ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 41 (2015), No. 2, pp. 429–451

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 2, pp. 429–451
Online ISSN: 1735-8515

ON CERTAIN SUBCLASSES OF UNIVALENT p-HARMONIC MAPPINGS

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(Communicated by Abbas Salemi)

ABSTRACT. In this paper, the main aim is to introduce the class $\mathcal{U}_p(\lambda, \alpha, \beta, k_0)$ of p-harmonic mappings together with its subclasses $\mathcal{U}_p(\lambda, \alpha, \beta, k_0) \cap \mathcal{T}_p$ and $\mathcal{U}_p(\lambda, \alpha, \beta, k_0) \cap \mathcal{T}_p^0$, and investigate the properties of the mappings in these classes. First, we give a sufficient condition for mappings to be in $\mathcal{U}_p(\lambda, \alpha, \beta, k_0)$ and also the characterization of mappings in $\mathcal{U}_p(\lambda, \alpha, \beta, k_0) \cap \mathcal{T}_p$ for $\max\{0, \frac{\lambda-\frac{1}{2}}{\lambda+1}\} \le \alpha \le \lambda$. Second, we consider the starlikeness of mappings in $\mathcal{U}_p(\lambda, \alpha, \beta, k_0) \cap \mathcal{T}_p^0$ for $\max\{0, \frac{\lambda-\frac{1}{2}}{\lambda+1}\} \le \alpha \le \lambda$. Third, extreme points of $\mathcal{U}_p(\lambda, \alpha, \beta, k_0) \cap \mathcal{T}_p$ for $\max\{0, \frac{\lambda-\frac{1}{2}}{\lambda+1}\} \le \alpha \le \lambda$ are found. The support points of $\mathcal{U}_p(\lambda, \alpha, \beta, k_0) \cap \mathcal{T}_p$ for $\max\{0, \frac{\lambda-\frac{1}{2}}{\lambda+1}\} \le \alpha \le \lambda$ and convolution of mappings in $\mathcal{U}_p(\lambda, \alpha, \beta, k_0) \cap \mathcal{T}_p$ for $\max\{0, \frac{\lambda-\frac{1}{2}}{\lambda+1}\} \le \alpha \le \lambda$ are also discussed.
Keywords: p-harmonic mapping, uniform convexity, uniform starlikeness, extreme point, support point.
MSC(2010): Primary: 30C65; Secondary: 30C45, 30C20.

1. Introduction

A $2p$ times continuously differentiable complex-valued function $F = u + iv$ in a domain $D \subseteq \mathbb{C}$ is *p-harmonic* if F satisfies the p-harmonic equation $\underbrace{\Delta \cdots \Delta}_{p} F = 0$, where $p\ (\ge 1)$ is an integer and Δ represents the complex Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Article electronically published on April 29, 2015.
Received: 22 January 2013, Accepted: 13 March 2014.
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A mapping F is p-harmonic in a simply connected domain D if and only if F has the following representation:

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z),$$

where each G_{p-k+1} is harmonic, i.e. $\Delta G_{p-k+1}(z) = 0$ for $k \in \{1, \cdots, p\}$ (cf. [10, Proposition 1]).

Obviously, when $p = 1$ (respectively, 2), F is harmonic (respectively, biharmonic). The properties of harmonic mappings have been investigated by many authors, see [6, 11, 14, 37], etc.

Biharmonic mappings arise in a lot of physical problems, particularly in fluid dynamics and elasticity problems, and have many important applications in engineering and biology. See [19, 26, 28] for the details. There exist many references on biharmonic mappings in literature, see [1–3].

For analytic functions, Goodman first considered uniformly convex functions ([16]). The class of k_0-uniformly convex functions was introduced and investigated by Kanas and Wiśniowska ([24]). Subsequently, both of them introduced and discussed the class of k_0-uniformly starlike functions ([25]). See [23,29,35] for other discussions. In [36], it was first to give the classes of k_0-uniformly convex functions of order β and k_0-uniformly starlike functions of order β have been given. Recently, authors considered a new subclass of k_0-uniformly convex functions with negative coefficients, $\check{\mathcal{U}}(\lambda, \alpha, \beta, k_0)$, which generalizes the class of uniformly convex functions (cf. [38]).

In section 2 of this paper, we mainly discuss properties of mappings which belong to the class $\mathcal{U}_p(\lambda, \alpha, \beta, k_0)$ and a generalization of $\check{\mathcal{U}}(\lambda, \alpha, \beta, k_0)$ for p-harmonic mappings. In fact, the class $\mathcal{U}_p(\lambda, \alpha, \beta, k_0)$ contains many well-known as well as new classes of harmonic univalent mappings. In particular, $\mathcal{U}_1(0, 0, \beta, 0) = S_H(\beta)$(cf. [21, 33]), $\mathcal{U}_1(0, 1, \beta, 0) = K_H(\beta)$ (cf. [20, 22]), $\mathcal{U}_1(0, 0, \beta, k_0) = S_H D(k_0, \beta)$ and $\mathcal{U}_1(0, 1, \beta, k_0) = K_H D(k_0, \beta)$, where $S_H D(k_0, \beta)$ (respectively, $K_H D(k_0, \beta)$) (see Section 2 for the definitions) consists of k_0-uniformly starlike (respectively, convex) harmonic mappings of order β which is a generalization of the corresponding one in [36] for harmonic mappings, and, in particular, $S_H D(k_0, 0)$ (respectively, $K_H D(k_0, 0)$) consisting of k_0-uniformly starlike (respectively, convex) harmonic mappings have been considered in [5].

In order to discuss the starlikeness, extreme points and support points of p-harmonic mappings, in Section 2, we will introduce the notations: $S_{H_p}(\beta)$, $S_H D(k_0, \beta)$, $K_H D(k_0, \beta)$, $\mathcal{U}_p(\lambda, \alpha, \beta, k_0)$, $\mathcal{T}_p$ and $\mathcal{T}_p^0$ for p-harmonic mappings, respectively. Other necessary notions and notations will also be presented in Section 2.

As the first aim of this paper, we prove a sufficient condition for p-harmonic mappings to be in $\mathcal{U}_p(\lambda, \alpha, \beta, k_0)$ in terms of their coefficients, and also give the characterization of mappings in $\mathcal{U}_p(\lambda,\alpha,\beta,k_0)\cap\mathcal{T}_p$ with $\max\{0,\frac{\lambda-\frac{1}{2}}{\lambda+1}\}\leq\alpha\leq\lambda$. So our first result is Theorem 3.1.

In Section 4, we discuss the starlikeness of mappings in $\mathcal{U}_p(\lambda,\alpha,\beta,k_0)\cap\mathcal{T}_p^0$ for $\max\{0,\frac{\lambda-\frac{1}{2}}{\lambda+1}\}\leq\alpha\leq\lambda$. It is proved that mappings in $\mathcal{U}_p(\lambda,\alpha,\beta,k_0)\cap\mathcal{T}_p^0$ with $\max\{0,\frac{\lambda-\frac{1}{2}}{\lambda+1}\}\leq\alpha\leq\lambda$ is starlike of order δ for the constant δ which depends only on λ,α,β and k_0. Our main result is Theorem 4.1.

There are many references concerning extreme points of analytic functions in $\mathbb{D}$ in the literature (see [4, 13, 18, 30, 32, 41]). Extreme points of harmonic mappings also have been discussed by many authors and it is known that some classes of harmonic mappings are convex hull of their corresponding extreme points (see [20, 22, 27, 39, 40]). In Section 5, we determine the extreme points of $\mathcal{U}_p(\lambda,\alpha,\beta,k_0)\cap\mathcal{T}_p$ for $\max\{0,\frac{\lambda-\frac{1}{2}}{\lambda+1}\}\leq\alpha\leq\lambda$. And it is proved that the mappings in $\mathcal{U}_p(\lambda,\alpha,\beta,k_0)\cap\mathcal{T}_p$ with $\max\{0,\frac{\lambda-\frac{1}{2}}{\lambda+1}\}\leq\alpha\leq\lambda$ can be expressed as the convex combination of extreme points. So our main result is Theorem 5.4.

Support points of analytic functions are critical in solving extremal problems. It is known that any compact analytic function family contains support points and the set of support points contains an extreme point at least. This fact plays an active role in solving extremal problems for various families of analytic functions (see [7,8,12,13,17,30,31]). No references on this topic have been in the literature for harmonic mappings. In this paper, we consider the support points of $\mathcal{U}_p(\lambda,\alpha,\beta,k_0)\cap\mathcal{T}_p$ with $\max\{0,\frac{\lambda-\frac{1}{2}}{\lambda+1}\}\leq\alpha\leq\lambda$ for p-harmonic mappings and get Theorem 6.1.

At the end, the convolution of p-harmonic mappings are discussed. We prove that if $F_1\in\mathcal{U}_p(\lambda_1,\alpha_1,\beta_1,k_{0,1})\cap\mathcal{T}_p$ for $\max\{0,\frac{\lambda_1-\frac{1}{2}}{\lambda_1+1}\}\leq\alpha_1\leq\lambda_1$ and $F_2\in\mathcal{U}_p(\lambda_2,\alpha_2,\beta_2,k_{0,2})\cap\mathcal{T}_p$ for $\frac{\lambda_2}{\lambda_2+1}\leq\alpha_2\leq\lambda_2$, then $F_1*F_2\in\mathcal{U}_p(\lambda_1,\alpha_1,\beta_1,k_{0,1})\cap\mathcal{T}_p$ for $\max\{0,\frac{\lambda_1-\frac{1}{2}}{\lambda_1+1}\}\leq\alpha_1\leq\lambda_1$. Our last result is Theorem 7.1.

2. Preliminaries

In [3], the properties of the linear complex operator $L(f)(z)=zf_z(z)-\bar{z}f_{\bar{z}}(z)$, which is defined on the class of complex-valued C^1 functions in the plane, are investigated. It is shown that harmonicity and biharmonicity are invariant under the linear operator L. Also it is easy to deduce that it preserves p-harmonicity. The operator L can be manipulated to express the conditions in the definitions of starlikeness and convexity in a convenient way.

Definition 2.1. *A univalent sense-preserving harmonic mapping f with $f(0)=f_z(0)-1=0$ is said to be starlike of order β $(0\leq\beta<1)$, written as $f\in S_H(\beta)$ if*

$$Re\big(\frac{L(f)(z)}{f(z)}\big)>\beta$$

for $z \neq 0$.

Proposition 2.2. *([34]) If* F *is univalent and sense-preserving,* $F(0) = 0$ *and* $\frac{d}{d\theta}(argF(re^{i\theta})) > \beta$ *for* $z = re^{i\theta} \neq 0$*, then* F *is starlike of order* β *with respect to the origin.*

Definition 2.3. *A univalent sense-preserving harmonic mapping* f *with* $f(0) = f_z(0) - 1 = 0$ *and* $L(f)(z) \neq 0$ *whenever* $z \neq 0$ *is said to be convex of order* β $(0 \leq \beta < 1)$*, written as* $f \in K_H(\beta)$ *if*

$$Re\big(\frac{L(L(f))(z)}{L(f)(z)}\big) > \beta$$

for any $z \neq 0$.

Properties of these mappings have been considered in [20–22,33]. If the constant $\beta = 0$ in above definitions, we obtain the definitions of starlike harmonic mappings and convex harmonic mappings (cf. [3,9] and [15]).

Definition 2.4. *If a univalent sense-preserving* p*-harmonic mapping* F *with* $F(0) = F_z(0) - 1 = 0$ *satisfying the inequality*

$$Re\big(\frac{L(F)(z)}{F(z)}\big) > k_0\Big|\frac{L(F)(z)}{F(z)} - 1\Big| + \beta, z \in \mathbb{D} \setminus \{0\}, k_0 \geq 0,\ 0 \leq \beta < 1, \tag{2.1}$$

then we say f *is a* k_0*-uniformly starlike harmonic mapping of order* β.

Let $k_0 = 0$ in (2.1). We obtain the definition of starlike p-harmonic mappings of order β and denote this class by $S_{H_p}(\beta)$,

In the case of $p = 1$, let $S_H D(k_0, \beta)$ be the class of univalent sense-preserving harmonic mappings which are k_0-uniformly starlike of order β.

Definition 2.5. *Let* F *be a univalent sense-preserving* p*-harmonic mapping with* $F(0) = F_z(0) - 1 = 0$ *and* $L(F)(z) \neq 0$ *for* $z \neq 0$*. If* F *satisfies*

$$Re(\frac{L(L(F))(z)}{L(F)(z)}) > k_0\Big|\frac{L(L(F))(z)}{L(F)(z)} - 1\Big| + \beta, z \in \mathbb{D} \setminus \{0\}, k_0 \geq 0,\ 0 \leq \beta < 1,$$

then F *is said to be* k_0*-uniformly convex of order* β.

Let $K_H D(k_0, \beta)$ be the class that consists of univalent sense-preserving harmonic mappings which are k_0-uniformly convex of order β.

The above two definitions generalize the corresponding ones in [36].

In the following, we give the generalized class of the corresponding one in [38] for p-harmonic mappings.

Definition 2.6. *Let*

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z) \tag{2.2}$$
$$= z + \sum_{n=2}^{\infty} a_{n,p} z^n + \sum_{n=1}^{\infty} \overline{b}_{n,p} \bar{z}^n$$
$$+ \sum_{k=2}^{p} |z|^{2(k-1)} \big(\sum_{n=1}^{\infty} a_{n,p-k+1} z^n + \sum_{n=1}^{\infty} \overline{b}_{n,p-k+1} \bar{z}^n \big)$$

be a univalent sense-preserving p-harmonic mapping and

$$\mathcal{F}(z) = \lambda\alpha L(L(F))(z) + (\lambda - \alpha - \lambda\alpha) L(F)(z) + (1 - \lambda + \alpha) F(z)$$

with $0 \leq \alpha \leq \lambda \leq 1$, $0 \leq \beta < 1$, $k_0 \geq 0$. We say F is in the class $\mathcal{U}_p(\lambda, \alpha, \beta, k_0)$ if $\mathcal{F}(z) \neq 0$ for $z \neq 0$ and for any $z \in \mathbb{D} \setminus \{0\}$,

$$Re\big(\frac{L(\mathcal{F})(z)}{\mathcal{F}(z)}\big) > k_0 \big| \frac{L(\mathcal{F})(z)}{\mathcal{F}(z)} - 1 \big| + \beta. \tag{2.3}$$

Through straight computation, we have

$$\begin{aligned}
\mathcal{F}(z) &= \sum_{k=1}^{p} \Big(\big(\lambda\alpha z^2 h''_{p-k+1}(z) + (\lambda - \alpha) z h'_{p-k+1}(z) \\
&\quad + (1 - \lambda + \alpha) h_{p-k+1}(z) \big) \\
&\quad + \big(\lambda\alpha z^2 \overline{g''_{p-k+1}(z)} - (\lambda - \alpha - 2\lambda\alpha) \overline{z g'_{p-k+1}(z)} \\
&\quad + (1 - \lambda + \alpha) \overline{g_{p-k+1}(z)} \big) \Big) \\
&= \sum_{k=1}^{p} |z|^{2(k-1)} \Big(\sum_{n=1}^{\infty} \big((n-1)(\lambda\alpha n + \lambda - \alpha) + 1 \big) a_{n,p-k+1} z^n \\
&\quad + \sum_{n=1}^{\infty} \big((n+1)(\lambda\alpha n - \lambda + \alpha) + 1 \big) \overline{b}_{n,p-k+1} \bar{z}^n \Big) \\
&= \sum_{k=1}^{p} |z|^{2(k-1)} \Big(\sum_{n=1}^{\infty} A_{n,p-k+1} z^n + \sum_{n=1}^{\infty} \overline{B}_{n,p-k+1} \bar{z}^n \Big).
\end{aligned}$$

Let $\mathcal{T}_p$ denote the subclass of univalent p-harmonic mapping family consisting of mappings given by

$$\begin{aligned} F(z) &= \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z) \\ &= z - \sum_{k=2}^{p} |z|^{2(k-1)} a_{1,p-k+1} z \\ &\quad - \sum_{k=1}^{p} |z|^{2(k-1)} \big(\sum_{n=2}^{\infty} a_{n,p-k+1} z^n - \sum_{n=1}^{\infty} b_{n,p-k+1} \bar{z}^n \big) \end{aligned}$$

such that $a_{1,p-k+1} \geq 0$ $(k \in \{2, \cdots, p\})$, $a_{n,p-k+1} \geq 0$ $(k \in \{1, \cdots, p\}, n \geq 2)$ and $b_{n,p-k+1} \geq 0$ $(k \in \{1, \cdots, p\}, n \geq 1)$, and the subclass

$$\mathcal{T}_p^0 = \{F \in \mathcal{T}_p : a_{1,p-k+1} = 0 \text{ for } k \in \{2, \cdots, p\}\}.$$

Denote S the class of univalent analytic functions f with the normalization $f(0) = f'(0) - 1 = 0$ and the subclass

$$\mathcal{T} = \mathcal{T}_1 = \{f \in S : f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0\}.$$

Furthermore, we introduce the following concepts for p-harmonic mappings.

Definition 2.7. *Let X be a topological vector space over the field of complex numbers, and let D be a set of X. A point $x \in D$ is called an extreme point of D if it has no representation of the form $x = ty + (1-t)z$ $(0 < t < 1)$ as a proper convex combination of two distinct points y and z in D.*

Definition 2.8. *Let X be a topological vector space over the field of complex numbers, and let D be a subset of X. A point $x \in D$ is called a support point of D if there is a continuous linear functional J, not constant on D, such that $Re(J(x)) \geq Re(J(y))$ for all $y \in D$.*

Finally, we give the convolution of two p-harmonic mappings.

Let

$$F_1(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \big(\sum_{n=1}^{\infty} a_{n,p-k+1} z^n + \sum_{n=1}^{\infty} \bar{b}_{n,p-k+1} \bar{z}^n \big)$$

and

$$F_2(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \big(\sum_{n=1}^{\infty} c_{n,p-k+1} z^n + \sum_{n=1}^{\infty} \bar{d}_{n,p-k+1} \bar{z}^n \big).$$

Then the convolution of F_1 and F_2 is defined to be the mapping

$$F_1 * F_2(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \big(\sum_{n=1}^{\infty} a_{n,p-k+1} c_{n,p-k+1} z^n + \sum_{n=1}^{\infty} \bar{b}_{n,p-k+1} \bar{d}_{n,p-k+1} \bar{z}^n \big).$$

3. Characterization

In this section, we give a sufficient condition for mappings to be in $\mathcal{U}_p(\lambda, \alpha, \beta, k_0)$ and also the characterization of mappings in $\mathcal{U}_p(\lambda, \alpha, \beta, k_0) \cap \mathcal{T}_p$. Our main result is the following theorem.

Theorem 3.1. *A univalent sense-preserving p-harmonic mapping F with the form* (2.2) *is in the class $\mathcal{U}_p(\lambda, \alpha, \beta, k_0)$ if*

(3.1)
$$\begin{aligned}
&\sum_{k=1}^{p}\sum_{n=2}^{\infty}\big(n(k_0+1)-(k_0+\beta)\big)\big((n-1)(\lambda\alpha n+\lambda-\alpha)+1\big)|a_{n,p-k+1}| \\
&+\sum_{k=1}^{p}\sum_{n=1}^{\infty}\big(n(k_0+1)+(k_0+\beta)\big)\big|(n+1)(\lambda\alpha n-\lambda+\alpha)+1\big||b_{n,p-k+1}| \\
&+\sum_{k=2}^{p}(1-\beta)|a_{1,p-k+1}| \\
&\leq 1-\beta.
\end{aligned}$$

Conversely, if $F \in \mathcal{U}_p(\lambda, \alpha, \beta, k_0) \cap \mathcal{T}_p$ for $\max\{0, \frac{\lambda-\frac{1}{2}}{\lambda+1}\} \leq \alpha \leq \lambda$, then (3.1) *holds.*

Proof. Assume that the univalent sense-preserving p-harmonic mapping F satisfies (3.1).

Since for any $z \neq 0$,

$$\begin{aligned}
&|z|\Big(1-\beta-\sum_{k=2}^{p}(1-\beta)|a_{1,p-k+1}| \\
&-\sum_{k=1}^{p}\sum_{n=2}^{\infty}\big(n(k_0+1)-(k_0+\beta)\big)\big((n-1)(\lambda\alpha n+\lambda-\alpha)+1\big)|a_{n,p-k+1}| \\
&-\sum_{k=1}^{p}\sum_{n=1}^{\infty}\big(n(k_0+1)+(k_0+\beta)\big)\big|(n+1)(\lambda\alpha n-\lambda+\alpha)+1\big||b_{n,p-k+1}|\Big) \\
&< (1-\beta)|z|-\sum_{k=2}^{p}(1-\beta)|a_{1,p-k+1}||z|^{2k-1} \\
&-\sum_{k=1}^{p}\sum_{n=2}^{\infty}(1-\beta)\big((n-1)(\lambda\alpha n+\lambda-\alpha)+1\big)|a_{n,p-k+1}||z|^{2(k-1)+n} \\
&-\sum_{k=1}^{p}\sum_{n=1}^{\infty}(1-\beta)\big|(n+1)(\lambda\alpha n-\lambda+\alpha)+1\big||b_{n,p-k+1}||z|^{2(k-1)+n} \\
&\leq (1-\beta)|\mathcal{F}(z)|,
\end{aligned}$$

we see that $\mathcal{F}(z) \neq 0$ whenever $z \neq 0$ by (3.1).

Next, we will prove that for any $z \neq 0$,

$$Re\Big(\frac{L(\mathcal{F})(z)}{\mathcal{F}(z)}\Big) > k_0\Big|\frac{L(\mathcal{F})(z)}{\mathcal{F}(z)} - 1\Big| + \beta$$

which is equivalent to

$$Re\Big(\frac{L(\mathcal{F})(z)}{\mathcal{F}(z)}(1 + k_0e^{i\theta}) - k_0e^{i\theta}\Big) > \beta \tag{3.2}$$

for each $\theta \in [0, 2\pi)$.

Let $\mathcal{G}(z) = L(\mathcal{F})(1 + k_0e^{i\theta}) - k_0e^{i\theta}\mathcal{F}(z)$. Then (3.2) is equivalent to

$$|\mathcal{G}(z) + (1-\beta)\mathcal{F}(z)| > |\mathcal{G}(z) - (1+\beta)\mathcal{F}(z)|.$$

Since

$$\begin{aligned}
& |\mathcal{G}(z) + (1-\beta)\mathcal{F}(z)| \\
= \; & |L(\mathcal{F}) + k_0e^{i\theta}(L(\mathcal{F}) - \mathcal{F}(z)) + (1-\beta)\mathcal{F}(z)| \\
= \; & \Big|\sum_{k=1}^{p}|z|^{2(k-1)}\Big(\sum_{n=1}^{\infty}(n+1-\beta)A_{n,p-k+1}z^n \\
& -\sum_{n=1}^{\infty}(n-1+\beta)\overline{B}_{n,p-k+1}\bar{z}^n \\
& +k_0e^{i\theta}\big(\sum_{n=2}^{\infty}(n-1)A_{n,p-k+1}z^n - \sum_{n=1}^{\infty}(n+1)\overline{B}_{n,p-k+1}\bar{z}^n\big)\Big)\Big| \\
\geq \; & (2-\beta)\big|z + \sum_{k=2}^{p}A_{1,p-k+1}|z|^{2(k-1)}z\big| \\
& -\sum_{k=1}^{p}\sum_{n=2}^{\infty}\big(n(k_0+1) - (k_0+\beta) + 1\big)|A_{n,p-k+1}||z|^{2(k-1)+n} \\
& -\sum_{k=1}^{p}\sum_{n=1}^{\infty}\big(n(k_0+1) + (k_0+\beta) - 1\big)|B_{n,p-k+1}||z|^{2(k-1)+n}
\end{aligned}$$

and

$$\begin{aligned}
& |\mathcal{G}(z)-(1+\beta)\mathcal{F}(z)| \\
= & |L(\mathcal{F})+k_0e^{i\theta}(L(\mathcal{F})-\mathcal{F}(z))-(1+\beta)\mathcal{F}(z)| \\
= & \Big|\sum_{k=1}^{p}|z|^{2(k-1)}\Big(-\beta A_{1,p-k+1}z+\sum_{n=2}^{\infty}(n-1-\beta)A_{n,p-k+1}z^n \\
& -\sum_{n=1}^{\infty}(n+1+\beta)\overline{B}_{n,p-k+1}\bar{z}^n \\
& +k_0e^{i\theta}\big(\sum_{n=2}^{\infty}(n-1)A_{n,p-k+1}z^n-\sum_{n=1}^{\infty}(n+1)\overline{B}_{n,p-k+1}\bar{z}^n\big)\Big)\Big| \\
\leq & \beta\big|z+\sum_{k=2}^{p}A_{1,p-k+1}|z|^{2(k-1)}z\big| \\
& +\sum_{k=1}^{p}\sum_{n=2}^{\infty}\big(n(k_0+1)-(k_0+\beta)-1\big)|A_{n,p-k+1}||z|^{2(k-1)+n} \\
& +\sum_{k=1}^{p}\sum_{n=1}^{\infty}\big(n(k_0+1)+(k_0+\beta)+1\big)|B_{n,p-k+1}||z|^{2(k-1)+n},
\end{aligned}$$

we see that

$$\begin{aligned}
& |\mathcal{G}(z)+(1-\beta)\mathcal{F}(z)|-|\mathcal{G}(z)-(1+\beta)\mathcal{F}(z)| \\
\geq & (2-2\beta)|z|-\sum_{k=2}^{p}(2-2\beta)|A_{1,p-k+1}||z|^{2k-1} \\
& -\sum_{k=1}^{p}\sum_{n=2}^{\infty}2\big(n(k_0+1)-(k_0+\beta)\big)|A_{n,p-k+1}||z|^{2(k-1)+n} \\
& -\sum_{k=1}^{p}\sum_{n=1}^{\infty}2\big(n(k_0+1)+(k_0+\beta)\big)|B_{n,p-k+1}||z|^{2(k-1)+n} \\
> & 0
\end{aligned}$$

by (3.1), which implies

$$Re\big(\frac{L(\mathcal{F})(z)}{\mathcal{F}(z)}\big)>k_0\Big|\frac{L(\mathcal{F})(z)}{\mathcal{F}(z)}-1\Big|+\beta.$$

Conversely, assume $F \in \mathcal{U}_p(\lambda, \alpha, \beta, k_0) \cap \mathcal{T}_p$ for $\max\{0, \frac{\lambda - \frac{1}{2}}{\lambda+1}\} \leq \alpha \leq \lambda$. Let $z = r \ \big(r \in (0,1)\big)$. Then (3.2) reduces to the following form

$$\begin{aligned}(3.3) \qquad & Re\Big(\frac{L(\mathcal{F})(r) - \beta\mathcal{F}(r) + k_0 e^{i\theta}(L(\mathcal{F})(r) - \mathcal{F}(r))}{\mathcal{F}(r)}\Big) \\ = \ & \frac{\mathrm{Re}A(r)}{C(r)} \\ > \ & 0,\end{aligned}$$

where

$$\begin{aligned} A(r) \ = \ & 1 - \beta - \sum_{k=2}^{p}(1-\beta)A_{1,p-k+1}r^{2(k-1)} \\ & - \sum_{k=1}^{p}\sum_{n=2}^{\infty}\big((n-\beta) + k_0 e^{i\theta}(n-1)\big)A_{n,p-k+1}r^{2(k-1)+n-1} \\ & - \sum_{k=1}^{p}\sum_{n=1}^{\infty}\big((n+\beta) + k_0 e^{i\theta}(n+1)\big)B_{n,p-k+1}r^{2(k-1)+n-1}\end{aligned}$$

and

$$\begin{aligned} C(r) \ = \ & 1 - \sum_{k=2}^{p}A_{1,p-k+1}r^{2(k-1)} - \sum_{k=1}^{p}\sum_{n=2}^{\infty}A_{n,p-k+1}r^{2(k-1)+n-1} \\ & - \sum_{k=1}^{p}\sum_{n=1}^{\infty}B_{n,p-k+1}r^{2(k-1)+n-1}.\end{aligned}$$

Since, for $\theta = 0$, we see that

$$\begin{aligned} Re(A(r)) \ = \ & B(r) \\ = \ & 1 - \beta - \sum_{k=2}^{p}(1-\beta)A_{1,p-k+1}r^{2(k-1)} \\ & - \sum_{k=1}^{p}\sum_{n=2}^{\infty}\big((n-\beta) + k_0(n-1)\big)A_{n,p-k+1}r^{2(k-1)+n-1} \\ & - \sum_{k=1}^{p}\sum_{n=1}^{\infty}\big((n+\beta) + k_0(n+1)\big)B_{n,p-k+1}r^{2(k-1)+n-1}.\end{aligned}$$

Then, by (3.3), we have

$$\frac{B(r)}{C(r)} > 0. \tag{3.4}$$

If the inequality (3.1) does not hold, then the numerator in (3.4) is negative for r sufficiently close to 1. Thus there exists a $z_0 = r_0$ in $(0,1)$ for which the quotient in (3.4) is negative, which contradicts the condition that $F \in \mathcal{U}_p(\lambda, \alpha, \beta, k_0)$. □

Take $\lambda = 0$, $\alpha = 0$ and $p = 1$ in Theorem 3.1, we can deduce the following corollary.

Corollary 3.2. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \bar{b}_n \bar{z}^n$ is a univalent sense-preserving harmonic mapping such that*

$$\sum_{n=2}^{\infty} \big(n(k_0+1) - (k_0+\beta)\big)|a_n| + \sum_{n=1}^{\infty} \big(n(k_0+1) + k_0+\beta\big)|b_n| < 1-\beta,$$

then $f \in S_H D(k_0, \beta)$.

Conversely, if $f \in S_H D(k_0, \beta) \cap \mathcal{T}_1$, then the above inequality also holds.

Remark 3.3. *If $f \in S$, that is $b_n = 0$ $(n \geq 1)$, [36, Theorem 2.1] follows from Corollary 3.2.*

Remark 3.4. *Take $k_0 = 0$ in Corollary 3.2, we obtain the characterization of mappings belonging to $S_H(\beta) \cap \mathcal{T}_1$ which is given in [21].*

Let $\lambda = 0$, $\alpha = 1$ and $p = 1$ in Theorem 3.1. We can get a sufficient condition for f to be in $K_H D(k_0, \beta)$.

Corollary 3.5. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \bar{b}_n \bar{z}^n$ is a univalent sense-preserving harmonic mapping such that*

$$\sum_{n=2}^{\infty} n\big(n(k_0+1) - (k_0+\beta)\big)|a_n| + \sum_{n=1}^{\infty} n\big(n(k_0+1) + k_0+\beta\big)|b_n| < 1-\beta,$$

then $f \in K_H D(k_0, \beta)$.

Conversely, if $f \in K_H D(k_0, \beta) \cap \mathcal{T}_1$, then the above inequality also holds.

Remark 3.6. *If $f \in S$, that is $b_n = 0$ $(n \geq 1)$, [36, Theorem 2.2] is the direct consequence of the above corollary.*

Corollary 3.7. *Suppose that $f(z) = z - \sum_{n=1}^{\infty} a_n z^n \in \mathcal{T}$. If*

$$\sum_{n=2}^{\infty} \big(n(k_0+1) - (k_0+\beta)\big)\big((n-1)(\lambda\alpha n + \lambda - \alpha) + 1\big)a_n \leq 1-\beta,$$

then $f(z) \in \check{\mathcal{U}}(\lambda, \alpha, \beta, k_0)$, where $\check{\mathcal{U}}(\lambda, \alpha, \beta, k_0)$ is defined in [38].

Remark 3.8. *This result coincides with [38, Theorem 1].*

4. **Starlikeness**

In this section, we consider the starlikeness of mappings in $\mathcal{U}_p(\lambda,\alpha,\beta,k_0)\cap\mathcal{T}_p^0$.

Theorem 4.1. *If $F\in\mathcal{U}_p(\lambda,\alpha,\beta,k_0)\cap\mathcal{T}_p^0$ for $\max\{0,\frac{\lambda-\frac{1}{2}}{\lambda+1}\}\leq\alpha\leq\lambda$, then $F\in S_{H_p}(\delta)$ where $\delta=\min\{m_1,m_2\}$ with*

$$m_1=1-\frac{1-\beta}{(2+k_0-\beta)(2\lambda\alpha+\lambda-\alpha+1)-(1-\beta)}$$

and

$$m_2=1-\frac{2(1-\beta)}{(2k_0+1+\beta)(2(\lambda\alpha-\lambda+\alpha)+1)+(1-\beta)}.$$

Proof. Assume

$$F(z)=z+\sum_{k=1}^{p}|z|^{2(k-1)}\big(-\sum_{n=2}^{\infty}a_{n,p-k+1}z^n+\sum_{n=1}^{\infty}b_{n,p-k+1}\bar{z}^n\big)$$

belongs to $\mathcal{U}_p(\lambda,\alpha,\beta,k_0)\cap\mathcal{T}_p^0$ for $\max\{0,\frac{\lambda-\frac{1}{2}}{\lambda+1}\}\leq\alpha\leq\lambda$ which implies

$$\begin{aligned}&\sum_{k=1}^{p}\sum_{n=2}^{\infty}\big(n(k_0+1)-(k_0+\beta)\big)\big((n-1)(\lambda\alpha n+\lambda-\alpha)+1\big)a_{n,p-k+1}\\&+\sum_{k=1}^{p}\sum_{n=1}^{\infty}\big(n(k_0+1)+(k_0+\beta)\big)\big((n+1)(\lambda\alpha n-\lambda+\alpha)+1\big)b_{n,p-k+1}\\&\leq 1-\beta.\end{aligned}$$

Let $r\in(0,1)$ and

$$F_r(z)=z+\sum_{k=1}^{p}r^{2(k-1)}\big(-\sum_{n=2}^{\infty}a_{n,p-k+1}z^n+\sum_{n=1}^{\infty}b_{n,p-k+1}\bar{z}^n\big).$$

Obviously, F_r is harmonic.

In [21], it has proved that if

$$\sum_{n=2}^{\infty}\frac{n-\delta_0}{1-\delta_0}|a_n|+\sum_{n=1}^{\infty}\frac{n+\delta_0}{1-\delta_0}|b_n|\leq 1, \tag{4.1}$$

then $f\in S_H(\delta_0)$.

Let

$$\varphi(n)=1-\frac{(n-1)(1-\beta)}{\big(n(k_0+1)-(k_0+\beta)\big)\big((n-1)(\lambda\alpha n+\lambda-\alpha)+1\big)-(1-\beta)}.$$

with $n\geq 2$. Then

$$\delta\leq m_1=\varphi(2)=1-\frac{1-\beta}{(2+k_0-\beta)(2\lambda\alpha+\lambda-\alpha+1)-(1-\beta)}.$$

Thus, It follows from

$$\varphi(n) \geq \varphi(2)\,(n \geq 2)$$

that

$$\begin{aligned}\delta &\leq \varphi(n)\\ &= 1 - \frac{(n-1)(1-\beta)}{\big(n(k_0+1)-(k_0+\beta)\big)\big((n-1)(\lambda\alpha n+\lambda-\alpha)+1\big)-(1-\beta)}\end{aligned}$$

which is equivalent to

$$\frac{n-\delta}{1-\delta} \leq \frac{\big(n(k_0+1)-(k_0+\beta)\big)\big((n-1)(\lambda\alpha n+\lambda-\alpha)+1\big)}{1-\beta}(n \geq 2).$$

Let

$$\Phi(n) = 1 - \frac{(n+1)(1-\beta)}{\big(n(k_0+1)+(k_0+\beta)\big)\big((n+1)(\lambda\alpha n-\lambda+\alpha)+1\big)+(1-\beta)}$$

with $n \geq 1$. It is easy to deduce that $\Phi(n) \geq m_2 \geq \delta$ for $n \geq 1$ which implies

$$\frac{n+\delta}{1-\delta} \leq \frac{\big(n(k_0+1)+k_0+\beta\big)\big((n+1)(\lambda\alpha n-\lambda+\alpha)+1\big)}{1-\beta}(n \geq 1).$$

Then by Theorem 3.1 and (4.1), we have $F_r \in S_H(\delta)$, that is

$$\frac{d}{d\theta}(\arg F_r(r_1e^{i\theta})) > \delta$$

for $r_1 \in (0,1)$. Let $r_1 = r$, we have $\frac{d}{d\theta}(\arg F(re^{i\theta})) > \delta$. By Proposition 2.2, we have $F \in S_{H_p}(\delta)$.

The proof is complete. □

5. **Extreme points**

We begin this section with two lemmas, one gives the distortion bounds for mappings in $\mathcal{U}_p(\lambda,\alpha,\beta,k_0)\cap\mathcal{T}_p$ and the other shows that this class is closed under the convex combination.

Lemma 5.1. *If* $F \in \mathcal{U}_p(\lambda,\alpha,\beta,k_0)\cap\mathcal{T}_p$ *for* $\max\{0,\frac{\lambda-\frac{1}{2}}{\lambda+1}\} \leq \alpha \leq \lambda$*, then*

$$\begin{aligned}|F(z)| \leq{} & \sum_{k=1}^{p}\big(a_{1,p-k+1}+b_{1,p-k+1}\big)|z|\\ & +m_0\big(1-\beta-\sum_{k=1}^{p}(2k_0+\beta+1)(2\lambda\alpha-2\lambda+2\alpha+1)b_{1,p-k+1}\\ & -\sum_{k=2}^{p}(1-\beta)a_{1,p-k+1}\big)|z|^2,\end{aligned}$$

where $m_0 = \max\{\frac{1}{(k_0+2-\beta)(2\lambda\alpha+\lambda-\alpha+1)},\frac{1}{(3k_0+\beta+2)(6\lambda\alpha-3\lambda+3\alpha+1)}\}$.

Proof. For

$$\begin{aligned} F(z) &= z - \sum_{n=2}^{\infty} a_{n,p} z^n + \sum_{n=1}^{\infty} b_{n,p} \bar{z}^n \\ &\quad - \sum_{k=2}^{p} |z|^{2(k-1)} \big(\sum_{n=1}^{\infty} a_{n,p-k+1} z^n - \sum_{n=1}^{\infty} b_{n,p-k+1} \bar{z}^n \big), \end{aligned}$$

obviously,

$$|F(z)| \leq \sum_{k=1}^{p} \big(a_{1,p-k+1} + b_{1,p-k+1}\big)|z| + \sum_{k=1}^{p}\sum_{n=2}^{\infty} \big(a_{n,p-k+1} + b_{n,p-k+1}\big)|z|^2.$$

By Theorem 3.1,

$$\begin{aligned} & \sum_{k=1}^{p}\sum_{n=2}^{\infty} \big(a_{n,p-k+1} + b_{n,p-k+1}\big) \\ \leq\ & \frac{1}{(k_0+2-\beta)(2\lambda\alpha+\lambda-\alpha+1)} \sum_{k=1}^{p}\sum_{n=2}^{\infty} \big(n(k_0+1) - (k_0+\beta)\big) \\ & \big((n-1)(\lambda\alpha n + \lambda - \alpha) + 1\big) a_{n,p-k+1} \\ & + \frac{1}{(3k_0+\beta+2)(6\lambda\alpha - 3\lambda + 3\alpha + 1)} \\ & \sum_{k=1}^{p}\sum_{n=2}^{\infty} \big(n(k_0+1) + (k_0+\beta)\big)\big((n+1)(\lambda\alpha n - \lambda + \alpha) + 1\big) b_{n,p-k+1} \\ \leq\ & m_0\big(1 - \beta - \sum_{k=2}^{p}(1-\beta)a_{1,p-k+1} \\ & - \sum_{k=1}^{p}(2k_0+\beta+1)(2\lambda\alpha - 2\lambda + 2\alpha + 1)b_{1,p-k+1}\big) \end{aligned}$$

with $m_0 = \max\{\frac{1}{(k_0+2-\beta)(2\lambda\alpha+\lambda-\alpha+1)}, \frac{1}{(3k_0+\beta+2)(6\lambda\alpha-3\lambda+3\alpha+1)}\}$. Hence

$$\begin{aligned} |F(z)| &\leq \sum_{k=1}^{p} \big(a_{1,p-k+1} + b_{1,p-k+1}\big)|z| \\ &\quad + m_0\big(1 - \beta - \sum_{k=1}^{p}(2k_0+\beta+1)(2\lambda\alpha - 2\lambda + 2\alpha + 1)b_{1,p-k+1} \\ &\quad - \sum_{k=2}^{p}(1-\beta)a_{1,p-k+1}\big)|z|^2. \end{aligned}$$

$\square$

Remark 5.2. *If $F \in \mathcal{U}_p(\lambda,\alpha,\beta,k_0)\cap\mathcal{T}_p$ for $\max\{0,\frac{\lambda-\frac{1}{2}}{\lambda+1}\}\le\alpha\le\lambda$, then for each $k\in\{1,\cdots,p\}$,*

$$\begin{aligned}
& |G_{p-k+1}(z)| \\
\le\ & \sum_{k=1}^{p}\big(a_{1,p-k+1}+b_{1,p-k+1}\big)|z| \\
& +m_0\big(1-\beta-\sum_{k=1}^{p}(2k_0+\beta+1)(2\lambda\alpha-2\lambda+2\alpha+1)b_{1,p-k+1} \\
& -\sum_{k=2}^{p}(1-\beta)a_{1,p-k+1}\big)|z|^2,
\end{aligned}$$

where $m_0=\max\{\frac{1}{(k_0+2-\beta)(2\lambda\alpha+\lambda-\alpha+1)},\frac{1}{(3k_0+\beta+2)(6\lambda\alpha-3\lambda+3\alpha+1)}\}$.

Lemma 5.3. *The family $\mathcal{U}_p(\lambda,\alpha,\beta,k_0)\cap\mathcal{T}_p$ is closed under convex combination for $\max\{0,\frac{\lambda-\frac{1}{2}}{\lambda+1}\}\le\alpha\le\lambda$.*

Proof. For $i=1,2,3,\cdots$, suppose that $F_i\in\mathcal{U}_p(\lambda,\alpha,\beta,k_0)\cap\mathcal{T}_p$ and

$$\begin{aligned}
F_i(z) &= z-\sum_{k=2}^{p}|z|^{2(k-1)}a_{i,1,p-k+1}z \\
&\quad -\sum_{k=1}^{p}|z|^{2(k-1)}\big(\sum_{n=2}^{\infty}a_{i,n,p-k+1}z^n-\sum_{n=1}^{\infty}b_{i,n,p-k+1}\bar{z}^n\big).
\end{aligned}$$

By Theorem 3.1,

$$\begin{aligned}
& \sum_{k=1}^{p}\sum_{n=2}^{\infty}\big(n(k_0+1)-(k_0+\beta)\big)\big((n-1)(\lambda\alpha n+\lambda-\alpha)+1\big)a_{i,n,p-k+1} \\
& +\sum_{k=1}^{p}\sum_{n=1}^{\infty}\big(n(k_0+1)+(k_0+\beta)\big)\big((n+1)(\lambda\alpha n-\lambda+\alpha)+1\big)b_{i,n,p-k+1} \\
& +\sum_{k=2}^{p}(1-\beta)a_{i,1,p-k+1} \\
& \le 1-\beta.
\end{aligned}$$

For $\sum_{i=1}^{\infty} t_i = 1$ with $t_i \in [0,1]$, from Lemma 5.1 and Remark 5.2, we have that $\sum_{i=1}^{\infty} t_i F_i$ is p-harmonic. Obviously,

$$\begin{aligned}\sum_{i=1}^{\infty} t_i F_i(z) \quad = \quad & z - \sum_{k=2}^{p} |z|^{2(k-1)} \big(\sum_{i=1}^{\infty} t_i a_{i,1,p-k+1}\big) z \\ & - \sum_{k=1}^{p} |z|^{2(k-1)} \Big(\sum_{n=2}^{\infty} \big(\sum_{i=1}^{\infty} t_i a_{i,n,p-k+1}\big) z^n \\ & - \sum_{n=1}^{\infty} \big(\sum_{i=1}^{\infty} t_i b_{i,n,p-k+1}\big) \bar{z}^n \Big).\end{aligned}$$

Then

$$\begin{aligned}& \sum_{k=1}^{p}\sum_{n=2}^{\infty} \big(n(k_0+1) - (k_0+\beta)\big)\big((n-1)(\lambda\alpha n + \lambda - \alpha) + 1\big) \\ & \big(\sum_{i=1}^{\infty} t_i a_{i,n,p-k+1}\big) \\ & + \sum_{k=1}^{p}\sum_{n=1}^{\infty} \big(n(k_0+1) + (k_0+\beta)\big)\big((n+1)(\lambda\alpha n - \lambda + \alpha) + 1\big) \\ & \big(\sum_{i=1}^{\infty} t_i b_{i,n,p-k+1}\big) + \sum_{k=2}^{p} (1-\beta)\big(\sum_{i=1}^{\infty} t_i a_{i,1,p-k+1}\big) \\ & \leq \sum_{i=1}^{\infty} t_i \Big(\sum_{k=1}^{p}\sum_{n=2}^{\infty} \big(n(k_0+1) - (k_0+\beta)\big)\big((n-1)(\lambda\alpha n + \lambda - \alpha) + 1\big) \\ & a_{i,n,p-k+1} \\ & + \sum_{k=1}^{p}\sum_{n=1}^{\infty} \big(n(k_0+1) + (k_0+\beta)\big)\big((n+1)(\lambda\alpha n - \lambda + \alpha) + 1\big) \\ & b_{i,n,p-k+1} + \sum_{k=2}^{p} (1-\beta) a_{i,1,p-k+1}\Big) \\ & \leq \sum_{i=1}^{\infty} t_i (1-\beta) \\ & = (1-\beta).\end{aligned}$$

Using Theorem 3.1, we have $\sum_{i=1}^{\infty} t_i F_i \in \mathcal{U}_p(\lambda, \alpha, \beta, k_0) \cap \mathcal{T}_p$. $\square$

Obviously, for $\max\{0, \frac{\lambda - \frac{1}{2}}{\lambda + 1}\} \leq \alpha \leq \lambda$, it follows from Remark 5.2 and [14, P_{80}] that $\mathcal{U}_p(\lambda, \alpha, \beta, k_0) \cap \mathcal{T}_p$ is normal. By Remark 5.2 and Lemma 5.3, this class is also convex and compact. It follows that this class contains some extreme points (cf. [13, P_{281}]). Now we are ready to find the forms of extreme points of $\mathcal{U}_p(\lambda, \alpha, \beta, k_0) \cap \mathcal{T}_p$ for $\max\{0, \frac{\lambda-\frac{1}{2}}{\lambda+1}\} \leq \alpha \leq \lambda$ and show that these mappings can be expressed as the convex combination of the extreme points.

Theorem 5.4. *If $F \in \mathcal{T}_p$, then, for $\max\{0, \frac{\lambda-\frac{1}{2}}{\lambda+1}\} \leq \alpha \leq \lambda$, $F \in \mathcal{U}_p(\lambda, \alpha, \beta, k_0)$ if and only if*

$$F(z) = \sum_{k=1}^{p} \sum_{n=1}^{\infty} \big(X_{kn} h_{kn}(z) + Y_{kn} g_{kn}(z)\big),$$

where

$$h_{kn}(z) = z - |z|^{2(k-1)} \frac{1-\beta}{\big(n(k_0+1) - (k_0+\beta)\big)\big((n-1)(\lambda\alpha n + \lambda - \alpha) + 1\big)} z^n$$

with $k \in \{2, \cdots, p\}$, $n \geq 1$,

$$h_{11}(z) = z,$$

$$h_{1n}(z) = z - \frac{1-\beta}{\big(n(k_0+1) - (k_0+\beta)\big)\big((n-1)(\lambda\alpha n + \lambda - \alpha) + 1\big)} z^n \ (n \geq 2),$$

$$g_{kn}(z) = z + |z|^{2(k-1)} \frac{1-\beta}{\big(n(k_0+1) + k_0+\beta\big)\big((n+1)(\lambda\alpha n - \lambda + \alpha) + 1\big)} \bar{z}^n$$

with $k \in \{1, \cdots, p\}$, $n \geq 1$, and

$$\sum_{k=1}^{p} \sum_{n=1}^{\infty} (X_{kn} + Y_{kn}) = 1 \ (X_{kn} \geq 0, \ Y_{kn} \geq 0).$$

In particular, the extreme points of $\mathcal{U}_p(\lambda, \alpha, \beta, k_0) \cap \mathcal{T}_p$ with $\max\{0, \frac{\lambda-\frac{1}{2}}{\lambda+1}\} \leq \alpha \leq \lambda$ are $\{h_{kn}\}$ and $\{g_{kn}\}$.

Proof. Since

$$\begin{aligned}
F(z) &= \sum_{k=1}^{p} \sum_{n=1}^{\infty} \big(X_{kn} h_{kn}(z) + Y_{kn} g_{kn}(z)\big) \\
&= z + \sum_{k=2}^{p} |z|^{2(k-1)} \\
&\quad \Big(- \sum_{n=2}^{\infty} \frac{(1-\beta) X_{kn}}{\big(n(k_0+1) - (k_0+\beta)\big)\big((n-1)(\lambda\alpha n + \lambda - \alpha) + 1\big)} z^n \\
&\quad + \sum_{n=1}^{\infty} \frac{(1-\beta) Y_{kn}}{\big(n(k_0+1) + k_0+\beta\big)\big((n+1)(\lambda\alpha n - \lambda + \alpha) + 1\big)} \bar{z}^n \Big) \\
&\quad - \sum_{k=2}^{p} \frac{(1-\beta) X_{k1}}{\big(n(k_0+1) - (k_0+\beta)\big)\big((n-1)(\lambda\alpha n + \lambda - \alpha) + 1\big)} z
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{k=1}^{p}\sum_{n=2}^{\infty}\big(n(k_0+1)-(k_0+\beta)\big)\big((n-1)(\lambda\alpha n+\lambda-\alpha)+1\big) \\
& \frac{(1-\beta)X_{kn}}{\big(n(k_0+1)-(k_0+\beta)\big)\big((n-1)(\lambda\alpha n+\lambda-\alpha)+1\big)} \\
& +\sum_{k=1}^{p}\sum_{n=1}^{\infty}\big(n(k_0+1)+(k_0+\beta)\big)\big((n+1)(\lambda\alpha n-\lambda+\alpha)+1\big) \\
& \frac{(1-\beta)Y_{kn}}{\big(n(k_0+1)+k_0+\beta\big)\big((n+1)(\lambda\alpha n-\lambda+\alpha)+1\big)} \\
& +\sum_{k=2}^{p}(1-\beta)X_{k1} \\
= \; & (1-\beta)(1-X_{11}) \\
\leq \; & 1-\beta,
\end{aligned}$$

it follows from Theorem 3.1 that $F\in\mathcal{U}_p(\lambda,\alpha,\beta,k_0)\cap\mathcal{T}_p$.

Conversely, assuming $F\in\mathcal{U}_p(\lambda,\alpha,\beta,k_0)\cap\mathcal{T}_p$ with $\max\{0,\frac{\lambda-\frac{1}{2}}{\lambda+1}\}\leq\alpha\leq\lambda$ and setting

$$X_{kn}=\frac{1-\beta}{\big(n(k_0+1)-(k_0+\beta)\big)\big((n-1)(\lambda\alpha n+\lambda-\alpha)+1\big)}a_{n,p-k+1}\ ,$$

with $2\leq k\leq p$ and $n\geq 1$,

$$X_{1n}=\frac{1-\beta}{\big(n(k_0+1)-(k_0+\beta)\big)\big((n-1)(\lambda\alpha n+\lambda-\alpha)+1\big)}a_{n,p}\ (n\geq 2),$$

$$Y_{kn}=\frac{1-\beta}{\big(n(k_0+1)+k_0+\beta\big)\big((n+1)(\lambda\alpha n-\lambda+\alpha)+1\big)}b_{n,p-k+1}$$

with $1\leq k\leq p$ and $n\geq 1$,

$$X_{11}=1-\sum_{k=1}^{p}\sum_{n=2}^{\infty}(X_{kn}+Y_{kn})-\sum_{k=2}^{p}(X_{k1}+Y_{k1})-Y_{11},$$

we obtain

$$F(z)=\sum_{k=1}^{p}\sum_{n=1}^{\infty}\big(X_{kn}h_{kn}(z)+Y_{kn}g_{kn}(z)\big).$$

The proof is complete. $\square$

6. **Support points**

Theorem 6.1. *Suppose* $F \in \mathcal{U}_p(\lambda, \alpha, \beta, k_0) \cap \mathcal{T}_p$ *for* $\max\{0, \frac{\lambda - \frac{1}{2}}{\lambda+1}\} \le \alpha \le \lambda$ *and*

$$\begin{aligned} F(z) &= \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z) \\ &= z - \sum_{k=2}^{p} |z|^{2(k-1)} a_{1,p-k+1} z \\ &\quad - \sum_{k=1}^{p} |z|^{2(k-1)} \Big(\sum_{n=2}^{\infty} a_{n,p-k+1} z^n - \sum_{n=1}^{\infty} b_{n,p-k+1} \bar{z}^n \Big). \end{aligned}$$

If there is some integer $m \ge 2$ *such that*

$$\begin{aligned} &\sum_{k=1}^{p} \sum_{n=2}^{m} \big(n(k_0+1) - (k_0+\beta) \big) \big((n-1)(\lambda\alpha n + \lambda - \alpha) + 1 \big) a_{n,p-k+1} \\ &+ \sum_{k=1}^{p} \sum_{n=1}^{m} \big(n(k_0+1) + (k_0+\beta) \big) \big| (n+1)(\lambda\alpha n - \lambda + \alpha) + 1 \big| b_{n,p-k+1} \\ &+ \sum_{k=2}^{p} (1-\beta) a_{1,p-k+1} \\ &= 1 - \beta, \end{aligned}$$

then F *is a support point of* $\mathcal{U}_p(\lambda, \alpha, \beta, k_0) \cap \mathcal{T}_p$ *for* $\max\{0, \frac{\lambda - \frac{1}{2}}{\lambda+1}\} \le \alpha \le \lambda$.

Proof. For $F_1 \in \mathcal{U}_p(\lambda, \alpha, \beta, k_0) \cap \mathcal{T}_p$ with $\max\{0, \frac{\lambda - \frac{1}{2}}{\lambda+1}\} \le \alpha \le \lambda$ and the expression:

$$\begin{aligned} F_1(z) &= \sum_{k=1}^{p} |z|^{2(k-1)} G^*_{p-k+1}(z) \\ &= z - \sum_{k=2}^{p} |z|^{2(k-1)} c_{1,p-k+1} z \\ &\quad - \sum_{k=1}^{p} |z|^{2(k-1)} \Big(\sum_{n=2}^{\infty} c_{n,p-k+1} z^n - \sum_{n=1}^{\infty} d_{n,p-k+1} \bar{z}^n \Big), \end{aligned}$$

let

$$\begin{aligned} & J_m(F_1) \\ &= \sum_{k=1}^{p}\sum_{n=2}^{m} \frac{\big(n(k_0+1)-(k_0+\beta)\big)\big((n-1)(\lambda\alpha n+\lambda-\alpha)+1\big)}{1-\beta} c_{n,p-k+1} \\ &+ \sum_{k=1}^{p}\sum_{n=1}^{m} \frac{\big(n(k_0+1)+(k_0+\beta)\big)\big|(n+1)(\lambda\alpha n-\lambda+\alpha)+1\big|}{1-\beta} d_{n,p-k+1} \\ &+ \sum_{k=2}^{p} c_{1,p-k+1}. \end{aligned}$$

It is easy to deduce that J_m is a continuous linear functional and not a constant on $\mathcal{U}_p(\lambda,\alpha,\beta,k_0)\cap\mathcal{T}_p$.

$$\begin{aligned} & Re(J_m(F_1)) \\ &= \sum_{k=1}^{p}\sum_{n=2}^{m} \frac{\big(n(k_0+1)-(k_0+\beta)\big)\big((n-1)(\lambda\alpha n+\lambda-\alpha)+1\big)}{1-\beta} c_{n,p-k+1} \\ &+ \sum_{k=1}^{p}\sum_{n=1}^{m} \frac{\big(n(k_0+1)+(k_0+\beta)\big)\big|(n+1)(\lambda\alpha n-\lambda+\alpha)+1\big|}{1-\beta} d_{n,p-k+1} \\ &+ \sum_{k=2}^{p} c_{1,p-k+1} \\ &\le 1. \end{aligned}$$

Obviously,

$$\begin{aligned} & Re\big(J_m(F)\big) \\ &= \sum_{k=1}^{p}\sum_{n=2}^{m} \frac{\big(n(k_0+1)-(k_0+\beta)\big)\big((n-1)(\lambda\alpha n+\lambda-\alpha)+1\big)}{1-\beta} a_{n,p-k+1} \\ &+ \sum_{k=1}^{p}\sum_{n=1}^{m} \frac{\big(n(k_0+1)+(k_0+\beta)\big)\big|(n+1)(\lambda\alpha n-\lambda+\alpha)+1\big|}{1-\beta} b_{n,p-k+1} \\ &+ \sum_{k=2}^{p} a_{1,p-k+1} \\ &= 1. \end{aligned}$$

Hence F is a support point of $\mathcal{U}_p(\lambda,\alpha,\beta,k_0)\cap\mathcal{T}_p$ for $\max\{0, \frac{\lambda-\frac{1}{2}}{\lambda+1}\} \le \alpha \le \lambda$. $\square$

7. Convolution

Theorem 7.1. *Suppose*

$$\begin{aligned} F_1(z) &= \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z) \\ &= z - \sum_{k=2}^{p} |z|^{2(k-1)} a_{1,p-k+1} z \\ &\quad - \sum_{k=1}^{p} |z|^{2(k-1)} \big(\sum_{n=2}^{\infty} a_{n,p-k+1} z^n - \sum_{n=1}^{\infty} b_{n,p-k+1} \bar{z}^n \big) \end{aligned}$$

belongs to $\mathcal{U}_p(\lambda_1, \alpha_1, \beta_1, k_{0,1}) \cap \mathcal{T}_p$ *for* $\max\{0, \frac{\lambda_1 - \frac{1}{2}}{\lambda_1 + 1}\} \leq \alpha_1 \leq \lambda_1$ *and*

$$\begin{aligned} F_2(z) &= \sum_{k=1}^{p} |z|^{2(k-1)} G^*_{p-k+1}(z) \\ &= z - \sum_{k=2}^{p} |z|^{2(k-1)} c_{1,p-k+1} z \\ &\quad - \sum_{k=1}^{p} |z|^{2(k-1)} \big(\sum_{n=2}^{\infty} c_{n,p-k+1} z^n - \sum_{n=1}^{\infty} d_{n,p-k+1} \bar{z}^n \big) \end{aligned}$$

is in the class $\mathcal{U}_p(\lambda_2, \alpha_2, \beta_2, k_{0,2}) \cap \mathcal{T}_p$ *for* $\frac{\lambda_2}{\lambda_2+1} \leq \alpha_2 \leq \lambda_2$. *Then* $F_1 * F_2 \in \mathcal{U}_p(\lambda_1, \alpha_1, \beta_1, k_{0,1}) \cap \mathcal{T}_p$ *for* $\max\{0, \frac{\lambda_1 - \frac{1}{2}}{\lambda_1 + 1}\} \leq \alpha_1 \leq \lambda_1$.

Proof. By Theorem 3.1 and assumptions, we know that F_1 satisfies (3.1) with constants λ_1, α_1, β_1, $k_{0,1}$, and F_2 subject to (3.1) with λ_2, α_2, β_2, $k_{0,2}$. Since $\frac{\lambda_2}{\lambda_2+1} \leq \alpha_2 \leq \lambda_2$, it follows that $c_{n,p-k+1} \leq 1$ and $d_{n,p-k+1} \leq 1$. Hence

$$\begin{aligned} & sum_{k=1}^{p} \sum_{n=2}^{\infty} \big(n(k_{0,1}+1) - (k_{0,1}+\beta_1) \big) \big((n-1)(\lambda_1 \alpha_1 n + \lambda_1 - \alpha_1) + 1 \big) \\ & a_{n,p-k+1} c_{n,p-k+1} \\ & + \sum_{k=1}^{p} \sum_{n=1}^{\infty} \big(n(k_{0,1}+1) + (k_{0,1}+\beta_1) \big) \big((n+1)(\lambda_1 \alpha_1 n - \lambda_1 + \alpha_1) + 1 \big) \\ & b_{n,p-k+1} d_{n,p-k+1} \\ & + \sum_{k=2}^{p} (1-\beta_1) a_{1,p-k+1} c_{1,p-k+1} \\ & \leq 1 - \beta_1. \end{aligned}$$

By Theorem 3.1, we have $F_1 * F_2 \in \mathcal{U}_p(\lambda_1, \alpha_1, \beta_1, k_{0,1}) \cap \mathcal{T}_p$ for $\max\{0, \frac{\lambda_1 - \frac{1}{2}}{\lambda_1 + 1}\} \leq \alpha_1 \leq \lambda_1$. ∎

Acknowledgments

The first author was supported by NSF of Hebei Science Foundation for Young Scientist (No. A2013201104). The second author was supported by NSF of Hebei Education Department (No. Q2012046).

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