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ON CONSTANT PRODUCTS OF ELEMENTS IN SKEW POLYNOMIAL RINGS

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(Communicated by Fariborz Azarpanah)

Abstract. Let R be a reversible ring which is α-compatible for an endomorphism α of R and $f(X) = a_0 + a_1X + \cdots + a_nX^n$ be a nonzero skew polynomial in $R[X;\alpha]$. It is proved that if there exists a nonzero skew polynomial $g(X) = b_0 + b_1X + \cdots + b_mX^m$ in $R[X;\alpha]$ such that $g(X)f(X) = c$ is a constant in R, then $b_0a_0 = c$ and there exist nonzero elements a and r in R such that $rf(X) = ac$. In particular, $r = ab_p$ for some p, $0 \leq p \leq m$, and a is either one or a product of at most m coefficients from $f(X)$. Furthermore, if b_0 is a unit in R, then $a_1, a_2, \cdots, a_n$ are all nilpotent. As an application of the above result, it is proved that if R is a weakly 2-primal ring which is α-compatible for an endomorphism α of R, then a skew polynomial $f(X)$ in $R[X;\alpha]$ is a unit if and only if its constant term is a unit in R and other coefficients are all nilpotent.
Keywords: Constant products, skew polynomial rings, reversible rings, weakly 2-primal rings.
MSC(2010): Primary: 16S36; Secondary: 16U60, 16N40.

1. Introduction

Throughout this note each ring R is associative with identity and a ring homomorphism preserves the identity. For a ring R, we use the symbol $N(R)$ to denote the set of nilpotent elements in R, $U(R)$ its unit group, $M_n(R)$ the ring of $n\times n$ matrices over R, and E_n the $n\times n$ identical matrix over R. The symbol $N_*(R)$ denotes the prime radical of a ring R, $N^*(R)$ its upper nil-radical, L-$rad(R)$ its Levitzki radical, and $J(R)$ its Jacobson radical, respectively.

Recall that a ring R is reduced if it has no nonzero nilpotent elements. A ring R is reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$. A ring R is semicommutative if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Obviously, a ring R is semicommutative if and only if its oppositive ring R^{op} is semicommutative. A

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ring R is 2-primal if $N(R) = N_*(R)$. A ring R is weakly 2-primal if $N(R) = L$-$rad(R)$. And a ring R is NI if $N(R) = N^*(R)$. It is known that reduced $\Rightarrow$ reversible $\Rightarrow$ semicommutative $\Rightarrow$ 2-primal $\Rightarrow$ weakly 2-primal $\Rightarrow$ NI, but the converse does not hold (see [3,8]). Let R be a ring and α be an endomorphism of R. The ring R is called α-compatible provided $ab = 0 \Leftrightarrow a\alpha(b) = 0$, and R is called weak α-compatible in case $ab \in N(R) \Leftrightarrow a\alpha(b) \in N(R)$ where $a, b \in R$. A ring R to be α-compatible is also said to satisfy α-condition in some literatures (see [3]). According to Krempa [6], an endomorphism α of a ring R is called rigid if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. It is proved in [6, Lemma 3.2] that if R is a reduced ring and α is an endomorphism of R, then α is rigid if and only if α is monomorphism persevering every minimal prime ideal of R if and only if $\alpha^{-1}(P) \subseteq P$ for any minimal prime P of R.

It is well known that a polynomial over a commutative ring R is a unit if and only if its constant term is a unit in R and other coefficients are nilpotent. This result in [2] has been extended to a 2-primal ring. However the conclusion is not true for a noncommutative ring in general (see [2, Example 2.8]). The aim of this note is to extend the main results in [2] to more general cases. We generalize the constant-product theorem for a commutative polynomial ring [4] to a skew polynomial ring $R[X;\alpha]$ where R is a reversible ring which is α-compatible for an endomorphism α of R. It follows that if R is a weakly 2-primal ring which is α-compatible for an endomorphism α of R, then a skew polynomial $f(X)$ in $R[X;\alpha]$ is a unit if and only if its constant term is a unit in R and other coefficients are nilpotent. For an NI-ring R, it is proved that if R is weak α-compatible then $f(X)$ in $R[X;\alpha]$ is a unit only if its constant term is a unit in R and the other coefficients are nilpotent, and that the stable range of $R[X;\alpha]$ is not equal to one. Moreover we define a ring R to be a UN-ring in case any $f(X)$ in $R[X]$ is a unit if and only if its constant term is a unit in R and other coefficients are nilpotent, and conclude that any NI-ring is a UN-ring if and only if Koethe's Conjecture has a positive solution.

2. **Constant products in skew polynomial rings**

We start this section with the following lemmas.

Lemma 2.1. *([3, Lemma 3.1]) Let R be a ring and α be an endomorphism of R. If R is α-compatible, then for any $n \geq 2$, $a_1a_2\cdots a_n = 0 \Leftrightarrow \alpha^{k_1}(a_1)\alpha^{k_2}(a_2)\cdots \alpha^{k_n}(a_n) = 0$ where $a_1, a_2, \cdots, a_n \in R$ and $k_1, k_2, \cdots, k_n$ are any nonnegative integers. In particular, $a_1a_2 \in N(R) \Leftrightarrow a_1\alpha^k(a_2) \in N(R)$ for any nonnegative integer k.*

Lemma 2.1 implies that if a ring R is α-compatible then it is weak α-compatible, and that if R is reversible and α-compatible then $rf(X) = 0$ if and only if $f(X)r = 0$ for $f(X)$ in $R[X;\alpha]$ and r in R.

Lemma 2.2. *([3, Theorem 3.1]) Let R be a weakly 2-primal ring and α be an endomorphism of R. If R is α-compatible, then the skew polynomial ring $R[X;\alpha]$ is also a weakly 2-primal ring. More specifically, $N(R)[X;\alpha] = N(R[X;\alpha]) = L\text{-}rad(R[X;\alpha])$.*

Lemma 2.3. *Let R be 2-primal ring which is α-compatible for an endomorphism α of R. If P is any minimal prime ideal of R, then both $\alpha(P)$ and $\alpha^{-1}(P)$ are contained in P.*

Proof. Since R is a 2-primal ring, $N(R) = N_*(R)$, and $\overline{R} = R/N_*(R)$ is reduced. By Lemma 2.1, α induces an endomorphism $\bar{\alpha}$ of $\overline{R}$ via $\bar{\alpha}(\bar{a}) = \overline{\alpha(a)}$ where $\bar{a} = a + N_*(R)$ for any $a \in R$. Now if $\bar{a}\bar{\alpha}(\bar{a}) = \bar{0}$ for $a \in R$, then $a\alpha(a) \in N_*(R)$. This gives $a^2 \in N_*(R)$ by Lemma 2.1, and so $a \in N_*(R)$. Hence $\bar{a} = \bar{0}$ and $\bar{\alpha}$ is a rigid endomorphism of $\overline{R}$. Note that $P/N_*(R)$ is a minimal prime ideal of $\overline{R}$. Using [6, Lemma 3.2], one has $\bar{\alpha}(P/N_*(R)) \subseteq P/N_*(R)$ and $\bar{\alpha}^{-1}(P/N_*(R)) \subseteq P/N_*(R)$. It yields that both $\alpha(P)$ and $\alpha^{-1}(P)$ are contained in P. □

By Lemma 2.3, α may induce an endomorphism of R/P naturally.

Recall that a prime ideal of a ring R is completely prime if R/P is a domain. It is known by [9, Proposition 1.11] that a ring R is 2-primal if and only if each minimal prime ideal is completely prime. Also note that for any ring R, the prime radical $N_*(R)$ is equal to the intersection of all minimal prime ideals of R (cf. [7, p. 180]).

The next two theorems are the counterparts of the main theorem of [4].

Theorem 2.4. *Let R be a reversible ring which is α-compatible for an endomorphism α of R and $f(X) = a_0 + a_1X + \cdots + a_nX^n$ be a nonzero skew polynomial in $R[X;\alpha]$. If there is nonzero skew polynomial $g(X) = b_0 + b_1X + \cdots + b_mX^m$ in $R[X;\alpha]$ such that $g(X)f(X) = c$ is a constant, then $b_0a_0 = c$ and there exist nonzero elements a and r in R such that $rf(X) = ac$. In particular, $r = ab_p$ for some p, $0 \le p \le m$, and a is either one or a product of at most m coefficients from $f(X)$. Furthermore, if b_0 is a unit in R, then $a_1, a_2, \cdots, a_n$ are all nilpotent.*

Proof. First we prove that the conclusion is true for any skew polynomial $f(X)$ of degree 0. By the assumption $f(X) = a_0 \neq 0$, and $g(X)a_0 = c$. This means $b_0a_0 = c$. If $b_0 \neq 0$, then $r = b_0$ and $a = 1$ are desired nonzero elements. If $b_0 = 0$, then $c = b_0a_0 = 0$. Assume that b_q is the least nonzero coefficient of $g(X)$. Then one has $g(X)a_0 = (b_qX^q + \cdots + b_mX^m)a_0 = 0$. This gives $b_q\alpha^q(a_0) = 0$, and so $b_qa_0 = 0$ by Lemma 2.1. Thus $r = b_q$ and $a = 1$ are desired nonzero elements.

Next we may assume that $f(X)$ is of degree $n \geq 1$. We proceed by induction on the degree of $g(X)$. If $m = 0$, then $g(X) = b_0 \neq 0$. From $g(X)f(X) = b_0(a_0 + a_1X + \cdots + a_nX^n) = c$, one has $c = b_0a_0$ and so $r = b_0, a = 1$ are desired. Assume that the conclusion is true for all skew polynomials of degree less then m. Let $g(X)f(X) = c$ for $g(X)$ of degree m, we show that $g(X)$ can be replaced by a skew polynomial of lower degree. If $b_0 = 0$, then $c = b_0a_0 = 0$, and $g(X) = b_qX^q + \cdots + b_mX^m$ where b_q is the least nonzero coefficient of $g(X)$. If $a_kg(X) = 0$ for all $1 \leq k \leq n$, then $a_kb_q = 0$, and so $b_qa_k = 0$ for such k. In this case, $b_qf(X) = 0 = c$, and one may take $r = b_q, a = 1$ as desired. Thus we may assume that k is the largest positive integer such that $a_kg(X) \neq 0$. In the case of $k = n$, then from $g(X)f(X) = 0 = c$ one has $b_m\alpha^m(a_n) = 0$, which implies $b_ma_n = a_nb_m = 0$ and so $a_ng(X)$ is of degree less than m satisfying $a_ng(X)f(X) = a_nc = 0$. In the case of $k < n$, then $a_sg(X) = 0 = g(X)a_s$ for $k+1 \leq s \leq n$. This means $g(X)(a_0 + a_1X + \cdots + a_kX^k) = g(X)f(X) = c = 0$. It yields that $b_m\alpha^m(a_k) = 0 = b_ma_k = a_kb_m$. This implies that $a_kg(X)$ is of degree less than m, and $a_kg(X)f(X) = a_kc = 0$. Thus induction hypothesis applied to $a_kg(X)$ and a_kc yields the desired conclusion. If $b_0 \neq 0$, and $a_kg(X) = 0$ for all $1 \leq k \leq n$, then $a_kb_0 = 0 = b_0a_k$ for such k. It follows that $b_0f(X) = c$, and so $b_0a_0 = c$. Clearly, $r = b_0$ and $a = 1$ satisfy the desired condition. Thus we assume that k is the largest positive integer such that $a_kg(X) \neq 0$. In the case of $k = n$, from $g(X)f(X) = c$ one has that $b_m\alpha^m(a_n) = 0$, which implies $b_ma_n = a_nb_m = 0 =$ and so $a_ng(X)$ is of degree less than m satisfying $a_ng(X)f(X) = a_nc$. In the case of $k < n$, then $a_sg(X) = 0 = g(X)a_s$ for $k+1 \leq s \leq n$. This means $g(X)(a_0 + a_1X + \cdots + a_kX^k) = g(X)f(X) = c$. It yields that $b_m\alpha^m(a_k) = 0 = b_ma_k = a_kb_m$, and thus $a_kg(X)$ is of degree less than m, and $a_kg(X)f(X) = a_kc$. Now induction hypothesis applied to $a_kg(X)$ and a_kc yields the desired conclusion.

Now we prove the nilpotency of $a_1, a_2, \cdots, a_n$. Assume that $g(X)f(X) = c$ and b_0 is a unit in R. Let P be any minimal prime ideal of R. By Lemma 2.3, one can define an endomorphism $\bar{\alpha}$ of $\overline{R} = R/P$ via $\bar{\alpha}(\bar{a}) = \overline{\alpha(a)}$ where $\bar{a} = a + P$ for any $a \in R$. Thus $\overline{R}[X; \bar{\alpha}]$ is a skew polynomial ring. Since R is a 2-primal ring, P is a completely prime ideal of R and so $\overline{R}$ is a domain. Clearly, $\overline{R}$ is a reversible ring. We prove that $\overline{R}$ is $\bar{\alpha}$-compatible. By Lemma 2.3, $\alpha(P) \subseteq P$ and $\alpha^{-1}(P) \subseteq P$ hold. If $\bar{a}\bar{b} = \bar{0}$ for $a, b \in R$, then $ab \in P$. This implies $a \in P$ or $b \in P$ since P is a completely prime ideal of R. It follows that $a\alpha(b) \in P$ by Lemma 2.3, that is, $\bar{a}\bar{\alpha}(\bar{b}) = \bar{0}$. Conversely, if $\bar{a}\bar{\alpha}(\bar{b}) = \bar{0}$ for $a, b \in R$, then $a\alpha(b) \in P$. This means $a \in P$ or $\alpha(b) \in P$. Again by Lemma 2.3, $a \in P$ or $b \in P$, this gives $ab \in P$ and so $\bar{a}\bar{b} = \bar{0}$. It is easy to check that there exists a natural ring epimorphism from $R[X; \alpha]$ onto $\overline{R}[X; \bar{\alpha}]$. It follows that $\bar{g}(X)\bar{f}(X) = \bar{c}$ in $\overline{R}[X; \bar{\alpha}]$. If $\bar{f}(X) = \bar{0}$, then clearly $a_i \in P$ for all $i \geq 1$. Since b_0 is a unit, $\bar{g}(X) \neq \bar{0}$. If $\bar{f}(X) \neq \bar{0}$, then $\bar{g}(X)\bar{f}(X) = \bar{c}$ implies that there exist $\bar{r}, \bar{a} \neq \bar{0}$ such that $\bar{r}\bar{f}(X) = \bar{a}\bar{c}$. This means $\bar{r}\bar{a}_i = \bar{0}$ for each $i \geq 1$. Noticing that R/P is a domain, one has $\bar{a}_i = \bar{0}$, and so $a_i \in P$. Therefore $a_i \in N_*(R)$ for all $i \geq 1$, since $N_*(R)$ is the intersection of all minimal prime ideals of R. □

Theorem 2.5. *Let R be a reversible ring which is α-compatible for an endomorphism α of R and $f(X) = a_0 + a_1X + \cdots + a_nX^n$ in $R[X;\alpha]$ be a nonzero skew polynomial. If there is nonzero skew polynomial $g(X) = b_0 + b_1X + \cdots + b_mX^m$ in $R[X;\alpha]$ such that $f(X)g(X) = c$ is a constant, then $a_0b_0 = c$ and there exist nonzero elements a and r in R such that $f(X)r = ca$. In particular, $r = b_pa$ for some p, $0 \leq p \leq m$, and a is either one or a product of at most m elements from $\{\alpha^k(a_i)|0 \leq k \leq m, 0 \leq i \leq n\}$. Furthermore, if b_0 is a unit in R, then $a_1, a_2, \cdots, a_n$ are all nilpotent.*

Proof. Similar to the proof of Theorem 2.4, it is easy for one to prove the conclusion for $f(X)$ of degree 0. Next we may assume that $f(X)$ is of degree $n \geq 1$. We proceed by induction on the degree of $g(X)$. If $m = 0$, then $g(X) = b_0 \neq 0$ and $f(X)b_0 = (a_0 + a_1X + \cdots + a_nX^n)b_0 = c$. Clearly, $a_0b_0 = c$ and one may take $r = b_0$ and $a = 1$ as desired. Assume that the conclusion is true for all skew polynomials of degree less than m. Let $f(X)g(X) = c$ for $g(X)$ of degree m, we show that $g(X)$ can be replaced by a skew polynomial of lower degree. If $b_0 = 0$, then $c = a_0b_0 = 0$. From $f(X)g(X) = f(X)g^*(X)X^q = c = 0$, one has $f(X)g^*(X) = 0 = c$ where $g^*(X) = b_q + b_{q+1}X + \cdots + b_mX^{m-q}$ and b_q is the least nonzero coefficient of $g(X)$. One may get desired nonzero elements a and r by the inductive assumption. If $b_0 \neq 0$, and $g(X)a_k = 0$ for all $1 \leq k \leq n$, then $b_0a_k = 0$ for such k. This means that $a_kb_0 = a_k\alpha^k(b_0) = 0$ for all $1 \leq k \leq n$, since R is reversible and α-compatible. It follows that $f(X)b_0 = c$ and $c = a_0b_0$, hence $r = b_0$ and $a = 1$ satisfy the desired condition. Thus we assume that k is the largest positive integer such that $g(X)a_k \neq 0$. In the case of $k = n$, from $f(X)g(X) = c$ one has that $a_n\alpha^n(b_m) = 0$, which implies $a_nb_m = 0 = b_ma_n$ and so $b_m\alpha^m(a_n) = 0$. That is, $g(X)a_n$ is of degree less than m satisfying $f(X)(g(X)a_n) = ca_n$. In the case of $k < n$, then $g(X)a_s = 0$ for $k + 1 \leq s \leq n$. This means $a_sg(X) = 0$ by Lemma 2.1, and so $a_sb_0 = a_sb_1 = \cdots = a_sb_m = 0$. It yields that $a_sX^sg(X) = 0$ by Lemma 2.1. Thus one may get $(a_0 + a_1X + \cdots + a_kX^k)g(X) = f(X)g(X) = c$. It follows that $a_k\alpha^k(b_m) = 0 = a_kb_m = b_ma_k$, and thus $b_m\alpha^m(a_k) = 0$. This implies that $g(X)a_k$ is of degree less than m, and $f(X)(g(X)a_k) = ca_k$. Now induction hypothesis applied to $g(X)a_k$ and ca_k yields the desired conclusion.

The proof regarding the nilpotency of $a_1, a_2, \cdots, a_n$ is very similar to that of Theorem 2.4, so we omit the detail. □

The next corollary is a direct result of Theorem 2.4 or Theorem 2.5.

Corollary 2.6. *Let R be a reversible ring which is α-compatible for an endomorphism α of R. A skew polynomial $f(X)$ in $R[X;\alpha]$ is a divisor of zero if and only if there exists a nonzero constant $r \in R$ such that $rf(X) = f(X)r = 0$.*

A ring R is called right McCoy if for two nonzero polynomials $f(X)$ and $g(X)$ in $R[X]$ whenever $f(X)g(X) = 0$, then there exists nonzero element r in

R such that $f(X)r = 0$. A left McCoy ring can be defined similarly. If a ring R is left and right McCoy, then it is called McCoy (see [8]).

Corollary 2.7. *([8, Theorem 2]) If R is a reversible ring, then R is a McCoy ring.*

Remark 2.8. *Nielsen [8] proved that there exists a semicommutative ring R which is not right McCoy. This means that one could not expect to extend Theorem 2.4 or 2.5 to a semicommutative ring since the oppositive ring of a semicommutative ring is semicommutative. On the other hand, let S be a reduced ring, then it is easy to check that the ring*

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in S \right\}$$

is reversible. Define α: $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$, then α is an endomorphism of R, and R is α-compatible. In this way, one can get noncommutative reversible rings being α-compatible.

Corollary 2.9. *Let R be a reversible ring which is α-compatible for an endomorphism α of R. A skew polynomial $f(X)$ in $R[X;\alpha]$ is a unit if and only if its constant term is a unit and other coefficients are nilpotent.*

Proof. If $f(X) = a_0 + a_1X + \cdots + a_nX^n$ in $R[X;\alpha]$ is a unit, then there exists $g(X) = b_0 + b_1X + \cdots + b_mX^m$ in $R[X;\alpha]$ such that $f(X)g(X) = g(X)f(X) = 1$. this means that b_0 is a unit, and so all a_k are nilpotent for $k \geq 1$ by Theorem 2.4. Conversely, if a_0 is a unit and each a_k is nilpotent for $k \geq 1$, then $a_1X + \cdots + a_nX^n \in N(R)[X;\alpha] = L\text{-}rad(R[X;\alpha]) \subseteq J(R[X;\alpha])$ by Lemma 2.2, concluding that $f(X)$ is a unit in $R[X;\alpha]$. □

The next two corollaries are the counterparts of the last corollary in [4].

Corollary 2.10. *Let R be a reversible ring which is α-compatible for an endomorphism α of R, $f(X) = a_0 + a_1X + \cdots + a_nX^n$ and $g(X) = b_0 + b_1X + \cdots + b_mX^m$ be nonzero skew polynomials in $R[X;\alpha]$ such that a_1 is a unit in R. If $f(g(X)) = 0$ and either b_0 or $a_2, a_3, \cdots, a_n$ are all nilpotent, then $b_1, b_2, \cdots, b_m$ are also nilpotent.*

Proof. Note that the condition $f(g(X)) = 0$ implies that $a_0 + a_1g + \cdots + a_ng^n = 0$. This means that $(a_1 + a_2g + \cdots + a_ng^{n-1})g = -a_0$. However the constant term of $a_1 + a_2g + \cdots + a_ng^{n-1}$ is $a_1 + a_2b_0 + a_3b_0^2 + \cdots + a_nb_0^{n-1}$ which is a unit since it is a sum of a unit and a nilpotent element contained in $N_*(R)$. By Theorem 2.5, $b_1, b_2, \cdots, b_m$ are all nilpotent. □

Corollary 2.11. *Let R be a reversible ring which is α-compatible for an endomorphism α of R, $f(X) = a_0 + a_1X + \cdots + a_nX^n$ and $g(X) = b_0 + b_1X + \cdots + b_mX^m$ be nonzero skew polynomials in $R[X;\alpha]$ such that b_1 is a unit* *in R. If $g(f(X)) = 0$ and either a_0 or $b_2, b_3, \cdots, b_m$ are all nilpotent, then $a_1, a_2, \cdots, a_n$ are also nilpotent.*

Proof. It is similar to the proof of Corollary 2.10. □

Corollary 2.12. *Let R be a reduced ring and α be an endomorphism of R. If R is α-compatible, then $f(X) = a_0 + a_1X + \cdots + a_nX^n$ in $R[X;\alpha]$ is a unit if and only if a_0 is a unit in R and a_i is zero for each $i \geq 1$.*

Corollary 2.13. *Let R be an NI ring and α be an endomorphism of R. If R is weak α-compatible, then $f(X) = a_0 + a_1X + \cdots + a_nX^n$ in $R[X;\alpha]$ is a unit only if a_0 is a unit and a_i is nilpotent for each $i \geq 1$.*

Proof. Since R is NI, $N(R)$ is an ideal of R and $\overline{R} = R/N(R)$ is reduced. It is easy to check that α induces an endomorphism $\bar{\alpha}$ of $\overline{R}$ via $\bar{\alpha}(\bar{a}) = \alpha(a) + N(R)$ since $\alpha(N(R)) \subseteq N(R)$ where $\bar{a} = a + N(R)$ for $a \in R$. Noticing that R is weak α-compatible, one has $\bar{a}\bar{\alpha}(\bar{b}) = \bar{0} \Leftrightarrow a\alpha(b) \in N(R) \Leftrightarrow ab \in N(R) \Leftrightarrow \bar{a}\bar{b} = \bar{0}$ for $a, b \in R$, that is, $\overline{R}$ is $\bar{\alpha}$-compatible. Since there exists a ring epimorphism from $R[X;\alpha]$ onto $\overline{R}[X;\bar{\alpha}]$, which sends $f(X)$ to $\bar{f}(X) = \bar{a}_0 + \bar{a}_1X + \cdots + \bar{a}_nX^n$, $\bar{f}(X)$ is a unit in $\overline{R}[X;\bar{\alpha}]$. Hence $\bar{a}_0$ is a unit in $R/N(R)$ and $\bar{a}_i$ is zero for each $i \geq 1$ by Corollary 2.12. Now it is easy to see that a_0 is a unit in R and a_i is nilpotent for each $i \geq 1$. □

Corollary 2.14. *Let R be a weakly 2-primal ring and α be an endomorphism of R. If R is α-compatible, then $f(X) = a_0 + a_1X + \cdots + a_nX^n$ in $R[X;\alpha]$ is a unit if and only if a_0 is a unit in R and a_i is nilpotent for each $i \geq 1$.*

Proof. Since a weakly 2-primal ring is NI and R being α-compatible implies being weak α-compatible, the only if part follows from Corollary 2.13. Conversely, let $f(X) = a_0 + a_1X + \cdots + a_nX^n$ in $R[X;\alpha]$ be such that a_0 is a unit in R and a_i is nilpotent for each $i \geq 1$. Then one has $a_1X + \cdots + a_nX^n \in L\text{-}rad(R[x;\alpha]) \subseteq J(R[X;\alpha])$ by Lemma 2.2, this implies that $f(X)$ is a unit in $R[X;\alpha]$. □

Corollary 2.15. *([2, Theorem 2.5]) Let R be a 2-primal. Then $f(X) = a_0 + a_1X + \cdots + a_nX^n$ in $R[x]$ is a unit if and only if a_0 is a unit in R and a_i is nilpotent for each $i \geq 1$.*

According to [5], a ring R is called unit-central if the units of R lies in its center. As an application of the above result, we show that a ring R is unit-central if and only if $R[X]$ is unit-central.

Proposition 2.16. *A ring R is unit-central if and only if $R[X]$ is unit-central.*

Proof. First we show that a unit-central ring R is 2-primal. Since the prime radical of R is the set of all strongly nilpotent elements in R, it is sufficient to show that every nilpotent element of R is strongly nilpotent. Assume that $a \in N(R)$ satisfies $a^n = 0$ for some positive integer n. We consider the m-sequence beginning with a, that is, $a_0 = a$, $a_{i+1} = a_i r_i a_i$ where $r_i \in R$ $(i = 0, 1, \cdots)$. Note that R is unit-central implies that a is central. One has $a_1 = a^2 r_0$, $a_2 = a^{2^2} r_0 r_1 r_0$, inductively, $a_n = a^{2^n} s$ for some $s \in R$. This means $a_n = 0$ and so a is strongly nilpotent. Now assume that $R[X]$ is unit-central. Then R is unit-central as a subring of $R[X]$. Conversely, assume that R is unit-central. Then R is 2-primal by the above argument. For any $f(X) = a_0 + a_1X + \cdots + a_nX^n$ in $U(R[X])$, then a_0 is a unit in R and $a_i \in N(R)$ for each $i \geq 1$ by Corollary 2.15. Since R is unit-central, a_i is central for all $i \geq 0$. It follows that $f(X)$ is also central in $R[X]$. □

Recall that a ring R is said to have stable range one, denoted by $S_r(R) = 1$, if for $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is a unit in R. This notion is very important in the study of algebraic K-theory. It is well known that for a commutative ring R, $S_r(R[X]) > 1$. Now we prove the conclusion is true for the skew polynomial ring $R[X; \alpha]$ over an NI-ring R which is weak α-compatible.

Proposition 2.17. *If R is an NI ring which is weak α-compatible for an endomorphism α of R, then $S_r(R[X; \alpha]) > 1$.*

Proof. Assume the contrary, then $X(-X)+1+X^2 = 1$ implies that there exists $f(X) \in R[X; \alpha]$ such that $X+(1+X^2)f(X)$ is a unit in $R[X; \alpha]$. Write $f(X) = a_0+a_1X+\cdots+a_nX^n$. In the case of $n = 0$, then $a_0+X+\alpha^2(a_0)X^2$ is a unit in $R[X; \alpha]$. This implies that 1 is nilpotent in R by Corollary 2.13, a contradiction. When $n = 1$, then $X+(1+X^2)f(X) = a_0+(1+a_1)X+\alpha^2(a_0)X^2+\alpha^2(a_1)X^3$ is a unit. Hence $\alpha^2(a_1) \in N(R)$, and so does a_1 by Lemma 2.1. It follows that $1 + a_1$ is nilpotent, this is impossible since $1 + a_1$ is a unit. When $n = 2k, k \geq 1$, then $X + (1 + X^2)f(X) = a_0 + (1 + a_1)X + (a_2 + \alpha^2(a_0))X^2 + (a_3+\alpha^2(a_1))X^3+\cdots+\alpha^2(a_{2k-1})X^{2k+1}+\alpha^2(a_{2k})X^{2k+2}$ is a unit. This implies $\alpha^2(a_{2k-1}), \alpha^2(a_{2k}) \in N(R)$, and so $a_{2k-1}, a_{2k} \in N(R)$. Inductively, we have $a_{2k}, a_{2k-1}, \cdots, a_4, a_3 \in N(R)$. Hence $a_3 + \alpha^2(a_1) \in N(R)$, and so does $\alpha^2(a_1)$. It follows that $a_1 \in N(R)$ and $1 + a_1 \in N(R)$, a contradiction. In the case of $n = 2k + 1, k \geq 1$, similar to the case of $n = 2k$, we can get a desired contradiction. The proof is complete. □

Corollary 2.18. *([2, Proposition 2.7]) If R is a 2-primal ring, then $S_r(R[X]) > 1$.*

In view of Corollary 2.13, it is natural to study the sufficient and necessary condition under which $f(X)$ in $R[X]$ is a unit. This problem is closely related to the famous Koethe's Conjecture whether a nil one sided ideal of any ring R is contained in its upper nil radical. It is known that Koethe's Conjecture has a positive solution if and only if for each nil algebra S over any countable field, the polynomial algebra $S[X]$ is Jacobson radical (see [10]). This is equivalent to saying that for any ring R, $J(R[X]) = Nil^*(R)[X]$ holds (cf. [7, p.181]).

Let R be a ring. Consider the following condition: $f(X) = \sum_{i=0}^{n} a_i X^i \in U(R[X]) \Leftrightarrow a_0 \in U(R)$ and $a_i \in N(R)$ for each $i \geq 1$.

Call a ring R to be a UN ring if R satisfies the above condition.

Theorem 2.19. *Koethe's Conjecture has a positive solution if and only if any NI ring R is a UN ring.*

Proof. Assume that Koethe's Conjecture has a positive solution. Then we have $J(R[X]) = N^*(R)[X]$ for any ring R (see [7, p.181]). We prove that any NI ring R is a UN ring. Let $f(X) = a_0 + a_1X + \cdots + a_nX^n \in R[X]$. If $f(X) \in U(R[X])$, then $a_0 \in U(R)$ and $a_i \in N(R)$ for each $i \geq 1$ by Corollary 2.13. Conversely, let $f(X) = a_0 + a_1X + \cdots + a_nX^n \in R[X]$ be such that $a_0 \in U(R)$ and $a_i \in N(R)$ for each $i \geq 1$. Write $g(X) = a_1X + \cdots + a_nX^n$. Then $g(X) \in N(R)[X] = N^*(R)[X] = J(R[X])$ since R is an NI ring. This means $f(X) = a_0 + g(X)$ is a unit in $R[X]$. It follows that R is a UN ring. Now assume that any NI-ring R is a UN-ring. We prove that for each nil algebra S over any countable F, $J(S[X]) = S[X]$. Let $R = F + S$ (the sum of algebras). Then R is an NI ring with $N(R) = S$. Since S is an ideal of R, $S[X]$ is an ideal of $R[X]$. Hence $J(S[X]) = J(R[X]) \bigcap S[X] \subseteq J(R[X])$. On the other hand, $J(R[X]) = I[X]$ for some nil ideal of R by [1, Theorem 1]. This means that $J(R[X]) \subseteq S[X] \bigcap J(R[X]) = J(S[X])$ and so $J(R[X]) = J(S[X])$. Now for any $h(X) \in S[X] = N(R)[X]$, $1 + h(X)$ is a unit since R is a UN ring. Hence $S[X]$ is a quasi-regular ideal of $R[X]$. It follows that $S[X] \subseteq J(R[X]) = J(S[X])$, and so $S[X] = J(S[X])$. $\square$

We conclude this note with the following proposition.

Proposition 2.20. *For any ring R and $n \geq 2$, $M_n(R)$ is not a UN-ring.*

Proof. We may canonically identify $M_n(R)[X]$ with $M_n(R[X])$. When $n = 2$, let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. Clearly, A is a unit and B is a nonzero idempotent. However $f(X) = A + BX = \begin{pmatrix} 1 & 1 \\ X & 1+X \end{pmatrix}$ is a unit in $M_2(R)[X]$ with the inverse $\begin{pmatrix} 1+X & -1 \\ -X & 1 \end{pmatrix}$. When $n > 2$, then $f_1(X) = \begin{pmatrix} E_{n-2} & O \\ O & A \end{pmatrix} + \begin{pmatrix} O_{n-2} & O \\ O & B \end{pmatrix} X = \begin{pmatrix} E_{n-2} & O \\ O & A+BX \end{pmatrix}$ is a unit in $M_n(R)[X]$ by the above argument, but the coefficient of X is a nonzero idempotent. Moreover, let $g(X) = E_2 + E_{12}X + E_{21}X^2 = \begin{pmatrix} 1 & X \\ X^2 & 1 \end{pmatrix}$ where E_{12}, E_{21} are 2×2 matrix units. Then $g(X)$ is not a unit, since $\begin{pmatrix} 1 & 0 \\ -X^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & X \\ X^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1-X^3 \end{pmatrix}$ is not a unit in

$M_2(R)[X]$. But the constant term of $g(X)$ is a unit and the other coefficients are nilpotent. In the case of $n > 2$, then $g_1(X) = \begin{pmatrix} E_2 & O \\ O & E_{n-2} \end{pmatrix} + \begin{pmatrix} E_{12} & O \\ O & O \end{pmatrix} X + \begin{pmatrix} E_{21} & O \\ O & O \end{pmatrix} X^2 = \begin{pmatrix} g(X) & O \\ O & E_{n-2} \end{pmatrix}$ is not a unit in $M_n(R)[X]$, however its constant term is a unit and othercoefficients are nilpotent. □

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