ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 41 (2015), No. 2, pp. 375–380

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 2, pp. 375–380
Online ISSN: 1735-8515

ALMOST POWER-HERMITIAN RINGS

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(Communicated by Rahim Zaare-Nahandi)

ABSTRACT. In this paper we define a new type of rings "almost power-hermitian rings" (a generalization of almost hermitian rings) and establish several sufficient conditions over a ring R such that, every regular matrix admits a diagonal power-reduction.
Keywords: Primary:16U60, 16D10, Secondary: 16S50, 16S34.
MSC(2010): Diagonal power-reduction, exchange ring, regular ring, power-substitution property

1. Introduction

The purpose of this paper is to investigate power-diagonalizability of regular matrices over some rings. Let us say that an $m \times n$ matrix A over a ring R admits a diagonal reduction if there exist invertible matrices C and D such that CAD is a diagonal matrix, where by the diagonal matrix, we mean a matrix $(a_{ij})_{m\times n}$, such that $a_{ij} = 0$ for all $i \neq j$, we also say that A admits a diagonal power-reduction provided that there exists a $t \in \mathbb{N}$, such that $(a_{ij} I_t)_{mt\times nt}$ is a diagonal matrix. In 1861, Smith [9] proved that over the ring of integers, every matrix admits a diagonal reduction. This subject was investigated by Dickson [3], Wedderborn [12], Warden [11] and Jacobson [5] over some commutative and non commutative Euclidean domains and commutative principal ideal domains. Teichmuller [10] extended it over noncommutative principal ideal domains. The following question was proposed by Kaplansky in 1973 [6]. Kaplansky asked for a ring R and a matrix A over R, is it possible to find invertible matrices B and C such that BAC is a diagonal matrix.
In 1974 Levy in [7] proved that the answer of Kaplansky's question is yes for square matrices over serial rings, even though we have some rectangular matrices that don't admit diagonal reduction. For example if R isn't a Bezout ring we have some $a, b \in R$ such that $aR + bR \neq cR$ for all $c \in R$ and so $\begin{pmatrix} a & b \end{pmatrix}$

Article electronically published on April 29, 2015.
Received: 27 August 2013, Accepted: 5 March 2014.
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isn't diagonalized. Menal and Moncasi in [8] proved that all rectangular matrices over a given regular ring R are equivalent to a diagonal matrix if and only if the following cancellation law holds, for all finitely generated projective right R-modules:

$$2R \oplus A \cong R \oplus B \Longrightarrow R \oplus A \cong B.$$

In 1997 Ara, Goodearl, O'mera and Pardo in [1] extended that from regular rings to exchange rings and showed that every regular matrix over an exchange ring R admits a diagonal reduction if and only if $2R \oplus A \cong R \oplus B$ implies that $R \oplus A \cong B$ for all finitely generated projective R-modules A and B.
Here we show that an $m \times n$ matrix $A = (a_{ij})_{m\times n}$ may not admit a diagonal reduction, but we may find some $t \in \mathbb{N}$ such that $(a_{ij}I_t)$ admits a diagonal reduction. For example over a regular Dedekind domain which is not Bezout, There exist regular matrices that don't admit diagonal reduction, while by Lemma 8, we show that every regular matrix over such a ring admits diagonal power reduction.

Throughout this paper R is an associative ring. Ideals are two sided ideals and modules are right R-modules. We also use $M_n(R)$ for the ring of $n \times n$ matrices over R with identity I_n, $GL_n(R)$ the invertible $n \times n$ matrices over R and $FP(R)$ the class of finitely generated projective R-modules.

2. **Almost power-hermitian ring**

Definition 2.1. *Following [2], We say that R satisfies power-substitution in the case where $aR + bR = R$ with $a, b \in R$ implies that there exist $n \in \mathbb{N}$ and $Y \in M_n(R)$ such that $aI_n + bY \in GL_n(R)$.*

As we see in the following example a ring R may satisfy power-substitution while $M_n(R)$ for some $n \in \mathbb{N}$ does not. Recall that a ring R satisfies stable power-substitution in the case that for any $n \in \mathbb{N}$, $M_n(R)$ satisfies power-substitution.

Example 2.2. *Let $X = [-1, 1]^4$. Then $C_R(X)$ satisfies power-substitution, while $M_3(C_R(X))$ does not. (see [2], Example 10.4.1)*

Kaplansky defined a ring R to be right(left) Hermite provided that every $1 \times 2(2 \times 1)$ matrix over R admits a diagonal reduction (See [1]).

Definition 2.3. *A ring R is said to be an almost hermitian ring provided that every regular matrix over R admits a diagonal reduction (see [2]).*

Recall that a ring R satisfies the n-stable range condition if and only if $ax + b = 1$ with $a \in R^n$, $x \in^n R$, $b \in R$ implies that there exists some $y \in R^n$ such that $a + by \in R^n$ is unimodular.

Proposition 2.4. *Let R be a separative exchange ring. Then the following are equivalent:*

(1) *R satisfies the finite stable range condition.*
(2) *R satisfies 2-stable range condition.*
(3) *For any $A, B \in FP(R)$, $2R \oplus A \cong R \oplus B$ implies that $R \oplus A \cong B$*

Proof. See [2], Proposition 12.1.12. □

Example 2.5. *Let V be an infinite-dimensional vector space over a division ring D and let $R = End_D(V)$. Then R is not an almost hermitian ring.*

As V is an infinite dimensional vector space, we have $V \cong V \oplus V$, so

$$End_D(V) = Hom_D(V,V) \cong Hom_D(V \oplus V, V) \cong 2Hom_D(V,V) = 2End_D(V)$$
$$\Longrightarrow 2R \cong R.$$

It is well known that R is a regular ring and satisfies general comparability, So R is a separative exchange ring. If $2R \oplus A \cong R \oplus B$ implies that $R \oplus A \cong B$ for all finitely generated projective R-modules A, B, we have $R \cong 0$, that is a contradiction. Then R is not an almost hermitian ring.

Definition 2.6. *A ring R is said an almost power-hermitian ring provided that every regular matrix over R admits a diagonal power-reduction.*

Proposition 2.7. *Let R be an exchange ring satisfying stable power-substitution. Then R is an almost power-hermitian ring.*

Proof. Following [2], Theorem 10.4.14, every regular matrix over R admits a diagonal reduction, so R is an almost power-hermitian ring. □

Corollary 2.8. *Let R be an exchange ring satisfies n-stable range condition such that $M_n(R)$ has power-substitution property. Then R is an almost power-hermitian ring.*

Proof. As R satisfies stable power-substitution property, so R is an almost power-hermitian ring. □

Corollary 2.9. *Let R be a commutative exchange ring having power-substitution property, then R is an almost power-hermitian ring.*

Proof. We know from [4], Proposition 2.9, that R has stable power-substitution property, so the result is obtained from Proposition 2.7. □

Lemma 2.10. *Let R be a regular ring satisfying 2-stable range condition, then every matrix over R admits a diagonal power-reduction.*

Proof. It suffices to show that every 1×2 matrix over R admits a diagonal power-reduction.
Let $A = (a_{ij})_{1\times 2}$, for positive integer $t = 2$, we have $(a_{ij}I_t)_{2\times 4}$ is a 2×4 matrix, now we have $|2-4| = 2$ and the stable range of R is 2, so we deduce from [13] that $A = (a_{ij})_{1\times 2}$ admits diagonal power-reduction. □

Example 2.11. *Let $R = Z[\sqrt{-5}]$, then R isn't a hermitian ring.*

Set $A = \begin{pmatrix} 2 & 1+\sqrt{-5} \end{pmatrix}$ A doesn't admit diagonal reduction since $2R+(1+\sqrt{-5})R$ is a right ideal of R that can't be generated by only one element, and it's hard to investigate wether R is an almost power-hermitian ring, however R is a Dedekind domain so its stable range is 2.

We use $F(Q)$ to denote the category of of all nQ for $n \geq 0$ and all morphisms from F_1 to F_2 for each $F_1, F_2 \in F(Q)$ and $M(Q)$ to denote the category $(M, morM(Q), \circ)$, where M is the set of all nonnegative integers and for every $m, n \geq 0$, $mor_M(Q)(n, m)$ is the set of all $m \times n$ matrices over $End_R(Q)$ and $\circ$ is the usual product of matrices.

Lemma 2.12. *Let Q be a right R-module. Then there exist covariant functor $F : F(Q) \longrightarrow M(Q)$ and $G : M(Q) \longrightarrow F(Q)$ such that $FG = I_{M(Q)}$, $GF = I_{F(Q)}$ both identity functors*

Proof. It follows from [2], Lemma 14.1.2. □

Recall that a right R-homomorphism $f : 2Q \longrightarrow Q$ is said to admits a diagonal reduction if $F(f) \in M_{1\times2}(End(Q))$ admits a diagonal reduction.
For a right R-module Q and a nonnegative integer n, we use λ_n to denote the injection homomorphism from Q to $2nQ$, ($\lambda_n : Q \longrightarrow 2nQ$) and P_n to stand for the projection homomorphism from $2nQ$ to $2Q$, ($P_n : 2nQ \longrightarrow 2Q$).

Definition 2.13. *Let Q be an R-module and $f : 2Q \longrightarrow Q$ be an R-homomorphism. We say f admits a diagonal power-reduction where there exists $n \in \mathbb{N}$ such that $F(\lambda_n f P_n) \in M_{1\times2}(End(2nQ))$ admits a diagonal reduction. Here $\lambda_n : Q \longrightarrow 2nQ$ is injection and $P_n : 2nQ \longrightarrow 2Q$ is projection map.*

Recall that an R-homomorphism $f \in Hom(mQ, nQ)$ for nonnegative integers n, m is said to be regular if there exists an R-homomorphism $g : nQ \longrightarrow mQ$ such that $fogof = f$.

Lemma 2.14. *Let Q be a right R-module, and let $f : 2Q \longrightarrow Q$ be regular. Then f admits a diagonal power-reduction if and only if there exist $n \in \mathbb{N}$ and decomposition $ker(\lambda_n f P_n) \cong K_1 \oplus K_2$ such that $K_1 \oplus im(\lambda_n f P_n) \cong nQ \cong K_2$.*

Proof. Assume there is a decomposition $ker(\lambda_n f P_n) \cong K_1 \oplus K_2$ and $n \in \mathbb{N}$ such that $K_1 \oplus im(\lambda_n f P_n) \cong nQ \cong K_2$. Since f is regular, so $\lambda_n f P_n$ is regular, then we have $K_1 \oplus im(\lambda_n f P_n) \cong nQ \cong im(\lambda_n f P_n) \oplus coker(\lambda_n f P_n)$. By Lemma 2.12, there is a regular $g : nQ \longrightarrow nQ$ such that $ker(g) \cong K_1$, $im(g) \cong im(f\lambda_n f P_n)$ and $coker(g) \cong coker(\lambda_n f P_n)$. So we have $(g, 0) : 2nQ \longrightarrow nQ$ such that $ker(g, 0) = ker(g) \oplus nQ \cong K_1 \oplus K_2 \cong ker(\lambda_n f P_n)$, $im(g, 0) = im(g) \cong im(\lambda_n f P_n)$ and $coker(g, 0) = coker(g) \cong coker(\lambda_n f P_n)$. By virtue of [2], Lemma 7.2.1, $\lambda_n f P_n$ admits a diagonal power-reduction. Conversely, assume now that $f : 2Q \longrightarrow Q$ admits a diagonal power-reduction. By definition we have some $n \in \mathbb{N}$ such that $\lambda_n f P_n$ admits a diagonal reduction. Consider

$(g, 0)$ with $g : nQ \longrightarrow nQ$. Then g is also regular; hence, $nQ \cong ker(g) \oplus im(g)$. By [2], Lemma 7.2.1 again, $ker(\lambda_n f P_n) \cong ker(g, 0)$, $im(\lambda_n f P_n) \cong im(g, 0)$ and $coker(\lambda_n f P_n) \cong coker(g, 0)$. It is easy to check that $ker(g, 0) \cong ker(g) \oplus nQ$, $im(g, 0) \cong im(g)$, $coker(g, 0) \cong coker(g)$. Set $K_1 = ker(g)$ and $K_2 = nQ$. Then $ker(\lambda_n f P_n) \cong K_1 \oplus K_2$, $K_1 \oplus im(\lambda_n f P_n) \cong ker(g) \oplus im(g) \cong nQ \cong K_2$, as required. □

Lemma 2.15. *Let Q be a right R-module, and let $f : 2Q \longrightarrow Q$ be regular. Then f admits a diagonal power-reduction if and only if $nQ \oplus coker(\lambda_n f P_n) \cong ker(\lambda_n f P_n)$ for some $n \in \mathbb{N}$.*

Proof. Assume that f admits a diagonal power-reduction. According to Lemma 2.14, there exists a decomposition $ker(\lambda_n f P_n) \cong K_1 \oplus K_2$ such that $K_1 \oplus im(\lambda_n f P_n) \cong nQ \cong K_2$. Therefore $ker(\lambda_n f P_n) \cong K_1 \oplus K_2 \cong K_1 \oplus nQ \cong K_1 \oplus (coker(\lambda_n f P_n) \oplus im(f \lambda_n f P_n)) \cong (K_1 \oplus im(\lambda_n f P_n)) \oplus coker(\lambda_n f P_n) \cong coker(\lambda_n f P_n) \oplus nQ$. Assume that $nQ \oplus coker(\lambda_n f P_n) \cong ker(\lambda_n f P_n)$. Set $K_1 = coker(\lambda_n f P_n)$ and $K_2 = nQ$. Then $ker(\lambda_n f P_n) = K_1 \oplus K_2$ and $K_1 \oplus im(\lambda_n f P_n) \cong nQ \cong K_2$. It follows from lemma 2.14 that f admits a diagonal power-reduction. □

Theorem 2.16. *Let Q be a right R-module having finite exchange property and let $E = End_R(Q)$, also assume that there exists some $s \in \mathbb{N}$ such that $2sQ \oplus A \cong sQ \oplus B$ for any $A, B \in FP(R)$ implies that $sQ \oplus A \cong B$, then E is an almost power-hermitian ring.*

Proof. Let $f : nQ \longrightarrow mQ$ be regular, so $\lambda_s f P_s$ is regular. Then $nsQ \cong ker(\lambda_s f P_s) \oplus I$ and $msQ \cong I \oplus coker(\lambda_s f P_s)$ for a right R-module I. This implies that $msQ \oplus ker(\lambda_s f P_s) \cong nsQ \oplus coker(\lambda_s f P_s)$. Assume that $m = n(\geq 2)$. Given any decompositions $nsQ \cong K \oplus I \cong I \oplus C$, then K has the finite exchange property. Hence, we have $K = X_1 \oplus X_2$, $I = Y_1 \oplus Y_2$ such that $X_1 \oplus Y_1 \cong I$ and $X_2 \oplus Y_2 \cong C$. Thus $nsQ \oplus Y_2 \cong nsQ \oplus X_1$. By hypothesis, we get $sQ \oplus Y_2 \cong sQ \oplus X_1$. Further, $sQ = R_1 \oplus R_2$, $Y_2 = C_1 \oplus C_2$ such that $R_1 \oplus C_1 \cong sQ$ and $R_2 \oplus C_2 \cong X_1$. This implies that $nsQ \cong R_2 \oplus (I \oplus X_2 \oplus C_2) \cong (I \oplus X_2 \oplus C_2) \oplus C_1$. One can easily check that $2sQ \oplus (n-2)sQ \oplus R_1 \cong sQ \oplus I \oplus X_2 \oplus C_2$. By hypothesis, we get $(n-1)sQ \oplus R_1 \cong I \oplus X_2 \oplus C_2$. Consequently, we have $R_2 \cong R_2 \oplus 0 \oplus \ldots \oplus 0$, $I \oplus X_2 \oplus C_2 \cong R_1 \oplus sQ \oplus \ldots \oplus sQ$, $C_1 \cong C_1 \oplus 0 \oplus \ldots \oplus 0$ with $R_2 \oplus R_1 \cong C_1 \oplus R_1 \cong sQ$. so $\lambda_s f P_s$ admits a diagonal reduction. If $n > m$, then $msQ \oplus ker(\lambda_s f P_s) \oplus 2sQ \oplus (n-2)sQ \oplus coker(\lambda_s f P_s)$. By hypothesis, $ker(\lambda_s f P_s) \cong (n-m)sQ \oplus coker(\lambda_s f P_s)$. If $n < m$, $2sQ \oplus (m-2)sQ \oplus ker(\lambda_s f P_s) \cong nsQ \oplus coker(\lambda_s f P_s)$. By hypothesis, $coker(\lambda_s f P_s) \cong (m-n)sQ \oplus ker(\lambda_s f P_s)$. Similar to [2], Proposition 7.2.11 and Lemma 14.1.4, $\lambda_s f P_s$ admits a diagonal reduction, as required. □

Acknowledgment

We thank Huanyin Chen for reading an earlier version of this manuscript and for his many helpful comments and suggestions.

References

[1] P. Ara, K. R. Goodearl, K. C. O'Meara and E. Pardo, Diagonalization of matrices over regular rings, *Linear Algebra Appl.* **265** (1997) 147–163.

[2] H. Chen, Rings related to stable range conditions, Series in Algebra, 11, World Scientific Publishing Co. Pte. Ltd., Hackensack, 2011.

[3] L. E. Dickson, Algebras and Their Arithmetics, Univercity of Chicago Press, Chicago, 1923.

[4] K. R. Goodearl, Power-cancellation of groups and modules, *Pacific J. Math.* **64** (1976), no. 2, 387–411.

[5] N. Jacobson, Pseudo-linear transformations, *Ann. of Math. (2)* **38** (1937), no. 2, 484–507.

[6] I. Kaplansky, Elementary divisors and modules, *Trans. Amer. Math. Soc.* **66** (1949) 464–491.

[7] L. S. Levy, Sometimes only square matrices can be diagonalized, *Proc. Amer. Math. Soc.* **52**(1975) 18–22.

[8] P. Menal and J. Moncasi, On regular rings with stable range 2, *J. Pure Appl. Algebra* **24** (1982), no. 1, 25–40.

[9] H. J. S. Smith, On systems of linear indeterminate equations and congruences, *Philos. Trans. Roy. Soc. London* **151** (1861) 293–326.

[10] O. Teichmuller, Der Elementeilsatz fur nicktkommutative Ringe, *Abh. Preuss. Acad. Wiss. Phys. Math. Kl.* (1973) 169–177.

[11] B. L. V. Waerden, Modern Algebra, Springer, Berlin-New York, 1930.

[12] J. H. M. Wedderburn, Non-commutative domains of integrity, *J. Reine Angew Math.* **167** (1932), 129–141.

[13] B. V. Zabavsky, Diagonalization of matrices, *Mat. Stud.* **23** (2005), no. 1, 3–10.

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