ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 41 (2015), No. 2, pp. 423–428

Title:

The Artinian property of certain graded generalized local chohomology modules

Author(s):

Y. Gu and L. Chu

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 2, pp. 423–428
Online ISSN: 1735-8515

THE ARTINIAN PROPERTY OF CERTAIN GRADED GENERALIZED LOCAL CHOHOMOLOGY MODULES

Y. GU* AND L. CHU

(Communicated by Rahim Zaare-Nahandi)

Abstract. Let $R = \oplus_{n\in\mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with local base ring $(R_0, \mathfrak{m}_0)$, M and N two finitely generated graded R-modules. Let t be the least integer such that $H^t_{R_+}(M, N)$ is not minimax. We prove that $H^j_{\mathfrak{m}_0 R}(H^t_{R_+}(M, N))$ is Artinian for $j = 0, 1$. Also, we show that if $\mathrm{cd}(R_+, M, N) = 2$ and $t \in \mathbb{N}_0$, then $H^t_{\mathfrak{m}_0 R}(H^2_{R_+}(M, N))$ is Artinian if and only if $H^{t+2}_{\mathfrak{m}_0 R}(H^1_{R_+}(M, N))$ is Artinian.

Keywords: Graded local cohomology modules, Artinian modules, minimax.
MSC(2010): Primary: 13D45; Secondary: 13E10.

1. Introduction

Throughout this paper, let $R = \oplus_{n\in\mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with local base ring $(R_0, \mathfrak{m}_0)$. So R_0 is a Noetherian ring and there are finitely many elements $l_1, l_2, \dots, l_r \in R_1$ such that $R = R_0[l_1, l_2, \dots, l_r]$. We denote $R_+ = \oplus_{n\in\mathbb{N}} R_n$ the irrelevant ideal of R and that $\mathfrak{m} = \mathfrak{m}_0 \oplus R_+$ the graded maximal ideal of R. Assume that $M = \oplus_{n\in\mathbb{Z}} M_n$ and $N = \oplus_{n\in\mathbb{Z}} N_n$ are finitely generated graded R-modules.(Here, $\mathbb{N}_0$, $\mathbb{N}$ denotes the set of non-negative and positive integers respectively; $\mathbb{Z}$ will denote the set of all integers.)

Let $H^i_{R_+}(M, N)$ be the i-th graded generalized local cohomology module of M and N with respect to R_+. As it has been shown in [12], for each $i \in \mathbb{N}_0$, $H^i_{R_+}(M, N)$ has a natural graded structure. For each $n \in \mathbb{Z}$, we denote the n-th homogeneous component of $H^i_{R_+}(M, N)$ by $H^i_{R_+}(M, N)_n$. Assume that $\mathrm{pd}M < \infty$. Then the R_0-modules $H^i_{R_+}(M, N)_n$ are finitely generated for all $n \in \mathbb{Z}$ and they are zero for $n \gg 0$.

Article electronically published on April 29, 2015.
Received: 26 September 2013, Accepted: 13 March 2014.
*Corresponding author.

In this article, we are interested in the Artinian property of the graded generalized local cohomology modules $H^i_{R_+}(M,N)$.

The minimax modules were introduced by Zöschinger [13]. A graded R-module L is said to be **minimax** if there is a finitely generated graded submodule K such that L/K is Artinian. The class of minimax modules includes all finitely generated modules and all Artinian modules.

Recall that a graded R-module $T = \oplus_{n\in\mathbb{Z}} T_n$ is said to be **tame** if there exists an integer n_0 such that either $T_n = 0$ for all $n < n_0$ or $T_n \neq 0$ for all $n < n_0$. Brodmann and Hellus in [2] proved that all finitely generated modules and Artinian modules are tame.

Mafi and Saremi in [8] proved that $H^j_{\mathfrak{m}_0 R}(H^t_{R_+}(M))$ is Artinian for $j = 0, 1$, where $t = \inf\{i \in \mathbb{N}_0 | H^i_{R_+}(M)$ is not finitely generated$\}$.

Here, we get a generalization of the above result.

Theorem 1.1. $H^j_{\mathfrak{m}_0 R}(H^t_{R_+}(M,N))$ *is Artinian for* $j = 0, 1$, *where* $t = \inf\{i \in \mathbb{N}_0 | H^i_{R_+}(M,N)$ *is not minimax*$\}$.

In [11] Sazeedeh showed that if $H^2_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ is Artinian, then $\Gamma_{\mathfrak{m}_0 R}(H^2_{R_+}(M))$ is Artinian. By Theorem 1.1, we establish a similar result for graded generalized local cohomology modules.

In [10] Sazeedeh proved that if $\operatorname{ara}(R_+) = 2$, then $H^t_{\mathfrak{m}_0 R}(H^2_{R_+}(M))$ is Artinian if and only if $H^{t+2}_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ is Artinian for each $t \in \mathbb{N}_0$. Mafi and Saremi [8] also proved that if $\operatorname{cd}(R_+, M) = 2$, then $H^t_{\mathfrak{m}_0 R}(H^2_{R_+}(M))$ is Artinian if and only if $H^{t+2}_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ is Artinian for each $t \in \mathbb{N}_0$.

In this paper, we get the following result.

Theorem 1.2. *Let* $\operatorname{cd}(R_+, M, N) = 2$ *and* $t \in \mathbb{N}_0$. *Then* $H^t_{\mathfrak{m}_0 R}(H^2_{R_+}(M,N))$ *is Artinian if and only if* $H^{t+2}_{\mathfrak{m}_0 R}(H^1_{R_+}(M,N))$ *is Artinian.*

In addition, we show that $H^t_{R_+}(M,N)$ is tame, where t is the least integer i such that $H^i_{R_+}(M,N)$ is not Artinian. This result extends [7, Theorem 2.1].

For notations and terminologies not explained in this paper, the reader is referred to [3] and [4] if necessary.

2. The results

Definition 2.1. We denote by $c := \operatorname{cd}(R_+, M, N)$ the **cohomological dimension** of M and N with respect to R_+ which is

$$c := \operatorname{cd}(R_+, M, N) = \sup\{i \in \mathbb{N}_0 | H^i_{R_+}(M,N) \neq 0\}.$$

One can easily see that $\operatorname{cd}(R_+, M, N) = \operatorname{cd}(R_+, N)$ if $M = R$.

Theorem 2.2. $H^j_{\mathfrak{m}_0 R}(H^t_{R_+}(M,N))$ *is Artinian for* $j = 0, 1$, *where* $t = \inf\{i \in \mathbb{N}_0 | H^i_{R_+}(M,N)$ *is not minimax*$\}$.

Proof. By [9, Theorem 11.38], we consider the Grothendieck spectral sequence

$$E_2^{p,q} := H^p_{\mathfrak{m}_0 R}(H^q_{R_+}(M,N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{m}}(M,N).$$

Thus, for each $n \geq 0$, there is a finite filtration of the module $H^n = H^n_{\mathfrak{m}}(M,N)$

$$0 = \phi^{n+1}H^n \subseteq \phi^n H^n \subseteq \cdots \subseteq \phi^1 H^n \subseteq \phi^0 H^n = H^n$$

such that $E_\infty^{i,n-i} \cong \phi^i H^n/\phi^{i+1}H^n$ for all $0 \leq i \leq n$.

For each $i \geq 2$ and $p, q \geq 0$, we consider the following exact sequence

$$0 \longrightarrow \mathrm{Ker} d_i^{p,q} \longrightarrow E_i^{p,q} \xrightarrow{d_i^{p,q}} E_i^{p+i,q-i+1}.$$

(Note that $E_{i+1}^{p,q} \cong \mathrm{Ker} d_i^{p,q}/\mathrm{Im} d_i^{p-i,q+i-1}$ and $E_i^{p,q} = 0$ for all $q < 0$.)

For each $i \geq 2$ and $j = 0, 1$, we have $E_{i+1}^{j,t} \cong \mathrm{Ker} d_i^{j,t}$ and $E_{t+2}^{j,t} \cong E_{t+3}^{j,t} \cong \cdots \cong E_\infty^{j,t}$. As a subquotient of $H^{j+t}_{\mathfrak{m}}(M,N)$, $E_\infty^{j,t}$ is Artinian for $j = 0, 1$. So $E_{t+2}^{j,t}$ and $\mathrm{Ker} d_{t+1}^{j,t}$ are Artinian for $j = 0, 1$. By the above exact sequence, we can deduce that $\mathrm{Ker} d_t^{j,t}$ is Artinian for $j = 0, 1$. By applying the above argument for finite steps, we get that $\mathrm{Ker} d_2^{j,t}$ is Artinian for $j = 0, 1$. Since $E_2^{j+2,t-1}$ is Artinian by [11, Lemma 3.2], it follows that $E_2^{j,t}$ is Artinian for $j = 0, 1$ by the above exact sequence. □

The next corollary is a generalization of the main result in [10].

Corollary 2.3. $\Gamma_{\mathfrak{m}_0 R}(H^1_{R_+}(M,N))$ *and* $H^1_{\mathfrak{m}_0 R}(H^1_{R_+}(M,N))$ *are Artinian.*

Proof. If $H^1_{R_+}(M,N)$ is minimax, then $\Gamma_{\mathfrak{m}_0 R}(H^1_{R_+}(M,N))$ and $H^1_{\mathfrak{m}_0 R}(H^1_{R_+}(M,N))$ are Artinian by [11, Lemma 3.2]. If $H^1_{R_+}(M,N)$ is not minimax, then we get the result by Theorem 2.2. □

The following example which has been presented in [10, Example 2.9] shows that $\Gamma_{\mathfrak{m}_0 R}(H^2_{R_+}(M,N))$ and $H^2_{\mathfrak{m}_0 R}(H^1_{R_+}(M,N))$ are not Artinian even if R_0 is a regular local ring.

Example 2.4. *[10, Example 2.9]* Let K be a field, let x, y, t be indeterminates, let $R_0 = K[x,y]_{(x,y)}$ and $\mathfrak{m}_0 = (x,y)R_0$. Moreover, let $R = R_0[\mathfrak{m}_0 t]$ be the Rees ring of $\mathfrak{m}_0$. One can easily see that $R_+ = (xt, yt)R$, hence $\mathrm{ara}(R_+) = 2$. It follows from [1, Example 4.2] that $\Gamma_{\mathfrak{m}_0 R}(H^2_{R_+}(R))$ is not Artinian. We deduce that $H^2_{\mathfrak{m}_0 R}(H^1_{R_+}(R))$ is not Artinian from [10, Corollary 2.4].

Next, we shall prove that $\Gamma_{\mathfrak{m}_0 R}(H^2_{R_+}(M,N))$ is Artinian in some cases. For that, we need the following lemma.

Lemma 2.5. $H^i_{\mathfrak{m}_0 R}(H^j_{R_+}(M,N))$ *is Artinian if and only if* $H^i_{\mathfrak{m}_0 R}(H^j_{R_+}(M, N/\Gamma_{R_+}(N)))$ *is Artinian for all* $i, j \in \mathbb{N}_0$.

Proof.

$$0 \longrightarrow \Gamma_{R_+}(N) \longrightarrow N \longrightarrow N/\Gamma_{R_+}(N) \longrightarrow 0,$$

we get the following long exact sequence

$$\begin{aligned} H^j_{R_+}(M,\Gamma_{R_+}(N)) \xrightarrow{f} H^j_{R_+}(M,N) &\xrightarrow{g} H^j_{R_+}(M,N/\Gamma_{R_+}(N)) \\ &\xrightarrow{h} H^{j+1}_{R_+}(M,\Gamma_{R_+}(N)) \end{aligned}$$

which deduce two short exact sequences

$$0 \longrightarrow \mathrm{Im}f \longrightarrow H^j_{R_+}(M,N) \longrightarrow \mathrm{Im}g \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Im}g \longrightarrow H^j_{R_+}(M,N/\Gamma_{R_+}(N)) \longrightarrow \mathrm{Im}h \longrightarrow 0.$$

For all $j \geq 0$, $H^j_{R_+}(M,\Gamma_{R_+}(N)) \cong \mathrm{Ext}^j_R(M,\Gamma_{R_+}(N))$ is finitely generated, it follows that $\mathrm{Im}f$ and $\mathrm{Im}h$ are finitely generated. Application of the functor $H^i_{\mathfrak{m}_0R}(-)$ to the above two short exact sequences induce the following exact sequences

$$H^i_{\mathfrak{m}_0R}(\mathrm{Im}f) \longrightarrow H^i_{\mathfrak{m}_0R}(H^j_{R_+}(M,N)) \longrightarrow H^i_{\mathfrak{m}_0R}(\mathrm{Im}g) \longrightarrow H^{i+1}_{\mathfrak{m}_0R}(\mathrm{Im}f)$$

and

$$H^{i-1}_{\mathfrak{m}_0R}(\mathrm{Im}h) \longrightarrow H^i_{\mathfrak{m}_0R}(\mathrm{Im}g) \longrightarrow H^i_{\mathfrak{m}_0R}(H^j_{R_+}(M,N/\Gamma_{R_+}(N))) \longrightarrow H^i_{\mathfrak{m}_0R}(\mathrm{Im}h).$$

Then $H^i_{\mathfrak{m}_0R}(H^j_{R_+}(M,N))$ is Artinian if and only if $H^i_{\mathfrak{m}_0R}(H^j_{R_+}(M,N/\Gamma_{R_+}(N)))$ is Artinian. □

Theorem 2.6. *Let $H^2_{\mathfrak{m}_0R}(H^1_{R_+}(M,N))$ is Artinian. Then $\Gamma_{\mathfrak{m}_0R}(H^2_{R_+}(M,N))$ is Artinian.*

Proof. If $\dim(N/\mathfrak{m}_0N) \leq 0$, then $N = \Gamma_{R_+}(N)$. Thus $H^2_{R_+}(M,N) \cong \mathrm{Ext}^2_R(M,N)$ is finitely generated, and $\Gamma_{\mathfrak{m}_0R}(H^2_{R_+}(M,N))$ is Artinian by [11, Lemma 3.2]. Now we may assume that $\dim(N/\mathfrak{m}_0N) > 0$ and that $\Gamma_{R_+}(N) = 0$ by Lemma 2.5, and so there exists a homogeneous element $a \in R_+$ which is non-zero divisor on N. Let $\deg(a) = t$. From the short exact sequence

$$0 \longrightarrow N(-t) \xrightarrow{a} N \longrightarrow N/aN \longrightarrow 0,$$

we get the following long exact sequence

$$\begin{aligned} 0 &\longrightarrow \Gamma_{R_+}(M,N/aN) \longrightarrow H^1_{R_+}(M,N)(-t) \xrightarrow{f} H^1_{R_+}(M,N) \\ &\xrightarrow{g} H^1_{R_+}(M,N/aN) \xrightarrow{h} H^2_{R_+}(M,N)(-t) \longrightarrow H^2_{R_+}(M,N) \longrightarrow \cdots . \end{aligned}$$

Set $X_1 = \mathrm{Im}h$, $X_2 = \mathrm{Im}g$ and $X_3 = \mathrm{Im}f$. Next, we will prove that $\Gamma_{\mathfrak{m}_0R}(X_1)$ is Artinian. Since $\Gamma_{\mathfrak{m}_0R}(H^1_{R_+}(M,N/aN))$ is Artinian by Corollary 2.3, it is enough to show that $H^1_{\mathfrak{m}_0R}(X_2)$ is Artinian. From the exact sequence

$$\cdots \longrightarrow H^1_{\mathfrak{m}_0R}(H^1_{R_+}(M,N)) \longrightarrow H^1_{\mathfrak{m}_0R}(X_2) \longrightarrow H^2_{\mathfrak{m}_0R}(X_3) \longrightarrow \cdots ,$$

and by Corollary 2.3, it suffices to show that $H^2_{\mathfrak{m}_0R}(X_3)$ is Artinian. The short exact sequence

$$0 \longrightarrow \Gamma_{R_+}(M, N/aN) \longrightarrow H^1_{R_+}(M,N)(-t) \longrightarrow X_3 \longrightarrow 0$$

yields the following exact sequence

$$H^2_{\mathfrak{m}_0R}(H^1_{R_+}(M,N))(-t) \longrightarrow H^2_{\mathfrak{m}_0R}(X_3) \longrightarrow H^3_{\mathfrak{m}_0R}(\Gamma_{R_+}(M, N/aN)),$$

it follows that $H^2_{\mathfrak{m}_0R}(X_3)$ is Artinian. Hence

$$\Gamma_{\mathfrak{m}_0R}(X_1) = \Gamma_{\mathfrak{m}_0R}(0 :_{H^2_{R_+}(M,N)} a)(-t)) = (0 :_{\Gamma_{\mathfrak{m}_0R}(H^2_{R_+}(M,N))} a)(-t)$$

is Artinian. In virtue of $\Gamma_{\mathfrak{m}_0R}(H^2_{R_+}(M,N))$ being a-torsion, then the result follows by [3, Theorem 7.1.2]. □

Next we give another main result of this paper.

Theorem 2.7. *Let* $\operatorname{cd}(R_+, M, N) = 2$ *and* $t \in \mathbb{N}_0$. *Then* $H^t_{\mathfrak{m}_0R}(H^2_{R_+}(M,N))$ *is Artinian if and only if* $H^{t+2}_{\mathfrak{m}_0R}(H^1_{R_+}(M,N))$ *is Artinian.*

Proof. By [9, Theorem 11.38], we consider the Grothendieck spectral sequence

$$E_2^{p,q} := H^p_{\mathfrak{m}_0R}(H^q_{R_+}(M,N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{m}}(M,N).$$

Since $E_2^{p,q} = 0$ for $q > 2$, we have the following exact sequence

$$0 \longrightarrow E_3^{t,2} \longrightarrow E_2^{t,2} \xrightarrow{d_2^{t,2}} E_2^{t+2,1} \longrightarrow G \longrightarrow 0,$$

where $G = E_2^{t+2,1}/\operatorname{Im} d_2^{t,2}$. We need just show that $E_3^{t,2}$ and G are Artinian for all $t \geq 0$. Note that $E_3^{t+3,0}$ is a subquotient of $E_2^{t+3,0}$, and thus is Artinian. So $E_3^{t,2}/\ker d_3^{t,2}$ is Artinian. Considering that $E_4^{t,2} \cong E_5^{t,2} \cong \cdots \cong E_\infty^{t,2}$ and $E_\infty^{t,2}$ is isomorphic to a subquotient of $H^{t+2}_{\mathfrak{m}}(M,N)$, then $E_4^{t,2}$ is Artinian. Hence $E_3^{t,2}$ is Artinian.

From the complex

$$E_2^{t,2} \xrightarrow{d_2^{t,2}} E_2^{t+2,1} \xrightarrow{d_2^{t+2,1}} E_2^{t+4,0},$$

we know that $E_2^{t+2,1}/\ker d_2^{t+2,1}$ is Artinian. Since $E_3^{t+2,1} \cong E_4^{t+2,1} \cong \cdots \cong E_\infty^{t+2,1}$, it follows that $E_3^{t+2,1}$ is Artinian. Then $\operatorname{Ker} d_2^{t+2,1}/\operatorname{Im} d_2^{t,2}$ is Artinian. Hence G is Artinian, as required. □

We get the following result which has been proved in [8, Theorem 2.7].

Corollary 2.8. *Let* $\operatorname{cd}(R_+, M) = 2$ *and* $i \in \mathbb{N}_0$. *Then* $H^i_{\mathfrak{m}_0R}(H^2_{R_+}(M))$ *is Artinian if and only if* $H^{i+2}_{\mathfrak{m}_0R}(H^1_{R_+}(M))$ *is Artinian.*

The next corollary provides a condition under which $H^2_{\mathfrak{m}_0R}(H^1_{R_+}(M,N))$ is Artinian if $\Gamma_{\mathfrak{m}_0R}(H^2_{R_+}(M,N))$ is Artinian.

Corollary 2.9. *Let* $\mathrm{cd}(R_+, M, N) = 2$*. Then* $\Gamma_{\mathfrak{m}_0 R}(H^2_{R_+}(M,N))$ *is Artinian if and only if* $H^2_{\mathfrak{m}_0 R}(H^1_{R_+}(M,N))$ *is Artinian.*

Proposition 2.10. *Let* $t = \inf\{i | H^i_{R_+}(M,N)$ *is not Artinian*$\}$*. Then* $H^t_{R_+}(M,N)$ *is tame.*

Proof. For all $i < t$, $H^i_{R_+}(M,N)$ is Artinian. Then $\mathrm{Hom}_R(R/R_+, H^t_{R_+}(M,N))$ is finitely generated by [5, Theorem 2.5]. So $H^t_{R_+}(M,N)$ is tame by [6, Lemma 4.2]. □

Corollary 2.11. ([7, Theorem 2.1]) *Let* $t = \inf\{i | H^i_{R_+}(M)$ *is not Artinian*$\}$*. Then* $H^t_{R_+}(M)$ *is tame.*

Acknowledgments

This research was supported by the National Natural Science Foundation of China (No. 11201326, 11271275), the Natural Science Foundation of Jiangsu Province (No. BK20140300). The authors would like to thank Professor Zhongming Tang for helpful discussions, and the anonymous referee for his or her carefully reading of this manuscript.

References

[1] M. P. Brodmann, S. Fumasoli and R. Tajarod, Local cohomology over homogeneous rings with one-dimensional local base ring, *Proc. Amer. Math. Soc.* **131** (2003), no. 10, 2977–2985.
[2] M. P. Brodmann and M. Hellus, Cohomological patterns of coherent sheaves over projective schemes, *J. Pure Appl. Algebra* **172** (2002), no. 2-3, 165–182.
[3] M. P. Brodmann and R. Y. Sharp, Local Cohomology- An Algebraic Introduction With Geometric Applications, Cambridge University Press, Cambridge, 1998.
[4] W. Bruns and J. Herzog, Cohen-Macaulay Rings, Cambridge University Press, Cambridge, 1993.
[5] L. Z. Chu, Cofiniteness and finiteness of generalized local cohomology modules, *Bull. Aust. Math. Soc.* **80** (2009), no. 2, 244–250.
[6] M. T. Dibaei and A. Nazari, Graded local cohomology: attached and associated primes, asymptotic behaviors, *Comm. Algebra* **35** (2007), no. 5, 1567–1576.
[7] C. S. Lim, A note on graded components of local cohomology modules at the earliest level of non-artinianess, *Bull. Malays. Math. Sci. Soc. (2)* **35** (2012), no. 1, 61–64.
[8] A. Mafi and H. Saremi, Artinianness of certain graded local cohomology modules, *Canad. Math. Bull.* **55** (2012), no. 1, 153–156.
[9] J. Rotman, An introduction to homological algebra, Academic Press, Inc., New York-London, 1979.
[10] R. Sazeedeh, Finiteness of graded local cohomology modules, *J. Pure Appl. Algebra* **212** (2008), no. 1, 275–280.
[11] R. Sazeedeh, Upper and lower bounds for finiteness of graded local cohomology modules, *Comm. Algebra* **37** (2009), no. 10, 3487–3499.
[12] N. Zamani, On the homogeneous pieces of graded generalized local cohomology modules, *Colloq. Math.* **97** (2003), no. 2, 181–188.
[13] H. Zöschinger, Minimax modules, *J. Algebra* **102** (1986), no. 1, 1–32.

(Yan Gu) School of Mathematical Sciences, Soochow University, P.O. Box 215006, Suzhou, China
E-mail address: guyan@suda.edu.cn

(Lizhong Chu) School of Mathematical Sciences, Soochow University, P.O. Box 215006, Suzhou, China
E-mail address: chulizhong@suda.edu.cn