)dH(s)$  exists. Hence, we obtain that  $(C_4)$  –  $(C_6)$  hold.

Applying Theorem 3.3, we can obtain that equation (4.1) has at least one solution. Furthermore, by using Theorem 3.5, one can see that the solution set is an  $R_\delta$ .

**Remark 4.2.** Let  $F$  be the primitive of  $f$ . Since  $F$  is continuous but not ordinarily differentiable in  $[7]$ ,  $f$  is neither Henstock-Kurzweil integrable nor Lebesgue integrable in the above example. So  $f$  does not satisfy the hypotheses (1) – (2) in Lemma 4.1. Hence Theorem 6.2 in [19] is not applicable for the case of Example 1.

Moreover, we can use a similar method to verify that Theorem 3.3 is a generalization of Theorem 1 in [5], where  $g(x, t)$  is an  $n \times n$  matrix and  $u \in \mathbb{R}^n$  is a vector.

Let us give another example to illustrate this.

**Example 2.** Consider the following MDE

$$(4.2) \quad Dx = r(t) + x(t) + A(t)x(t)Du, \quad x(a) = 0, \quad t \in [a, b],$$

where

$$(4.3) \quad x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_n(t) \end{pmatrix}, r(t) = \begin{pmatrix} r_1(t) \\ r_2(t) \\ \dots \\ r_n(t) \end{pmatrix},$$

$$(4.4) \quad A(t) = \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & e^t \end{pmatrix}, u = \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{pmatrix}$$

and  $x_i(t)$  is continuous,  $r_i(t)$  is the distributional derivative of the Riemann function  $R_i(t) = \sum_{n=1}^{\infty} \frac{\sin n^2 \pi t}{n^2}$ ,  $Du_i = \delta$  is the Dirac measure,  $i = 1, 2, \dots, n$ . Then (4.2) has at least one solution.

*Proof.* According to (4.3) and (4.4), let

$$(4.5) \quad f(x, t) = r(t) + x(t) = \begin{pmatrix} x_1(t) + r_1(t) \\ x_2(t) + r_2(t) \\ \dots \\ x_n(t) + r_n(t) \end{pmatrix},$$

$$(4.6) \quad g(x, t) = A(t)x(t) = \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & e^t \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_n(t) \end{pmatrix}.$$

Since  $x_i(t)$  is continuous, there exists  $N_i > 0$  such that  $-N_i \leq x_i(t) \leq N_i$ . Note that  $r_i(t)$  is  $D_{HK}$ -integrable, thus  $r_i(t) \pm N_i \in D_{HK}$ . Let  $N = \max_{1 \leq i \leq n} N_i$ , one has

$$r(t) - N \preceq f(x, t) \preceq r(t) + N, \quad t \in [a, b].$$

It is clear that  $g(x, t)$  is continuous in  $x$ ,  $e^t x_i(t)$  is Henstock-Kurzweil integrable on  $[a, b]$  and  $u$  is bounded variation on  $[a, b]$ ,  $f(x, t)$  and  $g(x, t)$  satisfy the assumptions of Theorem 3.3. So equation (4.2) has at least one solution. However, equation (4.2) does not satisfy the conditions of Theorem 1 in [5]. Hence, Theorem 1 in [5] is not applicable.  $\square$

It follows from Examples 1 and 2 that Theorem 3.3 in Section 3 generalizes not only Theorem 6.2 in [19] but also Theorem 1 in [5].

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