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UNCOUNTABLY MANY BOUNDED POSITIVE SOLUTIONS FOR A SECOND ORDER NONLINEAR NEUTRAL DELAY PARTIAL DIFFERENCE EQUATION

Z. LIU, Z. WU, J. S. UME* AND S. M. KANG

(Communicated by Behzad Djafari-Rouhani)

Abstract. In this paper we consider the second order nonlinear neutral delay partial difference equation

$$\Delta_n\Delta_m\big(x_{m,n}+a_{m,n}x_{m-k,n-l}\big)+f\big(m,n,x_{m-\tau,n-\sigma}\big)=b_{m,n},$$
$$m\geq m_0,\ n\geq n_0.$$

Under suitable conditions, by making use of the Banach fixed point theorem, we show the existence of uncountably many bounded positive solutions for the above partial difference equation. Three nontrivial examples are given to illustrate the advantages of our results.

Keywords: Uncountably many bounded positive solutions, second order nonlinear neutral delay partial difference equation, Banach fixed point theorem.
MSC(2010): Primary: 39A14; Secondary: 39A10.

1. Introduction and preliminaries

In recent years there has been much interest in the study of qualitative analysis of various first and second order difference and partial difference equations, for example, see [1–7] and the references therein. Tang [6] studied the existence of a bounded nonoscillatory solution for the second order linear delay difference equation

$$\Delta^2 x_n = p_n x_{n-k},\quad n\geq 0 \tag{1.1}$$

and

$$\Delta^2 x_n = \sum_{i=1}^{\infty} p_i(n)x_{n-k_i},\quad n\geq 0. \tag{1.2}$$

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Jinfa [1] utilized the contraction mapping principle to investigate the existence of a nonoscillatory solution for the second order neutral delay difference equation with positive and negative coefficients

$$\Delta^2(x_n + px_{n-m}) + p_n x_{n-k} - q_n x_{n-l} = 0, \quad n \geq n_0 \tag{1.3}$$

under the condition $p \in \mathbb{R} \setminus \{-1\}$. Migda and Migda [5] discussed the asymptotic behavior of the second order neutral difference equation

$$\Delta^2(x_n + px_{n-k}) + f(n, x_n) = 0, \quad n \geq 1. \tag{1.4}$$

Meng and Yan [4] investigated the sufficient and necessary conditions of the existence of bounded nonoscillatory solutions for the second order nonlinear neutral delay difference equation

$$\Delta^2(x_n - px_{n-k}) = \sum_{i=1}^{m} q_i f_i(x_{n-\sigma-i}), \quad n \geq n_0. \tag{1.5}$$

Karpuz and Ocalan [2] studied the first order linear partial difference equation

$$x_{m+1,n} + x_{m,n+1} - x_{m,n} + p_{m,n} x_{m-k,n-l} = 0, \quad (m,n) \in \mathbb{Z}_{0,0}, \tag{1.6}$$

where $\{p_{m,n}\}_{(m,n)\in\mathbb{Z}_{0,0}}$ is a nonnegative sequence and $k, l \in \mathbb{N}_1$, and got sufficient conditions under which every solution of Eq.(1.6) is oscillatory. Wong [7] discussed the existence of eventually positive and monotone decreasing solutions for the partial difference inequalities

$$\begin{aligned} &\Delta_m \Delta_n x_{m,n} + \sum_{i=1}^{r} p_i(m, n, x_{g_i(m),h_i(n)}) \geq \\ &(\leq) \sum_{i=1}^{r} Q_i(m, n, x_{g_i(m),h_i(n)}), m \geq m_0,\, n \geq n_0. \end{aligned} \tag{1.7}$$

where $g_i(m)$ and $h_i(m)$ are some deviating arguments for $1 \leq i \leq \tau$.

Our aim in the present paper is to investigate the following second order nonlinear neutral delay partial difference equation

$$\begin{aligned} &\Delta_n \Delta_m \big(x_{m,n} + a_{m,n} x_{m-k,n-l}\big) + f\big(m, n, x_{m-\tau,n-\sigma}\big) = b_{m,n}, \\ &m \geq m_0,\, n \geq n_0, \end{aligned} \tag{1.8}$$

where $m_0, n_0 \in \mathbb{N}_0, k, l, \tau, \sigma \in \mathbb{N}$, $\{a_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}, \{b_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ are real sequences with $a_{m,n} \neq \pm 1$ for $(m,n) \in \mathbb{N}_{m_0,n_0}$ and $f : \mathbb{N}_{m_0,n_0} \times \mathbb{R} \to \mathbb{R}$.

Utilizing the Banach fixed point theorem, we prove several existence results of uncountably many bounded positive solutions for Eq.(1.8). Three nontrivial examples are constructed to illustrate our results.

Throughout the paper, the forward partial difference operators Δ_m and Δ_n are defined by $\Delta_m x_{m,n} = x_{m+1,n} - x_{m,n}$ and $\Delta_n x_{m,n} = x_{m,n+1} - x_{m,n}$, respectively, the second partial difference operator is defined by $\Delta_n \Delta_m x_{m,n} =$

$\Delta_n(\Delta_m x_{m,n})$. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of all positive integers and integers, respectively,

$$\mathbb{N}_0 = \{0\} \cup \mathbb{N}, \quad \mathbb{N}_s = \{n : n \in \mathbb{N}_0 \text{ with } n \geq s\}, \quad s \in \mathbb{N}_0,$$
$$\mathbb{N}_{s,t} = \{(m,n) : m,n \in \mathbb{N}_0 \text{ with } m \geq s, n \geq t\}, \quad s,t \in \mathbb{N}_0,$$
$$\mathbb{Z}_{s,t} = \{(m,n) : m,n \in \mathbb{Z} \text{ with } m \geq s, n \geq t\}, \quad s,t \in \mathbb{Z},$$
$$\alpha = \min\{m_0 - k, m_0 - \tau\}, \quad \beta = \min\{n_0 - l, n_0 - \sigma\},$$

$l^\infty_{\alpha,\beta}$ represents the Banach space of all bounded sequences on $\mathbb{Z}_{\alpha,\beta}$ with the norm

$$\|x\| = \sup_{m,n \in \mathbb{Z}_{\alpha,\beta}} |x_{m,n}| \text{ for } x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in l^\infty_{\alpha,\beta}$$

and

$$A(N,M) = \big\{x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in l^\infty_{\alpha,\beta} : N \leq x_{m,n} \leq M, (m,n) \in \mathbb{Z}_{\alpha,\beta}\big\} \quad \text{for } M > N > 0.$$

It is easy to verify that $A(N,M)$ is a bounded closed convex subset of the Banach space $l^\infty_{\alpha,\beta}$. By a solution of Eq.(1.8), we mean a sequence $\{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}$ with positive integers $m_1 \geq m_0 + k + |\alpha|$ and $n_1 \geq n_0 + l + |\beta|$ such that Eq.(1.8) is satisfied for all $m \geq m_1$ and $n \geq n_1$.

2. Existence of uncountably many bounded positive solutions

Now we investigate the existence of uncountably many bounded positive solutions for Eq.(1.8).

Theorem 2.1. *Assume that there exist positive constants M and N, nonnegative constants a_1 and a_2 and nonnegative sequences $\{P_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ and $\{Q_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ satisfying*

$$a_1 + a_2 < 1, \quad N < \big[1 - (a_1 + a_2)\big]M; \tag{2.1}$$

$$-a_2 \leq a_{m,n} \leq a_1 \text{ eventually}; \tag{2.2}$$

$$|f(m,n,u) - f(m,n,\overline{u})| \leq P_{m,n}|u - \overline{u}|, \quad (m,n,u,\overline{u}) \in \mathbb{N}_{m_0,n_0} \times [N,M]^2; \tag{2.3}$$

$$|f(m,n,u)| \leq Q_{m,n}, \quad (m,n,u) \in \mathbb{N}_{m_0,n_0} \times [N,M]; \tag{2.4}$$

$$\sum_{i=m_0}^{\infty} \sum_{t=n_0}^{\infty} \max\big\{P_{i,t},\, Q_{i,t},\, |b_{i,t}|\big\} < +\infty. \tag{2.5}$$

Then Eq.(1.8) possesses uncountably many bounded positive solutions in $A(M,N)$.

Proof. Set $L \in (N + a_1 M, (1 - a_2)M)$. It follows from (2.1), (2.2) and (2.5) that there exist $\theta \in (0, 1)$, $m_1 \geq m_0 + k + |\alpha|$ and $n_1 \geq n_0 + l + |\beta|$ satisfying

$$\theta = a_1 + a_2 + \sum_{i=m_1}^{\infty} \sum_{t=n_1}^{\infty} P_{i,t}, \tag{2.6}$$

$$-a_2 \leq a_{m,n} \leq a_1, \quad (m, n) \in \mathbb{N}_{m_1,n_1}, \tag{2.7}$$

$$\sum_{i=m_1}^{\infty} \sum_{t=n_1}^{\infty} \big(Q_{i,t} + |b_{i,t}|\big) \leq \min\big\{(1 - a_2)M - L, L - a_1 M - N\big\}. \tag{2.8}$$

Define a mapping $T_L : A(N, M) \to l^{\infty}_{\alpha,\beta}$ by

$$T_L x_{m,n} = \begin{cases} L - a_{m,n} x_{m-k,n-l} - \sum_{i=m}^{\infty} \sum_{t=n}^{\infty} \big[f(i, t, x_{i-\tau,t-\sigma}) \\ \qquad\qquad -b_{i,t}\big], \quad (m, n) \in \mathbb{Z}_{m_1,n_1}, \\ T_L x_{m_1,n_1}, \quad (m, n) \in \mathbb{Z}_{\alpha,\beta} \setminus \mathbb{Z}_{m_1,n_1} \end{cases} \tag{2.9}$$

for each $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N, M)$. By using (2.1)∼(2.4) and (2.6)∼(2.9), we infer that for $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}, y = \{y_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N, M)$ and $(m, n) \in \mathbb{Z}_{m_1,n_1}$

$$\begin{aligned} &|T_L x_{m,n} - T_L y_{m,n}| \\ &= \left| a_{m,n}(x_{m-k,n-l} - y_{m-k,n-l}) + \sum_{i=m}^{\infty} \sum_{t=n}^{\infty} \big[f(i, t, x_{i-\tau,t-\sigma}) - f(i, t, y_{i-\tau,t-\sigma})\big] \right| \\ &\leq |a_{m,n}||x_{m-k,n-l} - y_{m-k,n-l}| + \sum_{i=m}^{\infty} \sum_{t=n}^{\infty} |f(i, t, x_{i-\tau,t-\sigma}) - f(i, t, y_{i-\tau,t-\sigma})| \\ &\leq (a_1 + a_2)\|x - y\| + \sum_{i=m}^{\infty} \sum_{t=n}^{\infty} P_{i,t}|x_{i-\tau,t-\sigma} - y_{i-\tau,t-\sigma}| \\ &\leq \left(a_1 + a_2 + \sum_{i=m_1}^{\infty} \sum_{t=n_1}^{\infty} P_{i,t} \right) \|x - y\| \\ &= \theta\|x - y\|, \end{aligned}$$

$$\begin{aligned}
T_L x_{m,n} &= L - a_{m,n} x_{m-k,n-l} - \sum_{i=m}^{\infty}\sum_{t=n}^{\infty}\big[f(i,t,x_{i-\tau,t-\sigma}) - b_{i,t}\big] \\
&\le L + a_2 M + \sum_{i=m}^{\infty}\sum_{t=n}^{\infty}\big[|f(i,t,x_{i-\tau,t-\sigma})| + |b_{i,t}|\big] \\
&\le L + a_2 M + \sum_{i=m_1}^{\infty}\sum_{t=n_1}^{\infty}\big(Q_{i,t} + |b_{i,t}|\big) \\
&\le L + a_2 M + \min\big\{(1-a_2)M - L, L - a_1 M - N\big\} \\
&\le M
\end{aligned}$$

and

$$\begin{aligned}
T_L x_{m,n} &= L - a_{m,n} x_{m-k,n-l} - \sum_{i=m}^{\infty}\sum_{t=n}^{\infty}\big[f(i,t,x_{i-\tau,t-\sigma}) - b_{i,t}\big] \\
&\ge L - a_1 M - \sum_{i=m}^{\infty}\sum_{t=n}^{\infty}\big[|f(i,t,x_{i-\tau,t-\sigma})| + |b_{i,t}|\big] \\
&\ge L - a_1 M - \sum_{i=m_1}^{\infty}\sum_{t=n_1}^{\infty}\big(Q_{i,t} + |b_{i,t}|\big) \\
&\ge L - a_1 M - \min\big\{(1-a_2)M - L, L - a_1 M - N\big\} \\
&\ge N,
\end{aligned}$$

which leads to

$$T_L(A(N,M)) \subseteq A(N,M), \quad \|T_L x - T_L y\| \le \theta\|x-y\|, \quad x,y \in A(N,M). \tag{2.10}$$

Consequently, (2.10) means that T_L is a contraction mapping in $A(N,M)$. Thus the Banach fixed point theorem ensures that T_L has a unique fixed point $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, which together with (2.9) give that

$$x_{m,n} = L - a_{m,n} x_{m-k,n-l} - \sum_{i=m}^{\infty}\sum_{t=n}^{\infty}\big[f\big(i,t,x_{i-\tau,t-\sigma}\big) - b_{i,t}\big], \quad (m,n) \in \mathbb{Z}_{m_1,n_1},$$

which yields that

$$\Delta_m(x_{m,n} + a_{m,n} x_{m-k,n-l}) = \sum_{t=n}^{\infty}\big[f\big(m,t,x_{m-\tau,t-\sigma}\big) - b_{m,t}\big], \quad (m,n) \in \mathbb{Z}_{m_1,n_1}$$

and

$$\Delta_n\Delta_m(x_{m,n} + a_{m,n} x_{m-k,n-l}) = -f(m,n,x_{m-\tau,n-\sigma}) + b_{m,n}, \quad (m,n) \in \mathbb{Z}_{m_1,n_1},$$

that is, $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}$ is a bounded positive solution of Eq.(1.8) in $A(N,M)$.

Finally we prove that Eq.(1.8) has uncountably many bounded positive solutions in $A(N,M)$. Let $L_1, L_2 \in (N + a_1M, (1-a_2)M)$ and $L_1 \neq L_2$. Similarly we infer that for each $j \in \{1,2\}$, there exist $\theta_j, m_{L_j}, n_{L_j}$ and T_{L_j} satisfying (2.6)∼(2.9), where θ, m_1, n_1, L and T_L are replaced by $\theta_j, m_{L_j}, n_{L_j}, L_j$ and T_{L_j}, respectively, and the mapping T_{L_j} has a fixed point $x^j = \{x^j_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, which is a bounded positive solution of Eq.(1.8), that is,

$$x^1_{m,n} = L_1 - a_{m,n}x^1_{m-k,n-l} - \sum_{i=m}^{\infty}\sum_{t=n}^{\infty}\big[f\big(i,t,x^1_{i-\tau,t-\sigma}\big) - b_{i,t}\big], \quad (m,n)\in\mathbb{Z}_{m_{L_1},n_{L_1}} \tag{2.11}$$

and

$$x^2_{m,n} = L_2 - a_{m,n}x^2_{m-k,n-l} - \sum_{i=m}^{\infty}\sum_{t=n}^{\infty}\big[f\big(i,t,x^2_{i-\tau,t-\sigma}\big) - b_{i,t}\big], \quad (m,n)\in\mathbb{Z}_{m_{L_2},n_{L_2}}. \tag{2.12}$$

In order to show that the set of all bounded positive solutions of Eq.(1.8) is uncountable, it is sufficient to prove that $x^1 \neq x^2$. It follows from (2.3), (2.6), (2.7), (2.11) and (2.12) that for $(m,n)\in\mathbb{Z}_{\max\{m_{L_1},m_{L_2}\},\max\{n_{L_1},n_{L_2}\}}$

$$\begin{aligned}
&\big|x^1_{m,n} - x^2_{m,n}\big| \\
&= \Big|L_1 - L_2 - a_{m,n}\big(x^1_{m-k,n-l} - x^2_{m-k,n-l}\big) \\
&\qquad - \sum_{i=m}^{\infty}\sum_{t=n}^{\infty}\big[f\big(i,t,x^1_{i-\tau,t-\sigma}\big) - f\big(i,t,x^2_{i-\tau,t-\sigma}\big)\big]\Big| \\
&\geq |L_1 - L_2| - |a_{m,n}|\big|x^1_{m-k,n-l} - x^2_{m-k,n-l}\big| \\
&\qquad - \sum_{i=m}^{\infty}\sum_{t=n}^{\infty}\big[\big|f\big(i,t,x^1_{i-\tau,t-\sigma}\big) - f\big(i,t,x^2_{i-\tau,t-\sigma}\big)\big|\big] \\
&\geq |L_1 - L_2| - (a_1+a_2)\big\|x^1 - x^2\big\| - \sum_{i=m}^{\infty}\sum_{t=n}^{\infty}P_{i,t}\big|x^1_{i-\tau,t-\sigma} - x^2_{i-\tau,t-\sigma}\big| \\
&\geq |L_1 - L_2| - \bigg(a_1 + a_2 + \sum_{i=m}^{\infty}\sum_{t=n}^{\infty}P_{i,t}\bigg)\big\|x^1 - x^2\big\| \\
&\geq |L_1 - L_2| - \bigg(a_1 + a_2 + \sum_{i=\max\{m_{L_1},m_{L_2}\}}^{\infty}\ \sum_{t=\max\{n_{L_1},n_{L_2}\}}^{\infty}P_{i,t}\bigg)\big\|x^1 - x^2\big\| \\
&\geq |L_1 - L_2| - \max\{\theta_1,\theta_2\}\big\|x^1 - x^2\big\|,
\end{aligned}$$

which implies that

$$\|x^1 - x^2\| \geq \frac{|L_1 - L_2|}{1 + \max\{\theta_1, \theta_2\}} > 0,$$

that is, $x^1 \neq x^2$. This completes the proof. □

Theorem 2.2. *Assume that there exist positive constants M and N, negative constants a_1 and a_2 and nonnegative sequences $\{P_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ and $\{Q_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ satisfying* (2.3)∼(2.5) *and*

$$a_1 < -1, \quad N(1+a_2) > M(1+a_1); \tag{2.13}$$

$$a_2 \leq a_{m,n} \leq a_1 \text{ eventually }. \tag{2.14}$$

Then Eq.(1.8) *possesses uncountably many bounded positive solutions in $A(M, N)$.*

Proof. Taking $L \in \big(M(1+a_1), N(1+a_2)\big)$, from (2.5), (2.13) and (2.14) we infer that there exist $\theta \in (0,1)$, $m_1 \geq m_0 + k + |\alpha|$ and $n_1 \geq n_0 + l + |\beta|$ satisfying

$$\theta = -\frac{1}{a_1}\bigg(1 + \sum_{i=m_1}^{\infty}\sum_{t=n_1}^{\infty} P_{i,t}\bigg), \tag{2.15}$$

$$a_2 \leq a_{m,n} \leq a_1, \quad (m,n) \in \mathbb{N}_{m_1,n_1}, \tag{2.16}$$

$$\begin{aligned}&\sum_{i=m_1}^{\infty}\sum_{t=n_1}^{\infty}\big(Q_{i,t} + |b_{i,t}|\big)\\ &\leq \min\bigg\{L - M(1+a_1), a_1 N\bigg(1 + \frac{1}{a_2}\bigg) - \frac{a_1 L}{a_2}\bigg\}.\end{aligned} \tag{2.17}$$

Define a mapping $T_L : A(N, M) \to l^{\infty}_{\alpha,\beta}$ by

$$T_L x_{m,n} = \begin{cases} \frac{L}{a_{m+k,n+l}} - \frac{x_{m+k,n+l}}{a_{m+k,n+l}} \\ -\frac{1}{a_{m+k,n+l}} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} [f(i,t,x_{i-\tau,t-\sigma}) \\ -b_{i,t}], \quad (m,n) \in \mathbb{Z}_{m_1,n_1}, \\ T_L x_{m_1,n_1}, \quad (m,n) \in \mathbb{Z}_{\alpha,\beta} \setminus \mathbb{Z}_{m_1,n_1} \end{cases} \tag{2.18}$$

for each $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N, M)$. It follows from (2.3), (2.4), (2.13), (2.14) and (2.15)∼(2.18) that for $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}, y = \{y_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in$

$A(N, M)$ and $(m, n) \in \mathbb{Z}_{m_1,n_1}$

$$
\begin{aligned}
&|T_L x_{m,n} - T_L y_{m,n}| \\
&= \Bigg| \frac{x_{m+k,n+l} - y_{m+k,n+l}}{a_{m+k,n+l}} \\
&\qquad + \frac{1}{a_{m+k,n+l}} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \big[f(i,t,x_{i-\tau,t-\sigma}) - f(i,t,y_{i-\tau,t-\sigma}) \big] \Bigg| \\
&\le - \frac{|x_{m+k,n+l} - y_{m+k,n+l}|}{a_{m+k,n+l}} \\
&\qquad - \frac{1}{a_{m+k,n+l}} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \big| f(i,t,x_{i-\tau,t-\sigma}) - f(i,t,y_{i-\tau,t-\sigma}) \big| \\
&\le - \frac{\|x-y\|}{a_1} - \frac{1}{a_1} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} P_{i,t} |x_{i-\tau,t-\sigma} - y_{i-\tau,t-\sigma}| \\
&\le - \frac{1}{a_1} \left(1 + \sum_{i=m_1}^{\infty} \sum_{t=n_1}^{\infty} P_{i,t} \right) \|x-y\| \\
&= \theta \|x-y\|,
\end{aligned}
$$

$$
\begin{aligned}
&T_L x_{m,n} \\
&= \frac{L}{a_{m+k,n+l}} - \frac{x_{m+k,n+l}}{a_{m+k,n+l}} - \frac{1}{a_{m+k,n+l}} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} [f(i,t,x_{i-\tau,t-\sigma}) - b_{i,t}] \\
&\le \frac{L}{a_1} - \frac{M}{a_1} - \frac{1}{a_1} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \big[|f(i,t,x_{i-\tau,t-\sigma})| + |b_{i,t}| \big] \\
&\le \frac{L}{a_1} - \frac{M}{a_1} - \frac{1}{a_1} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \big(Q_{i,t} + |b_{i,t}| \big) \\
&\le \frac{L}{a_1} - \frac{M}{a_1} - \frac{1}{a_1} \sum_{i=m_1}^{\infty} \sum_{t=n_1}^{\infty} \big(Q_{i,t} + |b_{i,t}| \big) \\
&\le \frac{L}{a_1} - \frac{M}{a_1} - \frac{1}{a_1} \min \left\{ L - M(1+a_1), a_1 N \left(1 + \frac{1}{a_2} \right) - \frac{a_1 L}{a_2} \right\} \\
&\le M
\end{aligned}
$$

and

$$
\begin{aligned}
&T_L x_{m,n}\\
&= \frac{L}{a_{m+k,n+l}} - \frac{x_{m+k,n+l}}{a_{m+k,n+l}} - \frac{1}{a_{m+k,n+l}} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} [f(i,t,x_{i-\tau,t-\sigma}) - b_{i,t}]\\
&\ge \frac{L}{a_2} - \frac{N}{a_2} + \frac{1}{a_1} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \big[|f(i,t,x_{i-\tau,t-\sigma})| + |b_{i,t}|\big]\\
&\ge \frac{L}{a_2} - \frac{N}{a_2} + \frac{1}{a_1} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \big(Q_{i,t} + |b_{i,t}|\big)\\
&\ge \frac{L}{a_2} - \frac{N}{a_2} + \frac{1}{a_1} \sum_{i=m_1}^{\infty} \sum_{t=n_1}^{\infty} \big(Q_{i,t} + |b_{i,t}|\big)\\
&\ge \frac{L}{a_2} - \frac{N}{a_2} + \frac{1}{a_1} \min\left\{L - M(1+a_1), a_1 N\left(1+\frac{1}{a_2}\right) - \frac{a_1 L}{a_2}\right\}\\
&\ge N,
\end{aligned}
$$

which imply that (2.10) holds. Consequently, the contraction mapping T_L has a unique fixed point $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, which together with (2.18) gives that

$$
\begin{aligned}
x_{m,n} =& \frac{L}{a_{m+k,n+l}}\\
&- \frac{x_{m+k,n+l}}{a_{m+k,n+l}} - \frac{1}{a_{m+k,n+l}} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \big[f(i,t,x_{i-\tau,t-\sigma}) - b_{i,t}\big],\\
&(m,n) \in \mathbb{Z}_{m_1,n_1},
\end{aligned}
$$

which yields that

$$
\begin{aligned}
&\Delta_m (x_{m,n} + a_{m,n} x_{m-k,n-l}) = \sum_{t=n}^{\infty} \big[f\big(m,t,x_{m-\tau,t-\sigma}\big) - b_{m,t}\big],\\
&(m,n) \in \mathbb{Z}_{m_1,n_1}
\end{aligned}
$$

and

$$
\begin{aligned}
&\Delta_n \Delta_m (x_{m,n} + a_{m,n} x_{m-k,n-l}) = -f(m,n,x_{m-\tau,n-\sigma}) + b_{m,n},\\
&(m,n) \in \mathbb{Z}_{m_1,n_1}
\end{aligned}
$$

that is, $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}$ is a bounded positive solution of Eq.(1.8) in $A(N,M)$.

Finally we prove that Eq.(1.8) has uncountably many bounded positive solutions in $A(N,M)$. Let $L_1, L_2 \in (M(1+a_1), N(1+a_2))$ and $L_1 \ne L_2$. Similarly we deduce that for each $j \in \{1,2\}$, there exist $\theta_j, m_{L_j}, n_{L_j}$ and T_{L_j} satisfying (2.15)$\sim$(2.18), where θ, m_1, n_1, L and T_L are replaced by $\theta_j, m_{L_j}, n_{L_j}, L_j$ and

T_{L_j}, respectively, and the mapping T_{L_j} has a fixed point $x^j = \{x^j_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, which is a bounded positive solution of Eq.(1.8), that is,

$$x^1_{m,n} = \frac{L_1}{a_{m+k,n+l}} - \frac{x^1_{m+k,n+l}}{a_{m+k,n+l}} - \frac{1}{a_{m+k,n+l}} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \left[f\big(i,t,x^1_{i-\tau,t-\sigma}\big) - b_{i,t}\right], \quad (m,n) \in \mathbb{Z}_{m_{L_1},n_{L_1}} \tag{2.19}$$

and

$$x^2_{m,n} = \frac{L_2}{a_{m+k,n+l}} - \frac{x^2_{m+k,n+l}}{a_{m+k,n+l}} - \frac{1}{a_{m+k,n+l}} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \left[f\big(i,t,x^2_{i-\tau,t-\sigma}\big) - b_{i,t}\right], \quad (m,n) \in \mathbb{Z}_{m_{L_2},n_{L_2}}. \tag{2.20}$$

In order to show that the set of bounded positive solutions of Eq.(1.8) is uncountable, it is sufficient to prove that $x^1 \neq x^2$. It follows from (2.3), (2.15), (2.16), (2.19) and (2.20) that for $(m,n) \in \mathbb{Z}_{\max\{m_{L_1},m_{L_2}\},\max\{n_{L_1},n_{L_2}\}}$

$$\begin{aligned}
\big|x^1_{m,n} - x^2_{m,n}\big|
&= \Bigg| \frac{L_1 - L_2}{a_{m+k,n+l}} - \frac{x^1_{m+k,n+l} - x^2_{m+k,n+l}}{a_{m+k,n+l}} \\
&\quad - \frac{1}{a_{m+k,n+l}} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \left[f\big(i,t,x^1_{i-\tau,t-\sigma}\big) - f(i,t,x^2_{i-\tau,t-\sigma})\right] \Bigg| \\
&\geq -\frac{|L_1 - L_2|}{a_{m+k,n+l}} + \frac{\big|x^1_{m+k,n+l} - x^2_{m+k,n+l}\big|}{a_{m+k,n+l}} \\
&\quad + \frac{1}{a_{m+k,n+l}} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \big|f\big(i,t,x^1_{i-\tau,t-\sigma}\big) - f(i,t,x^2_{i-\tau,t-\sigma})\big| \\
&\geq -\frac{|L_1 - L_2|}{a_2} + \frac{\big\|x^1 - x^2\big\|}{a_1} \\
&\quad + \frac{1}{a_1} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} P_{i,t} \big|x^1_{i-\tau,t-\sigma} - x^2_{i-\tau,t-\sigma}\big| \\
&\geq -\frac{|L_1 - L_2|}{a_2} + \frac{1}{a_1}\left(1 + \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} P_{i,t}\right)\big\|x^1 - x^2\big\|
\end{aligned}$$

$$\geq -\frac{|L_1 - L_2|}{a_2} + \frac{1}{a_1}\left(1 + \sum_{i=\max\{m_{L_1}, m_{L_2}\}}^{\infty} \sum_{t=\max\{n_{L_1}, n_{L_2}\}}^{\infty} P_{i,t}\right\}\Big)\|x^1 - x^2\| \geq -\frac{|L_1 - L_2|}{a_2} - \max\{\theta_1, \theta_2\}\|x^1 - x^2\|,$$

which implies that

$$\|x^1 - x^2\| \geq -\frac{|L_1 - L_2|}{a_2(1 + \max\{\theta_1, \theta_2\})} > 0,$$

that is, $x^1 \neq x^2$. This completes the proof. □

Theorem 2.3. *Assume that there exist positive constants M and N, nonnegative constants a_1 and a_2 and nonnegative sequences $\{P_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ and $\{Q_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ satisfying* (2.3) $\sim$ (2.5), (2.14) *and*

$$(2.21) \qquad 1 < a_2, \quad a_1 < a_2^2, \quad Ma_1(a_2^2 - a_1) > Na_2(a_1^2 - a_2).$$

Then Eq.(1.8) *possesses uncountably many bounded positive solutions in $A(M,N)$.*

Proof. Put $L \in \left(a_1N + \frac{a_1M}{a_2}, a_2M + \frac{a_2N}{a_1}\right)$. It follows from (2.5), (2.14) and (2.21) that there exist $\theta \in (0,1)$, $m_1 \geq m_0 + k + |\alpha|$ and $n_1 \geq n_0 + l + |\beta|$ satisfying (2.21)

$$(2.22) \qquad \theta = \frac{1}{a_2}\left(1 + \sum_{i=m_1}^{\infty}\sum_{t=n_1}^{\infty} P_{i,t}\right),$$

$$(2.23) \qquad \sum_{i=m_1}^{\infty}\sum_{t=n_1}^{\infty}\left(Q_{i,t} + |b_{i,t}|\right) \leq \min\left\{a_2M - L + \frac{a_2N}{a_1}, \frac{a_2L}{a_1} - M - a_2N\right\}.$$

Let the mapping $T_L : A(N,M) \to l^{\infty}_{\alpha,\beta}$ be defined by (2.18). It follows from (2.3), (2.4), (2.14), (2.16), (2.18) and (2.21)$\sim$(2.23) that for $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}$, $y = \{y_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$ and $(m,n) \in \mathbb{Z}_{m_1,n_1}$

$$
\begin{aligned}
|T_L x_{m,n} - T_L y_{m,n}|
&= \Bigg| \frac{x_{m+k,n+l} - y_{m+k,n+l}}{a_{m+k,n+l}} \\
&\quad + \frac{1}{a_{m+k,n+l}} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \left[f(i,t,x_{i-\tau,t-\sigma}) - f(i,t,y_{i-\tau,t-\sigma}) \right] \Bigg| \\
&\leq \frac{|x_{m+k,n+l} - y_{m+k,n+l}|}{a_{m+k,n+l}} \\
&\quad + \frac{1}{a_{m+k,n+l}} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \left| f(i,t,x_{i-\tau,t-\sigma}) - f(i,t,y_{i-\tau,t-\sigma}) \right| \\
&\leq \frac{\|x-y\|}{a_2} + \frac{1}{a_2} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} P_{i,t} |x_{i-\tau,t-\sigma} - y_{i-\tau,t-\sigma}| \\
&\leq \frac{1}{a_2} \left(1 + \sum_{i=m_1}^{\infty} \sum_{t=n_1}^{\infty} P_{i,t} \right) \|x-y\| \\
&= \theta \|x-y\|,
\end{aligned}
$$

$$
\begin{aligned}
T_L x_{m,n} &= \frac{L}{a_{m+k,n+l}} - \frac{x_{m+k,n+l}}{a_{m+k,n+l}} \\
&\quad - \frac{1}{a_{m+k,n+l}} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \left[f(i,t,x_{i-\tau,t-\sigma}) - b_{i,t} \right] \\
&\leq \frac{L}{a_2} - \frac{N}{a_1} + \frac{1}{a_2} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \left[|f(i,t,x_{i-\tau,t-\sigma})| + |b_{i,t}| \right] \\
&\leq \frac{L}{a_2} - \frac{N}{a_1} + \frac{1}{a_2} \sum_{i=m_1}^{\infty} \sum_{t=n_1}^{\infty} \left(Q_{i,t} + |b_{i,t}| \right) \\
&\leq \frac{L}{a_2} - \frac{N}{a_1} + \frac{1}{a_2} \min \left\{ a_2 M - L + \frac{a_2 N}{a_1}, \frac{a_2 L}{a_1} - M - a_2 N \right\} \\
&\leq M
\end{aligned}
$$

and

$$
\begin{aligned}
T_L x_{m,n} &= \frac{L}{a_{m+k,n+l}} - \frac{x_{m+k,n+l}}{a_{m+k,n+l}} \\
&\quad - \frac{1}{a_{m+k,n+l}} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \left[f(i,t,x_{i-\tau,t-\sigma}) - b_{i,t} \right] \\
&\ge \frac{L}{a_1} - \frac{M}{a_2} - \frac{1}{a_2} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \left[|f(i,t,x_{i-\tau,t-\sigma})| + |b_{i,t}| \right] \\
&\ge \frac{L}{a_1} - \frac{M}{a_2} - \frac{1}{a_2} \sum_{i=m_1}^{\infty} \sum_{i=m_1}^{\infty} \sum_{t=n_1}^{\infty} \left(Q_{i,t} + |b_{i,t}| \right) \\
&\ge \frac{L}{a_1} - \frac{M}{a_2} - \frac{1}{a_2} \min\left\{ a_2 M - L + \frac{a_2 N}{a_1}, \frac{a_2 L}{a_1} - M - a_2 N \right\} \\
&\ge N,
\end{aligned}
$$

which imply that (2.10) holds. Consequently (2.10) ensures that T_L is a contraction mapping and hence it has a unique fixed point $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, which gives that

$$
\begin{aligned}
x_{m,n} &= \frac{L}{a_{m+k,n+l}} - \frac{x_{m+k,n+l}}{a_{m+k,n+l}} \\
&\quad - \frac{1}{a_{m+k,n+l}} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \left[f(i,t,x_{i-\tau,t-\sigma}) - b_{i,t} \right], \\
&\quad (m,n) \in \mathbb{Z}_{m_1,n_1}.
\end{aligned}
$$

As in the proof of Theorem 2.2, it is easy to verify that $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}$ is a bounded positive solution of Eq.(1.8) in $A(N,M)$.

Finally we prove that Eq.(1.8) has uncountably many bounded positive solutions in $A(N,M)$. Let $L_1, L_2 \in \left(a_1 N + \frac{a_1 M}{a_2}, a_2 M + \frac{a_2 N}{a_1}\right)$ and $L_1 \neq L_2$. Similarly we deduce that for each $j \in \{1,2\}$, there exist $\theta_j, m_{L_j}, n_{L_j}$ and T_{L_j} satisfying (2.16), (2.18), (2.22) and (2.23), where θ, m_1, n_1, L and T_L are replaced by $\theta_j, m_{L_j}, n_{L_j}, L_j$ and T_{L_j}, respectively, and the mapping T_{L_j} has a fixed point $x^j = \{x^j_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, which is a bounded positive solution of Eq.(1.8) and satisfies (2.19) and (2.20). In order to show that the set of bounded positive solutions of Eq.(1.8) is uncountable, it is sufficient to prove that $x^1 \neq x^2$. It follows from (2.3), (2.16), (2.19), (2.20) and (2.22) that for $(m,n) \in \mathbb{Z}_{\max\{m_{L_1},m_{L_2}\},\max\{n_{L_1},n_{L_2}\}}$

$$\begin{aligned}
&\left|x^1_{m,n} - x^2_{m,n}\right| \\
&\quad = \Bigg| \frac{L_1 - L_2}{a_{m+k,n+l}} - \frac{x^1_{m+k,n+l} - x^2_{m+k,n+l}}{a_{m+k,n+l}} \\
&\qquad - \frac{1}{a_{m+k,n+l}} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \left[f\left(i,t,x^1_{i-\tau,t-\sigma}\right) - f\left(i,t,x^2_{i-\tau,t-\sigma}\right)\right] \Bigg| \\
&\quad \geq \frac{|L_1 - L_2|}{a_{m+k,n+l}} - \frac{\left|x^1_{m+k,n+l} - x^2_{m+k,n+l}\right|}{a_{m+k,n+l}} \\
&\qquad - \frac{1}{a_{m+k,n+l}} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} \left| f\left(i,t,x^1_{i-\tau,t-\sigma}\right) - f\left(i,t,x^2_{i-\tau,t-\sigma}\right)\right| \\
&\quad \geq \frac{|L_1 - L_2|}{a_1} - \frac{\left\|x^1 - x^2\right\|}{a_2} - \frac{1}{a_2} \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} P_{i,t}\left|x^1_{i-\tau,t-\sigma} - x^2_{i-\tau,t-\sigma}\right| \\
&\geq \frac{|L_1 - L_2|}{a_1} - \frac{1}{a_2}\left(1 + \sum_{i=m+k}^{\infty} \sum_{t=n+l}^{\infty} P_{i,t}\right)\left\|x^1 - x^2\right\| \\
&\geq \frac{|L_1 - L_2|}{a_1} - \frac{1}{a_2}\left(1 + \sum_{i=\max\{m_{L_1},m_{L_2}\}}^{\infty} \sum_{t==\max\{n_{L_1},n_{L_2}\}}^{\infty} P_{i,t}\right)\left\|x^1 - x^2\right\| \\
&\geq \frac{|L_1 - L_2|}{a_1} - \max\{\theta_1,\theta_2\}\left\|x^1 - x^2\right\|,
\end{aligned}$$

which implies that

$$\left\|x^1 - x^2\right\| \geq \frac{|L_1 - L_2|}{a_1(1 + \max\{\theta_1,\theta_2\})} > 0,$$

that is, $x^1 \neq x^2$. This completes the proof. □

3. **Examples**

Now we construct three examples to explain the results presented in Section 2. Note that none of the known results can be applied to these examples.

Example 3.1. Consider the second order nonlinear neutral delay partial difference equation

$$\begin{aligned}
&\Delta_n\Delta_m\left(x_{m,n} + \frac{(-1)^{m+n}}{3}x_{m-k,n-l}\right) + \frac{\sin(m^2n^5 - \ln n)}{m^3(n^2+1)}x^3_{m-\tau,n-\sigma} \\
&\qquad = \frac{(-1)^m\cos(m^3 - 3n)}{\sqrt{m^7n^6+2}}, \quad m \geq 1,\ n \geq 1,
\end{aligned} \tag{3.1}$$

where $k, l, \tau, \sigma \in \mathbb{N}$ are fixed. Let $m_0 = n_0 = 1$, $a_1 = a_2 = \frac{1}{3}$, $\alpha = \min\{1 - k, 1 - \tau\}$, $\beta = \min\{1 - l, 1 - \sigma\}$, M and N be two positive constants with $M > 3N$ and

$$\begin{aligned} a_{m,n} &= \frac{(-1)^{m+n}}{3}, \quad b_{m,n} = \frac{(-1)^m \cos(m^3 - 3n)}{\sqrt{m^7 n^6 + 2}}, \\ f(m,n,u) &= \frac{\sin(m^2 n^5 - \ln n)}{m^3(n^2+1)} u^3, P_{m,n} = \frac{3M^2}{m^3(n^2+1)}, \\ Q_{m,n} &= \frac{M^3}{m^3(n^2+1)}, \quad (m,n,u) \in \mathbb{N}_{m_0,n_0} \times \mathbb{R}. \end{aligned}$$

It is easy to verify that (2.1)∼(2.4) hold. Note that

$$\begin{aligned} &\sum_{i=m_0}^{\infty} \sum_{t=n_0}^{\infty} \max\left\{P_{i,t}, Q_{i,t}, |b_{i,t}|\right\} \\ &= \sum_{i=m_0}^{\infty} \sum_{t=n_0}^{\infty} \max\left\{\frac{3M^2}{i^3(t^2+1)}, \frac{M^3}{i^3(t^2+1)}, \frac{|\cos(i^3 - 3t)|}{\sqrt{i^7 t^6 + 2}}\right\} < +\infty. \end{aligned}$$

It is easy to see that the conditions of Theorem 2.1 are satisfied. Thus Theorem 2.1 implies that Eq.(3.1) possesses uncountably many bounded positive solutions in $A(N, M)$.

Example 3.2. Consider the second order nonlinear neutral delay partial difference equation

$$\begin{aligned} &\Delta_n \Delta_m \left(x_{m,n} - \frac{5m + 2m(-1)^n}{m+1} x_{m-k,n-l} \right) + \frac{x^2_{m-\tau,n-\sigma}}{m^3 n^2} \\ &= \frac{\sin(n^3 m - \sqrt{n})}{\sqrt{m^2+1}}, \quad m \geq 1, \, n \geq 1, \end{aligned} \tag{3.2}$$

where $k, l, \tau, \sigma \in \mathbb{N}$ are fixed. Let $m_0 = n_0 = 1, a_1 = -2$, $a_2 = -7$, $\alpha = \min\{1 - k, 1 - \tau\}$, $\beta = \min\{1 - l, 1 - \sigma\}$, M and N be two positive constants with $M > 6N$ and

$$\begin{aligned} a_{m,n} &= -\frac{5m + 2m(-1)^n}{m+1}, \quad b_{m,n} = \frac{\sin(n^3 m - \sqrt{n})}{\sqrt{m^2+1}}, \\ f(m,n,u) &= \frac{u^2}{m^3 n^2}, \quad P_{m,n} = \frac{2M}{m^3 n^2}, \\ Q_{m,n} &= \frac{M^2}{m^3 n^2}, \quad (m,n,u) \in \mathbb{N}_{m_0,n_0} \times \mathbb{R}. \end{aligned}$$

It is clear that (2.3), (2.4), (2.13) and (2.14) hold. Observe that

$$\begin{aligned}&\sum_{i=m_0}^{\infty}\sum_{t=n_0}^{\infty}\max\left\{P_{i,t},Q_{i,t},|b_{i,t}|\right\}\\&=\sum_{i=m_0}^{\infty}\sum_{t=n_0}^{\infty}\max\left\{\frac{2M}{i^3t^2},\frac{M^2}{i^3t^2},\frac{|\sin(t^3i-\sqrt{t})|}{\sqrt{i^2+1}}\right\}<+\infty.\end{aligned}$$

That is, the conditions of Theorem 2.2 are fulfilled. Therefore Theorem 2.2 ensures that Eq.(3.2) has uncountably many bounded positive solutions in $A(N,M)$.

Example 3.3. Consider the second order nonlinear neutral delay partial difference equation

$$\begin{aligned}&\Delta_n\Delta_m\left(x_{m,n}+\frac{3mn+4}{mn+1}x_{m-k,n-l}\right)+\frac{((-1)^n-\frac{1}{m})x^4_{m-\tau,n-\sigma}}{m^6n^5}\\&=\frac{(-1)^{n+m}(m^3-2n^2)}{m^5(n^9+1)},\quad m\geq 1,\ n\geq 1,\end{aligned}\tag{3.3}$$

where $k,l,\tau,\sigma\in\mathbb{N}$ are fixed. Let $m_0=n_0=1$, $a_1=4$, $a_2=3$, $\alpha=\min\{1-k,1-\tau\}$, $\beta=\min\{1-l,1-\sigma\}$, M and N be two positive constants with $M>\frac{39}{20}N$ and

$$\begin{aligned}&a_{m,n}=\frac{3mn+4}{mn+1},\quad b_{m,n}=\frac{(-1)^{n+m}(m^3-2n^2)}{m^5(n^9+1)},\\&f(m,n,u)=\frac{((-1)^n-\frac{1}{m})u^4}{m^6n^5}\quad P_{m,n}=\frac{3M^3}{m^6n^2},\\&Q_{m,n}=\frac{M^4}{m^6n^5},\quad (m,n,u)\in\mathbb{N}_{m_0,n_0}\times\mathbb{R}.\end{aligned}$$

Clearly (2.3), (2.4), (2.13) and (2.21) hold. Notice that

$$\begin{aligned}&\sum_{i=m_0}^{\infty}\sum_{t=n_0}^{\infty}\max\left\{P_{i,t},Q_{i,t},|b_{i,t}|\right\}\\&=\sum_{i=m_0}^{\infty}\sum_{t=n_0}^{\infty}\max\left\{\frac{3M^3}{i^6t^2},\frac{M^4}{i^6t^5},\frac{|i^3-2t^2|}{i^5(t^9+1)}\right\}<+\infty.\end{aligned}$$

It is clear that the conditions of Theorem 2.3 are fulfilled. Consequently Theorem 2.3 implies that Eq.(3.3) possesses uncountably many bounded positive solutions in $A(N,M)$.

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