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RANDOM COINCIDENCE POINT RESULTS FOR WEAKLY INCREASING FUNCTIONS IN PARTIALLY ORDERED METRIC SPACES

A. R. KHAN*, N. HUSSAIN, N. YASMIN AND N. SHAFQAT

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Abstract. The aim of this paper is to establish random coincidence point results for weakly increasing random operators in the setting of ordered metric spaces by using generalized altering distance functions. Our results present random versions and extensions of some well-known results in the current literature.
Keywords: Random coincidence point, altering distance function, partially ordered metric space.
MSC(2010): Primary: 47H10 ; Secondary: 47B80, 47H40.

1. Introduction

It is noted that in probabilistic functional analysis, random fixed point and random fixed point theorems for contractive mappings in Polish spaces are of fundamental importance. Their study was initiated in the work of Spacek [34] and Hans [11, 12]. Probabilistic functional analysis is an important mathematical discipline because of its applications to probabilistic models in applied problems. Random operator theory is needed for the study of various classes of random equations. The interest in this subject enhanced after publication of the survey paper by Bharucha Ried [7]. Random fixed point theorems play an important role in the theory of random differential and random integral equations [17,20–23]. Sehgal and Singh [33] have proved different stochastic versions of the well-known Schauder's fixed point theorem. Random coincidence point theorems are stochastic generalizations of classical coincidence point theorems. The existence of fixed points for various multi-valued contractive mappings has been studied by many authors under different conditions. Abbas et al. [1,2] and Hussain et al. [16] proved coupled fixed point and coupled coincidence points

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results for nonlinear contractions in partially ordered metric spaces. Recently, Ćirić and Lakshmikantham [10] and Hussain et al. [15], respectively proved some coupled random fixed point and coupled random coincidence results in partially ordered complete metric spaces.

Khan et al. [19] introduced the altering distance function and used it for solving fixed points problems in metric spaces. Recently, the authors in [4, 5, 9, 14, 30–32] used altering distance function and obtained some fixed point theorems. The altering distance functions and their variants have been employed for iterative solution of nonlinear operator equations by Berinde [6]. In 2005, Choudhury [8] introduced a generalized distance function in three variables and obtained a common fixed point theorem for a pair of self maps in a complete metric space. Nashine and Aydi [27] generalized the results of Nashine et al. [26] to the case of four variables. They obtained coincidence point and common fixed point theorems in complete ordered metric spaces for mappings satisfying a contractive condition which involves two generalized altering distance functions in four variables. In our results, random coincidence points for weakly contractive maps are established on an ordered metric space by making use of generalized altering distance function.

2. **Preliminaries**

Definition 2.1. *[19] A function $\phi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if and only if*

(1) *ϕ is continuous,*
(2) *ϕ is nondecreasing,*
(3) *$\phi(t) = 0 \Leftrightarrow t = 0$.*

Khan et al. [19] proved the following result.

Theorem 2.2. *Let (X, d) be a complete metric space, $\phi : [0, +\infty) \to [0, +\infty)$ an altering distance function and $T : X \to X$ a self-mapping which satisfies the following inequality*

$$\phi(d(Tx, Ty)) \leq c\phi(d(x, y))$$

for all $x, y \in X$ and for some $0 < c < 1$. Then T has a unique fixed point.

In 1997, Alber and Guerre-Delabriere [3] introduced the concept of weak contractions in Hilbert spaces. This concept was extended to metric spaces by Rhoades in [29].

Definition 2.3. *A mapping $T : X \to X$, where (X, d) is a metric space, is said to be weakly contractive if and only if*

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) \quad \forall\ x, y \in X$$

where $\phi : [0, +\infty) \to [0, +\infty)$ is an altering distance function.

In [8], Choudhury introduced the concept of a generalized distance function for three variables.

Definition 2.4. *[8] A function $\phi : [0,+\infty) \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is said to be a generalized altering distance function if and only if*

(1) *ϕ is continuous,*
(2) *ϕ is nondecreasing in all three variables,*
(3) *$\phi(x,y,z) = 0 \Leftrightarrow x = y = z = 0$.*

Define $\psi(x) = \phi(x,x,x)$ for $x \in [0,\infty)$. Clearly, $\psi(x) = 0$ if and only if $x = 0$. Examples of ϕ are

$$\phi(a,b,c) = k \max\{a,b,c\}, \text{ for } k > 0,$$
$$\phi(a,b,c) = a^p + b^q + c^r,\ p,q,r \geq 1,$$
$$\phi(a,b,c) = (a + \alpha b^q)\, r + c^s, \text{ where } p,q,r,s \geq 1 \text{ and } \alpha > 0.$$

Rao et al. [28] generalized the above definition for four variables as follows.

Definition 2.5. *A function $\phi : [0,+\infty) \times [0,+\infty) \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is said to be a generalized altering distance function if and only if*

(1) *ϕ is continuous,*
(2) *ϕ is nondecreasing in all four variables,*
(3) *$\phi(t_1,t_2,t_3,t_4) = 0 \Leftrightarrow t_1 = t_2 = t_3 = t_4 = 0$.*

Definition 2.6. *[18] Let (X,d) be a metric space and $f,g : X \to X$. If $\gamma = fx = gx$, for some $x \in X$, then x is called a coincidence point of f and g, and γ is called a point of coincidence of f and g. The pair $\{f,g\}$ is said to be compatible if and only if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.*

Let X be a nonempty set and $R : X \to X$ a given mapping. For every $x \in X$, we denote by $R^{-1}(x)$ the subset of X defined by

$$R^{-1}(x) = \{u \in X \mid Ru = x\}. \tag{2.1}$$

Definition 2.7. *[25] Let $(X,\leq)$ be a partially ordered set and $T,S,R : X \to X$ be given mappings such that $TX \subseteq RX$ and $SX \subseteq RX$. We say that S and T are weakly increasing with respect to R if and only if for all $x \in X$, we have*

$$Tx \leq Sy \quad \forall\, y \in R^{-1}(Tx) \ \text{ and } Sx \leq Ty \quad \forall\, y \in R^{-1}(Sx). \tag{2.2}$$

Definition 2.8. *Let $(X,d,\leq)$ be a partially ordered metric space. We say that X is regular if and only if the following hypothesis holds: if $\{z_n\}$ is a nondecreasing sequence in X with respect to $\leq$ such that $z_n \to z \in X$ as $n \to +\infty$, then $z_n \leq z$ for all $n \in \mathbb{N}$.*

Theorem 2.9. *[27] Let $(X,d,\leq)$ be an ordered complete metric space. Let $T,S,R : X \to X$ be given mappings satisfying the following inequality for every* *pair* $(x,y) \in X \times X$ *with* Rx *and* Ry *comparable,*

$$\phi_1\left(d\left(Sx,Ty\right)\right)$$
$$\leq \psi_1(d\left(Rx,Ry\right), d\left(Rx,Sx\right), d\left(Ry,Ty\right), \tfrac{1}{2}[d\left(Rx,Ty\right)+d\left(Ry,Sx\right)])$$
$$-\psi_2(d\left(Rx,Ry\right), d\left(Rx,Sx\right), d\left(Ry,Ty\right), \tfrac{1}{2}[d\left(Rx,Ty\right)+d\left(Ry,Sx\right)])$$

where ψ_1 *and* ψ_2 *are generalized altering distance functions and* $\phi_1(x) = \psi_1(x,x,x,x)$. *One assumes the following hypotheses:*

(1) *either* T, S *and* R *are continuous or* X *is regular,*
(2) $TX \subseteq RX, SX \subseteq RX$,
(3) T *and* S *are weakly increasing with respect to* R,
(4) *the pairs* $\{T,R\}$ *and* $\{S,R\}$ *are compatible.*

Then, $T, S,$ *and* R *have a coincidence point, that is, there exists* $u \in X$ *such that* $Ru = Tu = Su$.

3. **Random coincidence point results**

Let (Ω,Σ) be a measurable space with a Σ sigma algebra of subsets of Ω and let (X,d) be a metric space. A mapping $T:\Omega \to X$ is called $\Sigma-$ measurable if for any open subset β of X, $\zeta^{-1}(\beta) \in \Sigma$. A mapping $S:\Omega\times X \to X$ is said to be a random map if and only if for each fixed $x \in X$, the mapping $S(.,x):\Omega \to X$ is measurable. A random map $S:\Omega \times X \to X$ is continuous if for each $\omega \in \Omega$, the mapping $S(\omega,.) : X \to X$ is continuous. A measurable mapping $\zeta : \Omega \to X$ is a random fixed point of the random map $S:\Omega\times X \to X$ if and only if $S(\omega,\zeta(\omega)) = \zeta(\omega)$ for each $\omega \in \Omega$.

Definition 3.1. *A measurable mapping* $\zeta:\Omega \to K$, *where* K *be a Polish subspace of* X, *is said to be*

(i)a random fixed point of $R:\Omega\times K \to K$, *if for each* $\omega \in \Omega$,

$$\zeta(\omega) = R(\omega,\zeta(\omega)).$$

(ii) A random coincidence point of $R:\Omega\times K\to K$, $S:\Omega\times K \to K$ *and* $T:\Omega\times K \to K$ *if for each* $\omega\in\Omega$,

$$R(\omega,\zeta(\omega)) = S(\omega,\zeta(\omega)) = T(\omega,\zeta(\omega)).$$

Definition 3.2. *Let* (X,d) *be a separable metric space and* (Σ,Ω) *be a measurable space. The pair* $\{f,g\}$ *is said to be a compatible random operator if and only if*

$$\lim\nolimits_{n\to+\infty} d\left(f\left(\omega,g\left(\omega,x_n\right)\right), g\left(\omega,f\left(\omega,x_n\right)\right)\right) = 0$$

whenever $\{x_n\}$ *is a sequence in* X *such that*

$$\lim_{n\to+\infty} f(\omega,x_n) = \lim_{n\to+\infty} g(\omega,x_n) = t$$

for some $t \in X$ *and for each* $\omega \in \Omega$.

Now, we state and prove our main result.

Theorem 3.3. *Let $(X, d, \leq)$ be a separable ordered metric space and K be a nonempty Polish subspace of X. Let $T, S, R : \Omega \times K \to K$ be random operators satisfying the following inequality, for every pair $(x, y) \in K \times K$ with $R(\omega, x)$ and $R(\omega, y)$ comparable,*

(3.1)

$$\begin{aligned}
&\phi_1 \left(d\left(S\left(\omega, x\right), T\left(\omega, y\right)\right)\right)\\
&\leq \psi_1(d\left(R\left(\omega, x\right), R\left(\omega, y\right)\right), d\left(R\left(\omega, x\right), S\left(\omega, x\right)\right), d\left(R\left(\omega, y\right), T\left(\omega, y\right)\right),\\
&\tfrac{1}{2}[d\left(R\left(\omega, x\right), T\left(\omega, y\right)\right) + d\left(R\left(\omega, y\right), S\left(\omega, x\right)\right)])\\
&-\psi_2(d\left(R\left(\omega, x\right), R\left(\omega, y\right)\right), d\left(R\left(\omega, x\right), S\left(\omega, x\right)\right), d\left(R\left(\omega, y\right), T\left(\omega, y\right)\right),\\
&\tfrac{1}{2}[d\left(R\left(\omega, x\right), T\left(\omega, y\right)\right) + d\left(R\left(\omega, y\right), S\left(\omega, x\right)\right)])
\end{aligned}$$

where ψ_1 and ψ_2 are generalized altering distance functions and $\phi_1(x) = \psi_1(x, x, x, x)$. Assume that

1. *T, S and R are continuous random operators,*
2. *$T(\omega, K) \subseteq R(\omega, K)$ and $S(\omega, K) \subseteq R(\omega, K)$ for each $\omega \in \Omega$,*
3. *$T(\omega, .)$ and $S(\omega, .)$ are weakly increasing with respect to $R(\omega, .)$ for each $\omega \in \Omega$,*
4. *the pairs $\{T, R\}$ and $\{S, R\}$ are compatible random operators.*

Then there exists a measurable mapping $\zeta : \Omega \to K$ such that

$$R\left(\omega, \zeta\left(\omega\right)\right) = T\left(\omega, \zeta\left(\omega\right)\right) = S\left(\omega, \zeta\left(\omega\right)\right)$$

for each $\omega \in \Omega$.

Proof. Let $\zeta_0 : \Omega \to K$ be a given measurable map. Since $T(\omega, K) \subseteq R(\omega, K)$, there exists $\zeta_1 : \Omega \to K$ such that $R\left(\omega, \zeta_1\left(\omega\right)\right) = T\left(\omega, \zeta_0\left(\omega\right)\right)$. Since $S(\omega, K) \subseteq R(\omega, K)$, there exists $\zeta_2\left(\omega\right) \in \Omega$ such that $R\left(\omega, \zeta_2\left(\omega\right)\right) = S\left(\omega, \zeta_1\left(\omega\right)\right)$. Inductively, we construct a sequence of maps $\{\zeta_n\left(\omega\right)\}$ from Ω to K such that

(3.2)

$$R\left(\omega, \zeta_{2n+1}\left(\omega\right)\right) = T\left(\omega, \zeta_{2n}\left(\omega\right)\right) \text{ and } R\left(\omega, \zeta_{2n+2}\left(\omega\right)\right) = S\left(\omega, \zeta_{2n+1}\left(\omega\right)\right).$$

Since R, S and T are continuous random operators, by a result of Himmelberg [13], $\{\zeta_n\left(\omega\right)\}$ is a measurable sequence. Now, we claim that

$$R\left(\omega, \zeta_n\left(\omega\right)\right) \leq R\left(\omega, \zeta_{n+1}\left(\omega\right)\right) \text{ for all } n \in N. \tag{3.3}$$

Since $R\left(\omega, \zeta_1\left(\omega\right)\right) = T\left(\omega, \zeta_0\left(\omega\right)\right)$, therefore $\left(\omega, \zeta_1\left(\omega\right)\right) \in R^{-1}\left(T\left(\omega, \zeta_0\left(\omega\right)\right)\right)$. By the increasing property of the mappings $S(\omega, .)$ and $T(\omega, .)$ with respect to $R(\omega.,)$, we get

$$R\left(\omega, \zeta_1\left(\omega\right)\right) = T\left(\omega, \zeta_0\left(\omega\right)\right) \leq S\left(\omega, \zeta_1\left(\omega\right)\right) = R\left(\omega, \zeta_2\left(\omega\right)\right) \tag{3.4}$$

and

$$\begin{aligned}&R\left(\omega, \zeta_2\left(\omega\right)\right) = S\left(\omega, \zeta_1\left(\omega\right)\right) \leq T\left(\omega, \zeta_2\left(\omega\right)\right) = R\left(\omega, \zeta_3\left(\omega\right)\right)\\ &\text{for each } \omega \in \Omega.\end{aligned} \tag{3.5}$$

Hence, by induction (3.3) holds. Without loss of generality, we can assume that

$$R(\omega,\zeta_n(\omega)) \neq R(\omega,\zeta_{n+1}(\omega)) \text{ for all } n \in N \text{ and for each } \omega \in \Omega. \tag{3.6}$$

Now, we prove that

$$\lim_{n\to\infty} d(R(\omega,\zeta_{n+1}(\omega)), R(\omega,\zeta_{n+2}(\omega))) = 0. \tag{3.7}$$

From (3.1), we have

(3.8)

$$\begin{aligned}
&\phi_1(d(R(\omega,\zeta_{2n+2}(\omega)), R(\omega,\zeta_{2n+1}(\omega))))\\
&= \phi_1(d(S(\omega,\zeta_{2n+1}(\omega)), T(\omega,\zeta_{2n}(\omega))))\\
&\le \psi_1(d(R(\omega,\zeta_{2n+1}(\omega)), R(\omega,\zeta_{2n}(\omega))), d(R(\omega,\zeta_{2n+1}(\omega)), S(\omega,\zeta_{2n+1}(\omega))),\\
&d(R(\omega,\zeta_{2n}(\omega)), T(\omega,\zeta_{2n}(\omega))), \tfrac{1}{2}[d(R(\omega,\zeta_{2n+1}(\omega)), T(\omega,\zeta_{2n}(\omega))\\
&+d(R(\omega,\zeta_{2n}(\omega)), S(\omega,\zeta_{2n+1}(\omega)))]) - \psi_2(d(R(\omega,\zeta_{2n+1}(\omega)), R(\omega,\zeta_{2n}(\omega))),\\
&d(R(\omega,\zeta_{2n+1}(\omega)), S(\omega,\zeta_{2n+1}(\omega))), d(R(\omega,\zeta_{2n}(\omega)), T(\omega,\zeta_{2n}(\omega))),\\
&\tfrac{1}{2}[d(R(\omega,\zeta_{2n+1}(\omega)), T(\omega,\zeta_{2n}(\omega)) + d(R(\omega,\zeta_{2n}(\omega)), S(\omega,\zeta_{2n+1}(\omega)))]).\\
&= \psi_1(d(R(\omega,\zeta_{2n+1}(\omega)), R(\omega,\zeta_{2n}(\omega))), d(R(\omega,\zeta_{2n+1}(\omega)), R(\omega,\zeta_{2n+2}(\omega))),\\
&d(R(\omega,\zeta_{2n}(\omega)), R(\omega,\zeta_{2n+1}(\omega))), \tfrac{1}{2}[d(R(\omega,\zeta_{2n}(\omega)), R(\omega,\zeta_{2n+2}(\omega))])\\
&-\psi_2(d(R(\omega,\zeta_{2n+1}(\omega)), R(\omega,\zeta_{2n}(\omega))), d(R(\omega,\zeta_{2n+1}(\omega)), R(\omega,\zeta_{2n+2}(\omega))),\\
&d(R(\omega,\zeta_{2n}(\omega)), R(\omega,\zeta_{2n+1}(\omega))), \tfrac{1}{2}[d(R(\omega,\zeta_{2n}(\omega)), R(\omega,\zeta_{2n+2}(\omega))]).
\end{aligned}$$

Suppose, for some $n \in N$, that

(3.9)

$$d(R(\omega,\zeta_{2n}(\omega)), R(\omega,\zeta_{2n+1}(\omega))) < d(R(\omega,\zeta_{2n+1}(\omega)), R(\omega,\zeta_{2n+2}(\omega))).$$

Using (3.9) and triangle inequality, we have

$$\begin{aligned}
&\tfrac{1}{2}d(R(\omega,\zeta_{2n}(\omega)), R(\omega,\zeta_{2n+2}(\omega)))\\
&\le \tfrac{1}{2}(d(R(\omega,\zeta_{2n}(\omega)), R(\omega,\zeta_{2n+1}(\omega)))\\
&+d(R(\omega,\zeta_{2n+1}(\omega)), R(\omega,\zeta_{2n+2}(\omega))))\\
&< d(R(\omega,\zeta_{2n+1}(\omega)), R(\omega,\zeta_{2n+2}(\omega))).
\end{aligned} \tag{3.10}$$

Using (3.9) and (3.10) together with a property of the generalized altering distance function ψ_1, we get

(3.11)

$$\begin{aligned}
&\psi_1(d(R(\omega,\zeta_{2n+1}(\omega)), R(\omega,\zeta_{2n}(\omega))), d(R(\omega,\zeta_{2n+1}(\omega)), R(\omega,\zeta_{2n+2}(\omega))),\\
&d(R(\omega,\zeta_{2n}(\omega)), R(\omega,\zeta_{2n+1}(\omega))), \tfrac{1}{2}[d(R(\omega,\zeta_{2n}(\omega)), R(\omega,\zeta_{2n+2}(\omega))])\\
&\le \phi_1(d(R(\omega,\zeta_{2n+2}(\omega)), R(\omega,\zeta_{2n+1}(\omega))).
\end{aligned}$$

Hence, we obtain

(3.12)

$$\begin{aligned}
&\phi_1(d(R(\omega,\zeta_{2n+2}(\omega)), R(\omega,\zeta_{2n+1}(\omega))))\\
&\le \phi_1(d(R(\omega,\zeta_{2n+2}(\omega)), R(\omega,\zeta_{2n+1}(\omega)))\\
&-\psi_2(d(R(\omega,\zeta_{2n+1}(\omega)), R(\omega,\zeta_{2n}(\omega))), d(R(\omega,\zeta_{2n+1}(\omega)), R(\omega,\zeta_{2n+2}(\omega))),\\
&d(R(\omega,\zeta_{2n}(\omega)), R(\omega,\zeta_{2n+1}(\omega))), \tfrac{1}{2}[d(R(\omega,\zeta_{2n}(\omega)), R(\omega,\zeta_{2n+2}(\omega))])
\end{aligned}$$

which implies that
(3.13)
$$\psi_2(d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n}\left(\omega\right)\right)\right),d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),\\ d\left(R\left(\omega,\zeta_{2n}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)\right),\tfrac{1}{2}[d(R\left(\omega,\zeta_{2n}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)])\\ =0$$

Thus, we have

$$d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n}\left(\omega\right)\right)\right)=0 \tag{3.14}$$

which contradicts (3.6). Hence, we deduce that
(3.15)
$$d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right)\leq d\left(R\left(\omega,\zeta_{2n}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)\right)$$

for all $n\in\mathbb{N}$ and for each $\omega\in\Omega$. Again from (3.1) and (3.3), we have

$$\phi_1\left(d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right)\right)\right)\\ =\phi_1\left(d\left(S\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),T\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right)\right)\\ \leq\psi_1(d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),S\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)\right),\\ d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),T\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),\tfrac{1}{2}[d(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),T\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)+\\ d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),S\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)\right)]) - \psi_2(d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),\\ d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),S\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)\right),d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),T\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),\\ \tfrac{1}{2}[d(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),T\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)+d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),S\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)\right)])\\ =\psi_1(d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),\\ d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right)\right),\tfrac{1}{2}[d(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right)+\\ d(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)]) - \psi_2(d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),\\ d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right)\right),\\ \tfrac{1}{2}[d(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right)+d(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)]).$$

(3.16)
$$\phi_1\left(d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right)\right)\right)\\ \leq\psi_1(d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),\\ d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right)\right),\tfrac{1}{2}d(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right)\\ -\psi_2(d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),\\ d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right)\right),\tfrac{1}{2}d(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right).$$

Suppose, for some $n\in N$, that
(3.17)
$$d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right)<d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right)\right).$$

Then, by triangle inequality, we have

$$\begin{aligned}&\tfrac{1}{2}d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right)\right)\\ &\leq\tfrac{1}{2}(d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right)\\ &+d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right)\right))\\ &<d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right)\right).\end{aligned} \tag{3.18}$$

Hence, by (3.16), (3.17) and (3.18) together with the property of the generalized altering function ψ_1, we obtain

$$\begin{aligned}(3.19)\quad &\phi_1\left(d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right)\right)\right)\\ &\leq \phi_1\left(d(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right))\right)\\ &-\psi_2(d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),\\ &d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),\\ &d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right)\right),\\ &\tfrac{1}{2}d(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right))\end{aligned}$$

which implies that

$$\begin{aligned}(3.20)\quad &\psi_2(d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),\\ &d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right)\right),\tfrac{1}{2}d(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right))\\ &=0,\end{aligned}$$

which leads to $d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right)=0$. Hence, we obtain a contradiction to (3.6). So we deduce that

$$(3.21)\quad d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+3}\left(\omega\right)\right)\right)\leq d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right)$$

for all $n\in\mathbb{N}$ and for each $\omega\in\Omega$. Combining (3.15) and (3.21), we obtain

$$(3.22)\quad d\left(R\left(\omega,\zeta_{n+2}\left(\omega\right)\right),R\left(\omega,\zeta_{n+3}\left(\omega\right)\right)\right)\leq d\left(R\left(\omega,\zeta_{n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{n+2}\left(\omega\right)\right)\right)$$

for all $n\in\mathbb{N}$ and for each $\omega\in\Omega$.

Thus $\{R\left(\omega,\zeta_{n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{n+2}\left(\omega\right)\right)\}$ is a nonincreasing sequence of positive real numbers for each $\omega\in\Omega$. This implies that there exists $r\geq 0$ such that

$$(3.23)\quad \lim_{n\to\infty} d\left(R\left(\omega,\zeta_{n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{n+2}\left(\omega\right)\right)\right)=r.$$

By (3.8), we have

$$\begin{aligned}(3.24)\quad &\phi_1\left(d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)\right)\right)\\ &\leq \psi_1(d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n}\left(\omega\right)\right)\right),d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),\\ &d\left(R\left(\omega,\zeta_{2n}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)\right),\tfrac{1}{2}d(R\left(\omega,\zeta_{2n}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right))\\ &-\psi_2(d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n}\left(\omega\right)\right)\right),d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),\\ &d\left(R\left(\omega,\zeta_{2n}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)\right),\tfrac{1}{2}d(R\left(\omega,\zeta_{2n}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right))\\ &\leq \psi_1(d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n}\left(\omega\right)\right)\right),d\left(R\left(\omega,\zeta_{2n}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)\right),\\ &d(R\left(\omega,\zeta_{2n}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)),d(R\left(\omega,\zeta_{2n}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)))\\ &-\psi_2(d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n}\left(\omega\right)\right)\right),d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),\\ &d\left(R\left(\omega,\zeta_{2n}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)\right),0)\\ &=\phi_1\left(d\left(R\left(\omega,\zeta_{2n}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)\right)\right)\\ &-\psi_2(d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n}\left(\omega\right)\right)\right),d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),\\ &d\left(R\left(\omega,\zeta_{2n}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)\right),0).\end{aligned}$$

Letting $n \to +\infty$ in (3.24) and using the continuity of ϕ_1 and ψ_2, we obtain

$$\phi_1(r) \leq \phi_1(r) - \psi_2(r, r, r, 0) \tag{3.25}$$

which implies that $\psi_2(r, r, r, 0) = 0$, so $r = 0$. Hence

$$\lim_{n \to +\infty} d\left(R\left(\omega, \zeta_{n+1}(\omega)\right), R\left(\omega, \zeta_{n+2}(\omega)\right)\right) = 0 \text{ for each } \omega \in \Omega.$$

Thus (3.7) holds. Now, we claim that for $\omega \in \Omega$,$\{R(\omega, \zeta_n(\omega))\}$ is a Cauchy sequence. From (3.7), it will be sufficient to prove that $\{R(\omega, \zeta_{2n}(\omega))\}$ is a Cauchy sequence. We proceed by negation and suppose that $\{R(\omega, \zeta_{2n}(\omega))\}$ is not a Cauchy sequence. Then, there exists $\epsilon > 0$ for which we can find two sequences of positive integers $\{m_i\}$ and $\{n_i\}$ such that, for all positive integers i,

$$\begin{array}{l} n(i) > m(i), \quad d\left(R\left(w, \zeta_{2m(i)}(\omega)\right), R\left(w, \zeta_{2n(i)}(\omega)\right)\right) \geq \epsilon \\ \text{and} \\ d\left(R\left(w, \zeta_{2m(i)}(\omega)\right), R\left(w, \zeta_{2n(i)-2}(\omega)\right)\right) < \epsilon. \end{array} \tag{3.26}$$

From (3.26) and using the triangle inequality, we get

$$\begin{array}{l} \epsilon \leq d\left(R\left(w, \zeta_{2m(i)}(\omega)\right), R\left(w, \zeta_{2n(i)}(\omega)\right)\right) \\ \leq d\left(R\left(w, \zeta_{2m(i)}(\omega)\right), R\left(w, \zeta_{2n(i)-2}(\omega)\right)\right) \\ +d\left(R\left(w, \zeta_{2n(i)-2}(\omega)\right), R\left(w, \zeta_{2n(i)-1}(\omega)\right)\right) \\ +d\left(R\left(w, \zeta_{2n(i)-1}(\omega)\right), R\left(w, \zeta_{2n(i)}(\omega)\right)\right) \\ < \epsilon + d\left(R\left(w, \zeta_{2n(i)-2}(\omega)\right), R\left(w, \zeta_{2n(i)-1}(\omega)\right)\right) \\ +d\left(R\left(w, \zeta_{2n(i)-1}(\omega)\right), R\left(w, \zeta_{2n(i)}(\omega)\right)\right). \end{array} \tag{3.27}$$

Letting $i \to +\infty$ in the above inequality and using (3.7), we obtain

$$\lim_{i \to +\infty} d\left(R\left(w, \zeta_{2m(i)}(\omega)\right), R\left(w, \zeta_{2n(i)}(\omega)\right)\right) = \epsilon. \tag{3.28}$$

Again, triangle inequality gives

$$\begin{array}{l} \mid d\left(R\left(w, \zeta_{2n(i)}(\omega)\right), R\left(w, \zeta_{2m(i)-1}(\omega)\right)\right) \\ -d\left(R\left(w, \zeta_{2n(i)}(\omega)\right), R\left(w, \zeta_{2m(i)}(\omega)\right)\right) \mid \\ \leq d\left(R\left(w, \zeta_{2m(i)-1}(\omega)\right), R\left(w, \zeta_{2m(i)}(\omega)\right)\right). \end{array}$$

Letting $i \to +\infty$ in the above inequality and using (3.7) and (3.28), we get

$$\lim_{i \to +\infty} d\left(R\left(w, \zeta_{2n(i)}(\omega)\right), R\left(w, \zeta_{2m(i)-1}(\omega)\right)\right) = \epsilon. \tag{3.29}$$

On the other hand, we have

$$\begin{array}{l} \phi_1\left(d\left(R\left(w, \zeta_{2n(i)}(\omega)\right), R\left(w, \zeta_{2m(i)}(\omega)\right)\right)\right) \\ \leq \phi_1(d\left(R\left(w, \zeta_{2n(i)}(\omega)\right), R\left(w, \zeta_{2n(i)+1}(\omega)\right)\right) \\ +d\left(R\left(w, \zeta_{2n(i)+1}(\omega)\right), R\left(w, \zeta_{2m(i)}(\omega)\right)\right)) \\ = \phi_1(d\left(R\left(w, \zeta_{2n(i)}(\omega)\right), R\left(w, \zeta_{2n(i)+1}(\omega)\right)\right) \\ +d\left(T\left(w, \zeta_{2n(i)}(\omega)\right), S\left(w, \zeta_{2m(i)-1}(\omega)\right)\right)). \end{array} \tag{3.30}$$

Then, from (3.7), (3.28) and the continuity of ϕ_1, we get by letting $i \to \infty$ in the above inequality

$$\phi_1(\epsilon) \leq \lim_{i\to+\infty} \phi_1\left(d\left(T\left(w, \zeta_{2n(i)}(\omega)\right), S\left(w, \zeta_{2m(i)-1}(\omega)\right)\right)\right). \tag{3.31}$$

Now, using the contractive condition (3.1), we have

$$\begin{aligned}
&\phi_1\left(d\left(S\left(w, \zeta_{2m(i)-1}(\omega)\right), T\left(w, \zeta_{2n(i)}(\omega)\right)\right)\right)\\
&\leq \psi_1(d\left(R\left(w, \zeta_{2m(i)-1}(\omega)\right), T\left(w, \zeta_{2n(i)}(\omega)\right)\right),\\
&d\left(R\left(w, \zeta_{2m(i)-1}(\omega)\right), R\left(w, \zeta_{2m(i)}(\omega)\right)\right)\\
&d\left(R\left(w, \zeta_{2n(i)}(\omega)\right), R\left(w, \zeta_{2n(i)+1}(\omega)\right)\right),\\
&\tfrac{1}{2}[d\left(R\left(w, \zeta_{2m(i)-1}(\omega)\right), R\left(w, \zeta_{2n(i)+1}(\omega)\right)\right)+\\
&d\left(R\left(w, \zeta_{2n(i)}(\omega)\right), T\left(w, \zeta_{2m(i)}(\omega)\right)\right)])\\
&-\psi_2(d\left(R\left(w, \zeta_{2m(i)-1}(\omega)\right), R\left(w, \zeta_{2n(i)}(\omega)\right)\right),\\
&d\left(R\left(w, \zeta_{2m(i)-1}(\omega)\right), R\left(w, \zeta_{2m(i)}(\omega)\right)\right),\\
&d\left(R\left(w, \zeta_{2n(i)}(\omega)\right), R\left(w, \zeta_{2n(i)+1}(\omega)\right)\right),\\
&\tfrac{1}{2}[d\left(R\left(w, \zeta_{2m(i)-1}(\omega)\right), R\left(w, \zeta_{2n(i)+1}(\omega)\right)\right)\\
&+d\left(R\left(w, \zeta_{2n(i)}(\omega)\right), R\left(w, \zeta_{2m(i)}(\omega)\right)\right)]).
\end{aligned} \tag{3.32}$$

From (3.7), (3.29) and the continuity of ψ_1 and ψ_2, we get by letting $i \to +\infty$ in the above inequality

$$\begin{aligned}
&\lim_{i\to+\infty} \phi_1\left(d\left(S\left(w, \zeta_{2m(i)-1}(\omega)\right), T\left(w, \zeta_{2n(i)}(\omega)\right)\right)\right)\\
&\leq \psi_1(\epsilon, 0, 0, \epsilon) - \psi_2(\epsilon, 0, 0, \epsilon)\\
&\leq \phi_1(\epsilon) - \psi_2(\epsilon, 0, 0, \epsilon).
\end{aligned} \tag{3.33}$$

Now, combining (3.31) with (3.33), we get

$$\phi_1(\epsilon) \leq \phi_1(\epsilon) - \psi_2(\epsilon, 0, 0, \epsilon) \tag{3.34}$$

which implies that $\psi_2(\epsilon, 0, 0, \epsilon) = 0$, and contradicts the inequality $\epsilon > 0$. We deduce that for $\omega \in \Omega$, $\{R(\omega, \zeta_n(\omega))\}$ is a Cauchy sequence. Since $\{R(\omega, \zeta_n(\omega))\}$ is a Cauchy sequence in the complete metric space K, so there exists $\zeta : \Omega \to K$ such that

$$\lim_{n\to+\infty} R(\omega, \zeta_n(\omega)) = \zeta(\omega). \tag{3.35}$$

From (3.35) and the continuity of R, we get

$$\lim_{n\to+\infty} R(\omega, R(\omega, \zeta_n(\omega))) = R(\omega, \zeta(\omega)). \tag{3.36}$$

By the triangle inequality, we have

$$\begin{aligned}
&d(R(\omega, \zeta(\omega)), T(\omega, \zeta(\omega)))\\
&\leq d(R(\omega, \zeta(\omega)), R(\omega, R(\omega, \zeta_{2n+1}(\omega))))\\
&+d(R(\omega, T(\omega, \zeta_{2n}(\omega))), T(\omega, R(\omega, \zeta_{2n}(\omega))))\\
&+d(T(\omega, R(\omega, \zeta_{2n}(\omega))), T(\omega, \zeta(\omega))).
\end{aligned} \tag{3.37}$$

On the other hand, we have $R(\omega, \zeta_{2n}(\omega)) \to \zeta(\omega)$, $T(\omega, \zeta_{2n}(\omega)) \to \zeta(\omega)$ as $n \to \infty$. As R and T are compatible random mappings, so we have

$$\lim_{n\to+\infty} d(R(\omega, T(\omega, \zeta_{2n}(\omega))), T(\omega, R(\omega, \zeta_{2n}(\omega)))) = 0. \tag{3.38}$$

Now, from the continuity of T and (3.35), we have

$$\lim_{n\to+\infty} d\left(T\left(\omega, R\left(\omega, \zeta_{2n}\left(\omega\right)\right)\right), T\left(\omega, \zeta\left(\omega\right)\right)\right) = 0. \tag{3.39}$$

Combining (3.36), (3.38) and (3.39) and letting $n \to +\infty$ in (3.37), we have

$$d\left(R\left(\omega, \zeta\left(\omega\right)\right), T\left(\omega, \zeta\left(\omega\right)\right)\right) \leq 0 \tag{3.40}$$

that is,

$$R\left(\omega, \zeta\left(\omega\right)\right) = T\left(\omega, \zeta\left(\omega\right)\right). \tag{3.41}$$

Again, by triangle inequality, we have

$$\begin{aligned} & d\left(R\left(\omega, \zeta\left(\omega\right)\right), S\left(\omega, \zeta\left(\omega\right)\right)\right) \\ & \leq d\left(R\left(\omega, \zeta\left(\omega\right)\right), R\left(\omega, R\left(\omega, \zeta_{2n+2}\left(\omega\right)\right)\right)\right) \\ & + d\left(R\left(\omega, S\left(\omega, \zeta_{2n+1}\left(\omega\right)\right)\right), S\left(\omega, R\left(\omega, \zeta_{2n+1}\left(\omega\right)\right)\right)\right) \\ & + d\left(S\left(\omega, R\left(\omega, \zeta_{2n+1}\left(\omega\right)\right)\right), S\left(\omega, \zeta\left(\omega\right)\right)\right). \end{aligned} \tag{3.42}$$

On the other hand, we have $R\left(\omega, \zeta_{2n+1}\left(\omega\right)\right) \to \zeta\left(\omega\right)$, $S\left(\omega, \zeta_{2n+1}\left(\omega\right)\right) \to \zeta\left(\omega\right)$ as $n \to \infty$. Since R and S are compatible mappings, therefore we get

$$\lim_{n\to+\infty} d\left(R\left(\omega, S\left(\omega, \zeta_{2n+1}\left(\omega\right)\right)\right), S\left(\omega, R\left(\omega, \zeta_{2n+1}\left(\omega\right)\right)\right)\right) = 0. \tag{3.43}$$

Now, from the continuity of S and (3.35), we have

$$\lim_{n\to+\infty} d\left(S\left(\omega, R\left(\omega, \zeta_{2n+1}\left(\omega\right)\right)\right), S\left(\omega, \zeta\left(\omega\right)\right)\right) = 0. \tag{3.44}$$

Combining (3.36), (3.43) and (3.44) and letting $n \to \infty$ in (3.42), we obtain

$$d\left(R\left(\omega, \zeta\left(\omega\right)\right), S\left(\omega, \zeta\left(\omega\right)\right)\right) \leq 0 \tag{3.45}$$

that is,

$$R\left(\omega, \zeta\left(\omega\right)\right) = S\left(\omega, \zeta\left(\omega\right)\right). \tag{3.46}$$

Finally, from (3.41) and (3.46), we have

$$T\left(\omega, \zeta\left(\omega\right)\right) = R\left(\omega, \zeta\left(\omega\right)\right) = S\left(\omega, \zeta\left(\omega\right)\right)$$

that is, $\zeta\left(\omega\right)$ is a random coincidence point of T, S and R. □

Corollary 3.4. *Let $(X, d, \leq)$ be a separable ordered metric space and K be a nonempty Polish subspace of X. Let $T, S, R : \Omega \times K \to K$ be random operators satisfying (3.1). Assume that*

(1) *T, S and R are continuous random operators,*
(2) *$T(\omega, K) \subseteq R(\omega, K)$ and $S(\omega, K) \subseteq R(\omega, K)$ for each $\omega \in \Omega$,*
(3) *$T(\omega, .)$ and $S(\omega, .)$ are weakly increasing with respect to $R(\omega, .)$ for each $\omega \in \Omega$,*
(4) *the pairs $\{T, R\}$ and $\{S, R\}$ are commuting random operators.*

Then there exists a measurable mapping $\zeta : \Omega \to K$ such that

$$R\left(\omega, \zeta\left(\omega\right)\right) = T\left(\omega, \zeta\left(\omega\right)\right) = S\left(\omega, \zeta\left(\omega\right)\right)$$

for each $\omega \in \Omega$.

Corollary 3.5. *Let $(X, d, \leq)$ be a separable ordered metric space and K be a nonempty Polish subspace of X. Let $T, S : \Omega \times K \to K$ be random operators satisfying the following inequality, for every pair $(x, y) \in K \times K$ with x and y comparable,*

(3.47)

$$\begin{aligned}&\phi_1\left(d\left(S\left(\omega, x\right), T\left(\omega, y\right)\right)\right)\\ &\leq \psi_1(d\left(x, y\right), d\left(x, S\left(\omega, x\right)\right), d\left(y, T\left(\omega, y\right)\right), \tfrac{1}{2}[d\left(x, T\left(\omega, y\right)\right) + d\left(y, S\left(\omega, x\right)\right)])\\ &-\psi_2(d\left(x, y\right), d\left(x, S\left(\omega, x\right)\right), d\left(y, T\left(\omega, y\right)\right), \tfrac{1}{2}[d\left(x, T\left(\omega, y\right)\right) + d\left(y, S\left(\omega, x\right)\right)])\end{aligned}$$

where ψ_1 and ψ_2 are generalized altering distance functions and $\phi_1(x) = \psi_1(x, x, x, x)$. Assume that T and S are continuous random operators and $T(\omega, .)$ and $S(\omega, .)$ are weakly increasing for each $\omega \in \Omega$. Then, there exists a measurable mapping $\zeta : \Omega \to K$ such that

$$T\left(\omega, \zeta\left(\omega\right)\right) = S\left(\omega, \zeta\left(\omega\right)\right) \text{ for each } \omega \in \Omega.$$

Definition 3.6. *Let $(X, d, \leq)$ be a separable partially ordered metric space and K be a nonempty Polish subspace of X. We say that K is regular if and only if the following hypothesis holds: if $\{\zeta_n(\omega)\}$ is a nondecreasing sequence in K with respect to $\leq$ and $\zeta : \Omega \to K$ such that $\zeta_n(\omega) \to \zeta(\omega) \in K$ as $n \to +\infty$, then $\zeta_n(\omega) \leq \zeta(\omega)$ for all $n \in \mathbb{N}$ and for each $\omega \in \Omega$.*

Theorem 3.7. *Let $(X, d, \leq)$ be a separable ordered metric space and K be a nonempty Polish subspace of X. Let $T, S, R : \Omega \times K \to K$ be random operators satisfying inequality (3.1). Assume that*

1. *$T(\omega, K) \subseteq R(\omega, K)$ and $S(\omega, K) \subseteq R(\omega, K)$ for each $\omega \in \Omega$,*
2. *$T(\omega, .)$ and $S(\omega, .)$ are weakly increasing with respect to $R(\omega, .)$ for each $\omega \in \Omega$,*
3. *K is regular,*
4. *$R(\omega, K)$ is a complete subspace of K.*

Then there exists a measurable mapping $\zeta : \Omega \to K$ such that

$$R\left(\omega, \zeta\left(\omega\right)\right) = T\left(\omega, \zeta\left(\omega\right)\right) = S\left(\omega, \zeta\left(\omega\right)\right) \text{ for each } \omega \in \Omega.$$

Proof. As in the proof of Theorem 3.3, we have $\{R(\omega, \zeta_n(\omega))\}$ is a Cauchy sequence in the complete metric space $(R(\omega, K), d)$, therefore there exists $\theta(\omega) = R(\omega, \zeta(\omega))$, $\zeta(\omega) \in K$ such that

$$\lim_{n\to+\infty}\left\{R\left(\omega, \zeta_n\left(\omega\right)\right)\right\} = \theta\left(\omega\right) = R\left(\omega, \zeta\left(\omega\right)\right) \quad \text{for each } \omega \in \Omega. \tag{3.48}$$

Since $\{R(\omega, \zeta_n(\omega))\}$ is a nondecreasing sequence and K is regular, it follows from (3.48) that $R(\omega, \zeta_n(\omega)) \leq R(\omega, \zeta(\omega))$ for all $n \in \mathbb{N}$ and for each $\omega \in \Omega$.

So, by contractive condition (3.1), we have

$$\begin{aligned}
&\phi_1\left(d\left(S\left(\omega,\zeta\left(\omega\right)\right),R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)\right)\right)=\phi_1\left(d\left(S\left(\omega,\zeta\left(\omega\right)\right),T\left(\omega,\zeta_{2n}\left(\omega\right)\right)\right)\right)\\
&\leq\psi_1(d\left(R\left(\omega,\zeta\left(\omega\right)\right),R\left(\omega,\zeta_{2n}\left(\omega\right)\right)\right),d\left(R\left(\omega,\zeta\left(\omega\right)\right),S\left(\omega,\zeta\left(\omega\right)\right)\right),\\
&d\left(R\left(\omega,\zeta_{2n}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)\right),\tfrac{1}{2}[d\left(R\left(\omega,\zeta\left(\omega\right)\right),T\left(\omega,\zeta_{2n}\left(\omega\right)\right)\right)\\
&+d\left(R\left(\omega,\zeta_{2n}\left(\omega\right)\right),S\left(\omega,\zeta\left(\omega\right)\right)\right)])-\psi_2(d\left(R\left(\omega,\zeta\left(\omega\right)\right),R\left(\omega,\zeta_{2n}\left(\omega\right)\right)\right),\\
&d\left(R\left(\omega,\zeta\left(\omega\right)\right),S\left(\omega,\zeta\left(\omega\right)\right)\right),d\left(R\left(\omega,\zeta_{2n}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)\right),\\
&\tfrac{1}{2}[d\left(R\left(\omega,\zeta\left(\omega\right)\right),T\left(\omega,\zeta_{2n}\left(\omega\right)\right)\right)+d\left(R\left(\omega,\zeta_{2n}\left(\omega\right)\right),S\left(\omega,\zeta\left(\omega\right)\right)\right)]).
\end{aligned}$$

Letting $n\to+\infty$ in the above inequality and using (3.7), (3.48) and the properties of $\psi_1$and ψ_2, we obtain

$$\begin{aligned}
&\phi_1\left(d\left(S\left(\omega,\zeta\left(\omega\right)\right),R\left(\omega,\zeta\left(\omega\right)\right)\right)\right)\\
&\leq\psi_1(0,d\left(R\left(\omega,\zeta\left(\omega\right)\right),S\left(\omega,\zeta\left(\omega\right)\right)\right),0,\tfrac{1}{2}d\left(R\left(\omega,\zeta\left(\omega\right)\right),S\left(\omega,\zeta\left(\omega\right)\right)\right))\\
&-\psi_2(0,d\left(R\left(\omega,\zeta\left(\omega\right)\right),S\left(\omega,\zeta\left(\omega\right)\right)\right),0,\tfrac{1}{2}d\left(R\left(\omega,\zeta\left(\omega\right)\right),S\left(\omega,\zeta\left(\omega\right)\right)\right))\\
&\leq\phi_1\left(d\left(S\left(\omega,\zeta\left(\omega\right)\right),R\left(\omega,\zeta\left(\omega\right)\right)\right)\right)-\psi_2(0,d\left(R\left(\omega,\zeta\left(\omega\right)\right),S\left(\omega,\zeta\left(\omega\right)\right)\right),0,\\
&\tfrac{1}{2}d\left(R\left(\omega,\zeta\left(\omega\right)\right),S\left(\omega,\zeta\left(\omega\right)\right)\right).
\end{aligned}$$

This implies that

$$\psi_2(0,d\left(R\left(\omega,\zeta\left(\omega\right)\right),S\left(\omega,\zeta\left(\omega\right)\right)\right),0,\frac{1}{2}d\left(R\left(\omega,\zeta\left(\omega\right)\right),S\left(\omega,\zeta\left(\omega\right)\right)\right)=0,$$

which gives that

$$d\left(R\left(\omega,\zeta\left(\omega\right)\right),S\left(\omega,\zeta\left(\omega\right)\right)\right)=0,$$

that is,

$$R\left(\omega,\zeta\left(\omega\right)\right)=S\left(\omega,\zeta\left(\omega\right)\right). \tag{3.49}$$

Again, by (3.1), we have

$$\begin{aligned}
&\phi_1\left(d\left(R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right),T\left(\omega,\zeta\left(\omega\right)\right)\right)\right)=\phi_1\left(d\left(S\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),T\left(\omega,\zeta\left(\omega\right)\right)\right)\right)\\
&\leq\psi_1(d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta\left(\omega\right)\right)\right),d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),\\
&d\left(R\left(\omega,\zeta\left(\omega\right)\right),T\left(\omega,\zeta\left(\omega\right)\right)\right),\tfrac{1}{2}[d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),T\left(\omega,\zeta\left(\omega\right)\right)\right)\\
&+d\left(R\left(\omega,\zeta\left(\omega\right)\right),S\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)\right)])-\psi_2(d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta\left(\omega\right)\right)\right),\\
&d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),R\left(\omega,\zeta_{2n+2}\left(\omega\right)\right)\right),d\left(R\left(\omega,\zeta\left(\omega\right)\right),T\left(\omega,\zeta\left(\omega\right)\right)\right),\\
&\tfrac{1}{2}[d\left(R\left(\omega,\zeta_{2n+1}\left(\omega\right)\right),T\left(\omega,\zeta\left(\omega\right)\right)\right)+d\left(R\left(\omega,\zeta\left(\omega\right)\right),S\left(\omega,\zeta_{2n+1}\left(\omega\right)\right)\right)]).
\end{aligned}$$

Letting $n\to+\infty$ in the above inequality, we get

$$\begin{aligned}
&\phi_1\left(d\left(R\left(\omega,\zeta\left(\omega\right)\right),T\left(\omega,\zeta\left(\omega\right)\right)\right)\right)\\
&\leq\psi_1(0,0,d\left(R\left(\omega,\zeta\left(\omega\right)\right),T\left(\omega,\zeta\left(\omega\right)\right)\right),\tfrac{1}{2}d\left(R\left(\omega,\zeta\left(\omega\right)\right),T\left(\omega,\zeta\left(\omega\right)\right)\right))\\
&-\psi_2(0,0,d\left(R\left(\omega,\zeta\left(\omega\right)\right),T\left(\omega,\zeta\left(\omega\right)\right)\right),\tfrac{1}{2}d\left(R\left(\omega,\zeta\left(\omega\right)\right),T\left(\omega,\zeta\left(\omega\right)\right)\right))\\
&\leq\phi_1\left(d\left(R\left(\omega,\zeta\left(\omega\right)\right),T\left(\omega,\zeta\left(\omega\right)\right)\right)\right)-\psi_2(0,0,d\left(R\left(\omega,\zeta\left(\omega\right)\right),T\left(\omega,\zeta\left(\omega\right)\right)\right)\\
&\tfrac{1}{2}d\left(R\left(\omega,\zeta\left(\omega\right)\right),T\left(\omega,\zeta\left(\omega\right)\right)\right)).
\end{aligned}$$

This implies that

$$\psi_2(0,0,d\left(R\left(\omega,\zeta\left(\omega\right)\right),T\left(\omega,\zeta\left(\omega\right)\right)\right),\frac{1}{2}d\left(R\left(\omega,\zeta\left(\omega\right)\right),T\left(\omega,\zeta\left(\omega\right)\right)\right)=0$$

and so,

$$R(\omega, \zeta(\omega)) = T(\omega, \zeta(\omega)) . \tag{3.50}$$

Now, combining (3.49) and (3.50), we obtain

$$R(\omega, \zeta(\omega)) = T(\omega, \zeta(\omega)) = S(\omega, \zeta(\omega))$$

for each $\omega \in \Omega$. Hence, $\zeta(\omega)$ is a random coincidence point of T, S, and R □

Corollary 3.8. *Let $(X, d, \leq)$ be a separable ordered metric space and K be a nonempty Polish subspace of X. Let $T, S : \Omega \times K \to K$ be random operators satisfying the following inequality for every pair $(x, y) \in K \times K$ with x and y comparable,*

$$\begin{aligned}
&\phi_1(d(S(\omega, x), T(\omega, y))) \\
&\leq \psi_1(d(x, y), d(x, S(\omega, x)), d(y, T(\omega, y)), \tfrac{1}{2}[d(x, T(\omega, y)) + d(y, S(\omega, x))]) \\
&-\psi_2(d(x, y), d(x, S(\omega, x)), d(y, T(\omega, y)), \tfrac{1}{2}[d(x, T(\omega, y)) + d(y, S(\omega, x))])
\end{aligned}$$

where ψ_1 and ψ_2 are generalized altering distance functions and $\phi_1(x) = \psi_1(x, x, x, x)$. Assume that $T(\omega, .)$ and $S(\omega, .)$ are weakly increasing. If K is regular, then there exists a measurable mapping $\zeta : \Omega \to K$ such that

$$T(\omega, \zeta(\omega)) = S(\omega, \zeta(\omega)) \text{ for each } \omega \in \Omega.$$

Corollary 3.9. *Let $(X, d, \leq)$ be a separable ordered metric space and K be a nonempty Polish subspace of X. Let $T : \Omega \times K \to K$ be a random operator satisfying the following inequality, for every comparable pair $(x, y) \in K \times K$*

$$\begin{aligned}
&\phi_1(d(T(\omega, x), T(\omega, y))) \\
&\leq \psi_1(d(x, y), d(x, T(\omega, x)), d(y, T(\omega, y)), \tfrac{1}{2}[d(x, T(\omega, y)) + d(y, T(\omega, x))]) \\
&-\psi_2(d(x, y), d(x, T(\omega, x)), d(y, T(\omega, y)), \tfrac{1}{2}[d(x, T(\omega, y)) + d(y, T(\omega, x))])
\end{aligned}$$

where ψ_1 and ψ_2 are generalized altering distance functions and $\phi_1(x) = \psi_1(x, x, x, x)$. Suppose $T(\omega, .)$ is weakly increasing for each $\omega \in \Omega$. Assume that either T is continuous random operator or K is regular, then, T has a random fixed point, that is, there exists a measurable mapping $\zeta : \Omega \to K$ such that

$$\zeta(\omega) = T(\omega, \zeta(\omega)) \text{ for each } \omega \in \Omega.$$

4. **Results and discussion**

Our results provide random versions of Theorem 2.9 and corresponding results in [8] and [19]. Theorem 3.3 is a generalization of Theorem 2.2 in [24] for three maps considering generalized altering distance functions. Corollary 3.5 is an extension of Theorem 2.4 in [24]. Consequently, Our results present random versions, improvement, extension and generalization of recent results in the literature.

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