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ON SPECTRAL RADIUS OF STRONGLY CONNECTED DIGRAPHS

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Abstract. It is known that the directed cycle of order n uniquely achieves the minimum spectral radius among all strongly connected digraphs of order $n \geq 3$. In this paper, among others, we determine the digraphs which achieve the second, the third and the fourth minimum spectral radii respectively among strongly connected digraphs of order $n \geq 4$.
Keywords: Spectral radius, strongly connected digraph, nonnegative irreducible matrix, bicyclic digraph.
MSC(2010): Primary: 05C50; Secondary: 15A18.

1. Introduction

We consider digraphs without loops and multiple arcs. Let D be a digraph of order n with vertex set $V(D)$ and arc set $E(D)$. Let $V(D) = \{v_1, v_2, \dots, v_n\}$. The adjacency matrix of D is the $(0,1)$-matrix $A(D) = (a_{ij})$ of order n where $a_{ij} = 1$ if there is an arc from v_i to v_j, and $a_{ij} = 0$ otherwise. The eigenvalues of D are the eigenvalues of $A(D)$. The spectral radius of D is the largest modulus of an eigenvalue of D, denoted by $\rho(D)$. Obviously, the eigenvalues of D are the roots of the characteristic polynomial of D, denoted by $P(D, x)$, defined to be the characteristic polynomial of the matrix $A(D)$, which is $\det(xI_n - A(D))$, where I_n is the identity matrix of order n. The spectra of digraphs have been studied to some extent, see e.g., [3–5], and for a survey, see [1].

A digraph D is strongly connected if for every pair $x, y \in V(D)$, there exists a directed path from x to y and a directed path from y to x. D is called a strongly connected bicyclic digraph if D is strongly connected with $|E(D)| = |V(D)|+1$. For $n \geq 3$, let $\mathbb{B}_n$ be the set of strongly connected bicyclic digraphs of order n.

Note that D is strongly connected if and only if $A(D)$ is irreducible. It follows from the Perron-Frobenius Theorem that if D is strongly connected,

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then $\rho(D)$ is an eigenvalue of D and there is a corresponding eigenvector whose coordinates are all positive.

Let P_n be a directed path of order n. If $P_n = u_1u_2\ldots u_n$, then u_1 is the initial vertex, and u_n is the terminal vertex of P_n. The θ-digraph with parameters a, b and c with $a \le b$, denoted by $\theta(a,b,c)$, consists of three directed paths P_{a+2}, P_{b+2} and P_{c+2} such that the initial vertex of P_{a+2} and P_{b+2} is the terminal vertex of P_{c+2}, and the initial vertex of P_{c+2} is the terminal vertex of P_{a+2} and P_{b+2}. These three directed paths are called the basic directed paths of $\theta(a,b,c)$. A ∞-digraph with parameters k and l with $k \le l$, denoted by $\infty(k,l)$, consists of two directed cycles of lengths k and l respectively with exactly one vertex in common. Note that any digraph in B_n is a θ-digraph or a ∞-digraph.

Recently, Lin and Shu [4] showed that $\theta(0,1,n-3)$ (respectively, $\infty(2,n-1)$) is the unique digraph in $\mathbb{B}_n$ which achieve the minimum (respectively, maximum) spectral radius for $n \ge 4$. Let C_n be the directed cycle of order n. Note that C_n uniquely achieves the minimum spectral radius among all strongly connected digraphs of order $n \ge 3$. This is because the spectral radius of an irreducible nonnegative matrix is bounded below by the minimum row sum, and it is attained if and only if all row sums are equal [6]. Lin and Shu [4] proposed the following problem.

Problem 1.1. *Does $\theta(0,1,n-3)$ achieve the second minimum spectral radius among all n-vertex strongly connected digraphs for $n \ge 4$?*

In this paper, we determine the unique digraphs which achieve the second, the third and the fourth minimum spectral radii respectively among strongly connected digraphs of order $n \ge 4$, and thus we answer Problem 1.1 affirmatively. To do this, we also determine the unique digraphs in $\mathbb{B}_n$ with the second and the third minimum spectral radii respectively for $n \ge 4$. Finally, we determine the unique digraph in $\mathbb{B}_n$ with the second maximum spectral radius for $n \ge 4$.

2. Preliminaries

We list some lemmas that will be used in our proofs.

Lemma 2.1. *[4] For $n \ge 4$, $\theta(0,1,n-3)$ is the unique digraph in $\mathbb{B}_n$ which achieves the minimum spectral radius, and $\infty\left(\left\lfloor \frac{n+1}{2}\right\rfloor, \left\lceil \frac{n+1}{2}\right\rceil\right)$ is the unique ∞-digraph in $\mathbb{B}_n$ which achieves the minimum spectral radius among ∞-digraphs.*

The following lemma was proved in [4] for $c \ge 1$. However, its proof also holds for $c = 0$.

Lemma 2.2. *[4] If $b \ge 1$, then $\rho(\theta(a,b,c)) > \rho(\theta(a,b-1,c+1))$. If $a \ge 1$, then $\rho(\theta(a,b,c)) > \rho(\theta(a-1,b,c+1))$.*

The following lemma was given in [4]. It is a consequence of the well known coefficients theorem for digraphs, see e.g., [2, Theorem 1.2, p. 36].

Lemma 2.3. *$P(\theta(a,b,c),x) = x^n - x^a - x^b$ with $n = a+b+c+2$, and $P(\infty(k,l),x) = x^n - x^{k-1} - x^{l-1}$ with $n = k+l-1$.*

Lemma 2.4. *For $n \geq 4$, $\rho(\infty(\lfloor \frac{n+1}{2} \rfloor, \lceil \frac{n+1}{2} \rceil)) > \rho(\theta(0,2,n-4))$.*

Proof. Let $D_1 = \theta(0,2,n-4)$ and $D_2 = \infty(\lfloor \frac{n+1}{2} \rfloor, \lceil \frac{n+1}{2} \rceil)$. By Lemma 2.3, $P(D_1,x) = x^n - x^2 - 1$ and $P(D_2,x) = x^n - x^{\lfloor \frac{n-1}{2} \rfloor} - x^{\lceil \frac{n-1}{2} \rceil}$. For $x \geq \rho(D_2) > 1$, $P(D_1,x) - P(D_2,x) = -x^2 - 1 + 2x^{\frac{n-1}{2}} \geq -x^2 - 1 + 2x^2 = x^2 - 1 > 0$ if n is odd and $P(D_1,x) - P(D_2,x) = -x^2 - 1 + x^{\frac{n}{2}-1} + x^{\frac{n}{2}} \geq -x^2 - 1 + x + x^2 = x - 1 > 0$ if n is even. Thus $\rho(D_2) > \rho(D_1)$. $\square$

Lemma 2.5. *$\rho(\theta(0,2,n-4))$ is strictly decreasing in $n \geq 4$.*

Proof. Suppose that $n_1 > n_2 \geq 4$. By Lemma 2.3, $P(\theta(0,2,n_1-4),x) - P(\theta(0,2,n_2-4),x) = x^{n_1} - x^{n_2} > 0$ for $x \geq \rho(\theta(0,2,n_2-4)) > 1$. Thus $\rho(\theta(0,2,n_1-4)) < \rho(\theta(0,2,n_2-4))$. $\square$

Recall that the spectral radius of a nonnegative irreducible matrix B is larger than that of a principal submatrix of B and it increases when an entry of B increases [6, p. 16, 38]. Thus we have the following well known lemma.

Lemma 2.6. *Let D be a strongly connected digraph and H a strongly connected proper subdigraph of D. Then $\rho(D) > \rho(H)$.*

Lemma 2.7. *$\rho(\theta(0,n-2,0))$ is strictly decreasing in $n \geq 4$.*

Proof. Suppose that $n_1 > n_2 \geq 4$. By Lemma 2.3, $P(\theta(0,n_1-2,0),x) - P(\theta(0,n_2-2,0),x) = (x^{n_2} - x^{n_2-2})(x^{n_1-n_2} - 1) > 0$ for $x \geq \rho(\theta(0,n_2-2,0)) > 1$. Thus $\rho(\theta(0,n_1-2,0)) < \rho(\theta(0,n_2-2,0))$. $\square$

Lemma 2.8. *[4] For $n \geq 4$, $\infty(2,n-1)$ is the unique digraph in $\mathbb{B}_n$ which achieves the maximum spectral radius, and $\theta(0,n-2,0)$ is the unique θ-digraph in $\mathbb{B}_n$ which achieves the maximum spectral radius among θ-digraphs.*

Lemma 2.9. *[4] If $k \geq 1$, then $\rho(\infty(k-1,l+1)) > \rho(\infty(k,l))$.*

3. Results

To determine the unique digraphs with the second, the third and the fourth minimum spectral radii respectively among strongly connected digraphs of order $n \geq 4$, we need first to determine the unique digraphs in $\mathbb{B}_n$ with the second and the third minimum spectral radii respectively for $n \geq 4$.

Theorem 3.1. *For $n \geq 4$, $\theta(1,1,n-4)$ and $\theta(0,2,n-4)$ are the unique digraphs in $\mathbb{B}_n$ which achieve the second and the third minimum spectral radii respectively.*

Proof. Let $D \in \mathbb{B}_n$ with $D \neq \theta(0,1,n-3)$. Then D is a θ-digraph or a ∞-digraph. Suppose that D is a θ-digraph and $D \neq \theta(1,1,n-4)$. By Lemma 2.2, we have $\rho(D) \geq \rho(\theta(0,2,n-4))$ with equality only if $D = \theta(0,2,n-4)$. By Lemma 2.3, $P(\theta(1,1,n-4),x) - P(\theta(0,2,n-4),x) = -2x + x^2 + 1 = (x-1)^2 > 0$ for $x \geq \rho(\theta(0,2,n-4)) > 1$. Thus $\rho(D) \geq \rho(\theta(0,2,n-4)) > \rho(\theta(1,1,n-4))$. If D is a ∞-digraph, then by Lemmas 2.1 and 2.4,

$$\rho(D) \geq \rho\left(\infty\left(\left\lfloor\frac{n+1}{2}\right\rfloor, \left\lceil\frac{n+1}{2}\right\rceil\right)\right) > \rho(\theta(0,2,n-4)).$$

Now the result follows from the first part of Lemma 2.1. $\square$

Theorem 3.2. *Let D be a strongly connected digraph of order $n \geq 4$ that is neither a bicyclic digraph nor C_n. Then $\rho(D) > \rho(\theta(0,2,n-4))$.*

Proof. Let C be a shortest directed cycle in G. Obviously, $V(C) \neq V(D)$. There is a vertex $u \in V(D) \setminus V(C)$ such that there is an arc from u to some vertex, say v, on C. Also, there is a directed path from some vertex on C to u. Let w be a vertex on C such that the distance from w to u in D is as small as possible. Let P be such a directed path. Then C and P have exactly one common vertex w. If $w \neq v$, then D has a proper θ-subdigraph, and if $w = v$, then D has a proper ∞-subdigraph.

If D has a proper ∞-subdigraph, say $\infty(k,l)$ with $k+l = n_1+1$ and $n_1 \leq n$, then by Lemma 2.6, the second part of Lemma 2.1, and Lemmas 2.4 and 2.5, we have

$$\begin{aligned}
\rho(D) &> \rho(\infty(k,l)) \\
&\geq \rho\left(\infty\left(\left\lfloor\frac{n_1+1}{2}\right\rfloor, \left\lceil\frac{n_1+1}{2}\right\rceil\right)\right) \\
&> \rho(\theta(0,2,n_1-4)) \\
&\geq \rho(\theta(0,2,n-4)).
\end{aligned}$$

Suppose that D has a proper θ-subdigraph, say $\theta(a,b,c)$ with $a+b+c = n_2-2$ and $n_2 \leq n$.

Case 1. $n_2 \leq n-1$. By Lemma 2.6 and the first part of Lemma 2.1, we have

$$\rho(D) > \rho(\theta(a,b,c)) \geq \rho(\theta(0,1,n_2-3)).$$

By Lemma 2.3, $P(\theta(0,2,n-4),x) - P(\theta(0,1,n_2-3),x) = x^n - x^{n_2} - x^2 + x = x^{n_2}(x^{n-n_2}-1) - x(x-1) \geq x^{n_2}(x-1) - x(x-1) = (x^{n_2}-x)(x-1) > 0$ for $x \geq \rho(\theta(0,1,n_2-3)) > 1$. Thus $\rho(\theta(0,1,n_2-3)) > \rho(\theta(0,2,n-4))$. Hence $\rho(D) > \rho(\theta(0,2,n-4))$.

Case 2. $n_2 = n$ and $\theta(a,b,c) \neq \theta(0,1,n-3)$ and $\theta(1,1,n-4)$. By Lemma 2.6, the first part of Lemma 2.1, and Theorem 3.1,

$$\rho(D) > \rho(\theta(a,b,c)) \geq \rho(\theta(0,2,n-4)).$$

Case 3. $n_2 = n$ and the θ-subdigraph of D can only be $\theta(0,1,n-3)$ or $\theta(1,1,n-4)$. Without loss of generality suppose that D has a θ-subdigraph $\theta(0,1,n-3)$ (the proof is similar if D has a θ-subdigraph $\theta(1,1,n-4)$). Let vw, vu_1w and $wu'_1u'_2\dots u'_{n-3}v$ be the basic directed paths of the subdigraph $\theta(0,1,n-3)$. We consider the possible arc(s) in D (except the arcs in $\theta(0,1,n-3)$) as follows.

(i) $wv \notin E(D)$; Otherwise, D has a θ-subdigraph $\theta(0,n-3,0)$, a contradiction.

(ii) $u_1v \notin E(D)$ and $wu_1 \notin E(D)$; Otherwise, D has a θ-subdigraph $\theta(0,n-2,0)$, a contradiction.

(iii) $u_1u'_k \notin E(D)$ and $u'_{n-k-2}u_1 \notin E(D)$ for $2 \le k \le n-3$; Otherwise, D has a θ-subdigraph $\theta(0,k,n-k-2)$, a contradiction.

(iv) $vu'_k \notin E(D)$ and $u'_{n-k-2}w \notin E(G)$ for $1 \le k \le n-3$; Otherwise, D has a θ-subdigraph $\theta(0,k+1,n-k-3)$, a contradiction.

(v) $u'_kv \notin E(D)$ and $wu'_{n-k-2} \notin E(D)$ for $1 \le k \le n-4$; Otherwise, D has a θ-subdigraph $\theta(0,1,k)$, a contradiction.

(vi) $u'_lu'_k \notin E(D)$ for $1 \le k < l \le n-3$; Otherwise, D has a θ-subdigraph $\theta(0,n-l+k-2,l-k-1)$, a contradiction.

(vii) $u'_ku'_l \notin E(D)$ for $1 \le k < l-1 \le n-4$; Otherwise, D has a θ-subdigraph $\theta(0,1,n-2-(l-k))$, a contradiction.

(viii) $\{u_1u'_1, u'_{n-3}u_1\} \nsubseteq E(D)$; Otherwise, D has a θ-subdigraph $\theta(0,1,n-4)$, a contradiction.

From (i)–(viii), we find that besides these arcs in $\theta(0,1,n-3)$, D contains one additional arc $u_1u'_1$ or $u'_{n-3}u_1$. Thus D is isomorphic to the digraph D' obtained from $\theta(0,1,n-3)$ by adding the arc $u_1u'_1$. Besides the empty union and C_n, $\mathcal{C}(D')$ contains two directed cycles on $n-1$ vertices. Thus $P(D,x) = P(D',x) = x^n - 2x - 1$. Obviously, $P(D,1) < 0$, $P(D,2) > 0$, and $P(D,x)$ is strictly increasing for $x \ge 1$. Thus $1 < \rho(D) < 2$. Similarly, $1 < \rho(\theta(0,2,n-4)) < 2$ by Lemma 2.3. Note that $P(\theta(0,2,n-4),x) - P(D,x) = -x^2 + 2x > 0$ for $1 < x < 2$. Thus $\rho(D) > \rho(\theta(0,2,n-4))$. $\square$

From Lemma 2.1 and Theorems 3.1 and 3.2, we have the following theorem.

Theorem 3.3. *Among the strongly connected digraphs of order $n \ge 4$, $\theta(0,1,n-3)$, $\theta(1,1,n-4)$ and $\theta(0,2,n-4)$ are the unique digraphs that achieve the second, the third and the fourth minimum spectral radii respectively.*

Thus we answer Problem 1.1 affirmatively.

Finally, we determine the unique digraphs in $\mathbb{B}_n$ with the second maximum spectral radius for $n \ge 4$.

Theorem 3.4. *For $n \ge 5$, $\infty(3,n-2)$ for $5 \le n \le 7$, and $\theta(0,n-2,0)$ for $n = 4$ and $n \ge 8$ are the unique digraphs in $\mathbb{B}_n$ which achieve the second maximum spectral radius.*

Proof. Obviously, $\mathbb{B}_4 = \{\infty(2,3), \theta(0,2,0), \theta(1,1,0), \theta(0,1,1)\}$. By Lemmas 2.1 and 2.8, and Theorem 3.1, we have $\rho(\infty(2,3)) > \rho(\theta(0,2,0)) > \rho(\theta(1,1,0)) > \rho(\theta(0,1,1))$. Thus the result follows for $n = 4$.

Suppose that $n \geq 5$. Let $D \in \mathbb{B}_n$ and $D \neq \infty(3, n-2)$, $\theta(0, n-2, 0)$. If D is a θ-digraph, then by the second part of Lemma 2.8, $\rho(D) < \rho(\theta(0, n-2, 0))$. If D is a ∞-digraph and $D \neq \infty(2, n-1)$, then by Lemma 2.9, $\rho(D) < \rho(\infty(3, n-2))$. Now by the first part of Lemma 2.8, the second maximum spectral radius of digraphs in $\mathbb{B}_n$ is $\max\{\rho(\theta(0, n-2, 0)), \rho(\infty(3, n-2))\}$, which is only achieved by $\theta(0, n-2, 0)$ or $\infty(3, n-2)$.

If $5 \leq n \leq 7$, then by direct calculation using maple, we have $\rho(\theta(0, n-2, 0)) < \rho(\infty(3, n-2))$.

Suppose that $n \geq 8$. Let $\rho = \rho(\theta(0, n-2, 0))$. Obviously, $\rho > 1$. By Lemma 2.7 and direct calculation using maple, we have $\rho \leq \rho(\theta(0,6,0)) = 1.1748\ldots < 1.175$. For $1 < x < 1.175$, let $h(x) = 1 + x + x^2 - x^3 - x^4$. Since $h'(x) = 1 + 2x - 3x^2 - 4x^3 < 0$, $h(x)$ is strictly decreasing. Thus $h(\rho) > h(1.175) = 0.027265 > 0$. By Lemma 2.3, $\rho^{n-2} = \frac{1}{\rho^2 - 1}$, and thus

$$\begin{aligned} P(\infty(3, n-2), \rho) &= P(\infty(3, n-2), \rho) - P(\theta(0, n-2, 0), \rho) \\ &= \rho^{n-2} + 1 - \rho^{n-3} - \rho^2 \\ &= (\rho - 1)(\rho^{n-3} - \rho - 1) \\ &= (\rho - 1)\left(\frac{1}{\rho(\rho^2 - 1)} - \rho - 1\right) \\ &= \frac{h(\rho)}{\rho(\rho + 1)} \\ &> 0. \end{aligned}$$

Obviously, $P(\infty(3, n-2), 1) = -1 < 0$ and $P(\infty(3, n-2), x)$ is strictly increasing for $x > 1$. Thus $\rho(\infty(3, n-2)) < \rho = \rho(\theta(0, n-2, 0))$. $\square$

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