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CONFORMAL MAPPINGS PRESERVING EINSTEIN TENSOR OF WEYL MANIFOLDS

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Abstract. In this paper, we obtain a necessary and sufficient condition for a conformal mapping between two Weyl manifolds to preserve Einstein tensor. Then we prove that some basic curvature tensors of W_n are preserved by such a conformal mapping if and only if the covector field of the mapping is locally a gradient. Also, we obtained the relation between the scalar curvatures of the Weyl manifolds related by a conformal mapping preserving the Einstein tensor with a gradient covector field. Then, we prove that a Weyl manifold W_n and a flat Weyl manifold $\tilde{W}_n$, which are in a conformal correspondence preserving the Einstein tensor are Einstein-Weyl manifolds. Moreover, we show that an isotropic Weyl manifold is an Einstein-Weyl manifold with zero scalar curvature and we obtain that a Weyl manifold W_n and an isotropic Weyl manifold related by the conformal mapping preserving the Einstein tensor are Einstein-Weyl manifolds.
Keywords: Weyl manifold, Einstein tensor, conformal mapping, flat Weyl manifold, isotropic Weyl manifold.
MSC(2010): Primary: 53A30; Secondary: 53B15.

1. Introduction

Conformal mappings of Riemannian manifolds were studied by many authors [3, 11–13]. Weyl and Schouten studied conformal mappings of Riemannian spaces onto a flat space [11, 13]. In [3], the authors obtained a necessary and sufficient condition for a Riemannian space V_n to admit a conformal mapping preserving the Einstein tensor onto some Riemannian space $\tilde{V}_n$. In [4], Gribacheva obtained necessary and sufficient conditions for a conformal flat Weyl space to admit a conformal mapping onto a flat Weyl space and in [1], the authors studied geodesic mappings preserving the Einstein tensor of Weyl spaces.

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The purpose of the present paper is to consider the conformal mappings preserving the Einstein tensor of Weyl manifolds. So, the results of the paper generalize some of the results in [3].

2. **Preliminaries**

An n-dimensional manifold with a conformal metric g and a symmetric connection ∇ satisfying the compatibility condition

$$\nabla g - 2\, g \otimes T = 0 \tag{2.1}$$

or, in local coordinates

$$\nabla_k\, g_{ij} - 2\, T_k\, g_{ij} = 0\ , \tag{2.2}$$

is called a Weyl space, where T is a 1-form. Such a Weyl space will be denoted by $W_n(g, T)$ [5, 7].

Under the renormalization

$$\tilde{g} = \lambda^2\, g \tag{2.3}$$

of the metric tensor g, T is transformed by the rule

$$\tilde{T}_k = T_k + \partial_k(\ln \lambda), \tag{2.4}$$

where $\partial_k = \dfrac{\partial}{\partial x^k}$ and λ is a scalar function [5, 7].

If under the renormalization (2.3) of the metric tensor g, a quantity A is changed according to the rule

$$\tilde{A} = \lambda^p A, \tag{2.5}$$

then A is called a satellite of g of weight $\{p\}$.

The prolonged covariant derivative of the satellite A with respect to ∇ is defined by

$$\dot{\nabla}_k\, A = \nabla_k\, A - p\, T_k\, A. \tag{2.6}$$

By writting (2.2) and expanding it we find that

$$\partial_k g_{ij} - g_{hj}\, \Gamma^h_{ik} - g_{ih}\, \Gamma^h_{jk} - 2\, T_k\, g_{ij} = 0, \tag{2.7}$$

where Γ^i_{jk} are the connection coefficients of the form

$$\Gamma^i_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} - (\, \delta^i_j \, T_k + \delta^i_k \, T_j - g_{jk} \, g^{ih} \, T_h \,). \tag{2.8}$$

$R(X,Y)\,Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ denotes the curvature tensor associated with the connection ∇ and in local coordinates, the curvature tensor R^h_{ijk} with weight $\{0\}$ is defined by

$$(\nabla_j \nabla_k - \nabla_k \nabla_j)\, v^h = v^i \, R^h_{ijk}, \tag{2.9}$$

which implies that

$$R^h_{ijk} = \partial_j \, \Gamma^h_{ik} - \partial_k \, \Gamma^h_{ij} + \Gamma^h_{mj} \, \Gamma^m_{ik} - \Gamma^h_{mk} \, \Gamma^m_{ij}. \tag{2.10}$$

The tensor defined by

$$R_{ijkl} = g_{ih} \, R^h_{jkl} \tag{2.11}$$

is called the covariant curvature tensor. It is clear R_{ijkl} is of weight $\{2\}$. The Ricci tensor of weight $\{0\}$ and the scalar curvature tensor of weight $\{-2\}$ are defined, respectively, by

$$R_{ij} = R^h_{ijk}, \quad (R_{ij} = g^{kl} \, R_{kijl}) \tag{2.12}$$

and

$$R = g^{ih} \, R_{ih}. \tag{2.13}$$

The tensor

$$E_{ij} = R_{(ij)} - \frac{R}{n} \, g_{ij} \tag{2.14}$$

is defined as the Einstein tensor of $W_n(g,T)$, where $R_{(ij)}$ denotes the symmetric part of the Ricci tensor.

The conformal mapping of Weyl spaces satisfying the condition

$$\tilde{E}_{ij} = E_{ij} \tag{2.15}$$

is said to be the conformal mapping preserving the Einstein tensor.

A Weyl manifold is an Einstein-Weyl manifold, when the symmetric part of the Ricci tensor is proportional to the metric tensor. In this case, the Einstein tensor vanishes. Hence, for an Einstein-Weyl manifold

$$E_{ij} = R_{(ij)} - \frac{R}{n}\, g_{ij} = 0. \tag{2.16}$$

3. **Conformal mappings preserving the Einstein tensor of Weyl spaces**

Let τ be a conformal mapping of Weyl manifold $W_n(g,T)$ onto another Weyl manifold $\tilde{W_n}(\tilde{g},\tilde{T})$. At corresponding points of the Weyl manifolds $W_n(g,T)$ and $\tilde{W}_n(\tilde{g},\tilde{T})$ it can be taken that [10],

$$g = \tilde{g}. \tag{3.1}$$

Let ∇ and $\bar{\nabla}$ be Weyl connections of Weyl manifolds $W_n(g,T)$ and $\tilde{W}_n(\tilde{g},\tilde{T})$, respectively. Then, from (2.2), (2.8) and (3.1) we have

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\, P_k + \delta^i_k\, P_j - g^{im}\, g_{jk}\, P_m, \tag{3.2}$$

where

$$P_i = T_i - \tilde{T}_i \tag{3.3}$$

is the covector field of the conformal mapping of weight zero.

Suppose that R^h_{ijk} and $\tilde{R}^h_{ijk}$ are the mixed curvature tensors of the Weyl connection coefficients Γ^h_{ij} and $\tilde{\Gamma}^h_{ij}$, respectively. Then, from (2.10) and (3.2) the equality

$$\begin{aligned} \tilde{R}^h_{ijk} \;&=\; R^h_{ijk} - 2\,\delta^h_i\,\nabla_{[j}\,P_{k]} + \delta^h_k\,P_{ij} - \delta^h_j\,P_{ik} \\ &\quad +g^{hl}\,g_{ij}\,P_{lk} - g^{hl}\,g_{ik}\,P_{lj} \end{aligned} \tag{3.4}$$

holds, where

$$P_{ij} = \nabla_j\,P_i - P_i\,P_j + \frac{1}{2}\,g^{mh}\,P_m\,P_h\,g_{ij} \tag{3.5}$$

and brackets indicate the antisymmetrization.

Contracting (3.4) with respect to h and k we get

$$\tilde{R}_{ij} = R_{ij} + 2\,\nabla_{[j}\,P_{i]} + (n-2)\,P_{ij} + g_{ij}\,P^h_h, \tag{3.6}$$

where $P^h_h = g^{ij} P_{ij}$.

Transvecting (3.6) by g^{ij} and using (3.1) we obtain

$$\tilde{R} = R + 2\,(n-1)\,P^h_h, \tag{3.7}$$

which implies

$$P_h^h = \frac{\tilde{R} - R}{2\,(n-1)}. \tag{3.8}$$

On the other hand, it can be easily seen from (3.6) that the antisymmetric parts of the Ricci tensors of $W_n(g,T)$ and $\tilde{W}_n(\tilde{g},\tilde{T})$ are related by

$$\tilde{R}_{[ij]} = R_{[ij]} + n\,\nabla_{[j}\,P_{i]}. \tag{3.9}$$

By virtue of (3.8) and (3.9), (3.6) reduces to,

$$\begin{aligned} \tilde{R}_{ij} \quad &= \quad R_{ij} + \frac{2}{n}\,(\tilde{R}_{[ij]} - R_{[ij]}) + (n-2)\,P_{ij} \\ &\quad + \frac{1}{2(n-1)}\,(\tilde{R} - R)\,g_{ij}, \end{aligned} \tag{3.10}$$

from which it follows that

$$\begin{aligned} P_{ij} \quad &= \quad \frac{1}{n(n-2)}\,[(n-1)\,(\tilde{R}_{ij} - R_{ij}) + (\tilde{R}_{ji} - R_{ji}) \\ &\quad - \frac{n}{2(n-1)}\,g_{ij}\,(\tilde{R} - R)]. \end{aligned} \tag{3.11}$$

Substituting (3.9) and (3.11) into (3.4) we obtain an invariant tensor denoted by

$$C_{ijk}^h = \tilde{C}_{ijk}^h, \tag{3.12}$$

where

$$\begin{aligned} C_{ijk}^h \quad &= \quad R_{ijk}^h + \frac{2}{n}\,\delta_i^h\,R_{[jk]} + \delta_k^h\,L_{ij} - \delta_l^h\,L_{ik} \\ &\quad + g^{hl}\,g_{ij}\,L_{lk} - g^{hl}\,g_{ij}\,L_{ij} \end{aligned} \tag{3.13}$$

and

$$L_{ij} = \frac{1}{(n-2)}\left[-R_{ij} + \frac{2}{n}\,R_{[ij]} + \frac{1}{2(n-1)}\,g_{ij}\,R\right]. \tag{3.14}$$

The tensor denoted by C_{ijk}^h is analogous to the conformal curvature tensor of Riemann manifolds and called the conformal curvature tensor of the Weyl manifold $W_n(g,T)$ [4].

If a conformal mapping from a Weyl manifold to another Weyl manifold preserves the generalized circles then such a conformal mapping is said to be a generalized concircular mapping [8].

A tensor denoted by Z^h_{ijk}, which is an invariant with respect to the generalized concircular mapping of the Weyl manifold is defined by

$$Z^h_{ijk} = R^h_{ijk} - \frac{R}{n(n-1)}\,(g_{ij}\delta^h_k - g_{ik}\,\delta^h_j) \tag{3.15}$$

and it is called the concircular curvature tensor of the Weyl manifold.

Contraction on the indices h and k in (3.15) gives the tensor

$$Z_{ij} = R_{ij} - \frac{R}{n}\,g_{ij} \tag{3.16}$$

of weight $\{0\}$.

Besides the concircular curvature tensor, the other important curvature tensor in differential geometry is the projective curvature tensor. It is defined by

$$\begin{aligned} W^h_{ijk} \;=\; & R^h_{ijk} + \frac{2}{(n+1)}\,\delta^h_i\,R_{[jk]} + \frac{1}{(n-1)}\,(\delta^h_j\,R_{ik} - \delta^h_k\,R_{ij}) \\ & + \frac{2}{(n^2-1)}\,(\delta^h_k\,R_{[ij]} - \delta^h_j\,R_{[ik]}) \end{aligned} \tag{3.17}$$

and preserved by the projective transformation from a Weyl manifold onto another Weyl manifold.

We now proceed to study the problem of the invariance of the Einstein tensor and then, the concircular curvature tensor and finally the projective curvature tensor under the conformal transformation of a Weyl manifold onto another Weyl manifold.

Let $\tau : W_n(g,T) \to \tilde{W}_n(\tilde{g},\tilde{T}),\quad (n > 2)$ be a conformal mapping. By considering the symmetric part of the Ricci tensor of $\tilde{W}_n(\tilde{g},\tilde{T})$ and by using (3.6),(3.7) and (3.8) we obtain

$$\begin{aligned} E_{ij} &= R_{(ij)} - \frac{R}{n}\,g_{ij} \\ &= \tilde{E}_{ij} - (n-2)\,P_{(ij)} + \frac{(n-2)}{n}\,g_{ij}\,P^h_h \\ &= \tilde{E}_{ij} - (n-2)\left[P_{(ij)} - g_{ij}\,\frac{(\tilde{R}-R)}{2n(n-1)}\right]. \end{aligned} \tag{3.18}$$

It can be easily seen that $E_{ij} = \tilde{E}_{ij}$ for $n = 2$. Moreover, it is known that any 2-dimensional Weyl manifold is an Einstein-Weyl manifold [10]. Since the Einstein tensor of an Einstein-Weyl manifold vanishes, the conformal mapping between two Einstein-Weyl manifolds of 2-dimensional preserves the Einstein tensor. So, in this section we assume that $n > 2$.

Suppose that the Einstein tensor of the Weyl manifold $W_n(g,T)(n > 2)$ is preserved by τ . Then we have

$$\tilde{E}_{ij} = E_{ij} \tag{3.19}$$

from which it follows that

$$P_{(ij)} = \frac{g_{ij}\,(\tilde{R} - R)}{2n(n-1)}. \tag{3.20}$$

Conversely, suppose that condition (3.20) is valid. By (3.18) it is clear that

$$\tilde{E}_{ij} = E_{ij} \tag{3.21}$$

Thus, we proved that

Theorem 3.1. *The conformal mapping of $W_n(g,T)$ onto $\tilde{W}_n(\tilde{g},\tilde{T})$ $(n > 2)$ preserves the Einstein tensor, if and only if the condition*

$$P_{(ij)} = \frac{1}{2n(n-1)}\, g_{ij}\,(\tilde{R} - R) \tag{3.22}$$

holds.

Substituting (3.4), (3.7) into (3.15) we obtain

$$\tilde{Z}^h_{ijk} = Z^h_{ijk} + X^h_{ijk}, \tag{3.23}$$

where

$$\begin{aligned} X^h_{ijk} \;&=\; 2\,\delta^h_i\,P_{[kj]} + \delta^h_k\,P_{ij} - \delta^h_j\,P_{ik} \\ &\quad -g_{ik}\,g^{hm}\,P_{mj} + g_{ij}\,g^{hm}\,P_{mk} \\ &\quad -\frac{2}{n}\,P^h_h\,(\delta^h_k\,g_{ij} - \delta^h_j\,g_{ik}) = 0. \end{aligned} \tag{3.24}$$

Let $\tilde{Z}^h_{ijk} = Z^h_{ijk}$. Then we have $X^h_{ijk} = 0$. By contracting h and i in (3.24) we get

$$P_{[kj]} = 0, \tag{3.25}$$

which implies that P_k is a gradient.

By similar calculations, and by using (3.4), (3.6), (3.9) and (3.17) we obtain

$$\tilde{W}^h_{ijk} = W^h_{ijk} + P^h_{ijk}, \tag{3.26}$$

where

$$\begin{aligned} P^h_{ijk} = \frac{2}{(n+1)}\,\delta^h_i\,P_{[kj]} \quad &+ \quad \frac{1}{(n-1)}\,\delta^h_k\,\left[P_{ij} - \frac{2}{(n+1)}\,P_{[ij]} - g_{ij}\,P^h_h\right] \\ &- \quad \frac{1}{(n-1)}\,\delta^h_j\,\left[P_{ik} - \frac{2}{(n+1)}\,P_{[ik]} - g_{ik}\,P^h_h\right] \\ &- \quad g_{ik}\,g^{hl}\,P_{lj} + g_{ij}\,g^{hl}\,P_{lk}. \end{aligned} \tag{3.27}$$

Suppose that $\tilde{W}^h_{ijk} = W^h_{ijk}$. Then we have $P^h_{ijk} = 0$. Contraction on h and i in (3.27) gives that

$$P_{[kj]} = 0, \tag{3.28}$$

or P_k is a gradient. Then we have

Theorem 3.2. *Let $\tau : W_n(g,T) \to \tilde{W}_n(\tilde{g},\tilde{T})$ be a conformal mapping preserving the concircular curvature tensor or the projective curvature tensor of the Weyl manifold $W_n(g,T)$ then the covector field P is a gradient.*

Assume that τ be a conformal mapping preserving the Einstein tensor. Then we have

$$P_{(ij)} = \frac{g_{ij}\,(\tilde{R} - R)}{2n(n-1)}. \tag{3.29}$$

Under this condition X^h_{ijk} and P^h_{ijk} reduce to

$$\begin{aligned} X^h_{ijk} \quad = \quad & 2\,\delta^h_i\,P_{[kj]} + \delta^h_k\,P_{[ij]} - \delta^h_j\,P_{[ik]} \\ & - g_{ik}\,g^{hm}\,P_{[mj]} + g_{ij}\,g^{hm}\,P_{[mk]} \\ & - \frac{2}{n}\,g^{mh}\,P_{[mh]}\,(\delta^h_k\,g_{ij} - \delta^h_j\,g_{ik}) \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} P^h_{ijk} \quad = \quad & \frac{2}{(n+1)}\,\delta^h_i\,P_{[kj]} + \frac{1}{(n-1)}\,\left[\delta^h_k\,P_{[ij]} - \delta^h_j\,P_{[ik]}\right] \\ & - g_{ik}\,g^{hl}\,P_{[lj]} + g_{ij}\,g^{hl}\,P_{lk}, \end{aligned} \tag{3.31}$$

respectively.

It can be easily seen that, condition (3.28) implies that

$$X^h_{ijk} = 0 \quad and \quad P^h_{ijk} = 0 \tag{3.32}$$

or, we have

$$\tilde{Z}^h_{ijk} = Z^h_{ijk} \quad and \quad \tilde{W}^h_{ijk} = W^h_{ijk} \tag{3.33}$$

Then, we can state the following theorem

Theorem 3.3. *Let $\tau : W_n(g, T) \to \tilde{W}_n(\tilde{g}, \tilde{T})$ be a conformal transformation preserving the Einstein tensor. Then the following cases are equivalent:*

(1) *The concircular curvature tensor is an invariant.*
(2) *The covector field of the mapping is a locally gradient.*
(3) *The projective curvature tensor is an invariant.*

Corollary 3.4. *Let $\tau : W_n(g, T) \to \tilde{W}_n(\tilde{g}, \tilde{T})$, be a conformal mapping preserving the Einstein tensor of the Weyl manifold $W_n(g, T)$. If the concircular or the projective curvature tensors are preserved by τ, then the scalar curvatures R and $\tilde{R}$ of the Weyl manifolds $W_n(g, T)$ and $\tilde{W}_n(\tilde{g}, \tilde{T})$ are related by*

$$\begin{aligned} \tilde{R} = R \quad + \quad & 2(n-1)[\Delta f + \frac{(n-2)}{2}(|\nabla f|^2 \\ & -2\, g(T, \nabla f))], \quad (n > 2) \end{aligned} \tag{3.34}$$

where $f \in C^2(W_n)$ and $|\nabla f|$ denotes the length of ∇f and Δf is the Laplacian of f.

Proof. Suppose that (3.33) holds. According to Theorems 3.2 and 3.3, P is a gradient. Then we have

$$P = \nabla f \tag{3.35}$$

for any scalar $f \in C^2(W_n)$.

Transvection (3.5) with g^{ij} gives

$$P_{ij}\, g^{ij} = (\dot{\nabla}_j\, P_i)\, g^{ij} - P_i\, P_j\, g^{ij} + \frac{n}{2}\, g^{mh}\, P_m\, P_h. \tag{3.36}$$

By using (3.20) and setting

$$\begin{aligned} g^{ij}\, \dot{\nabla}_j\, P_i &= \dot{\nabla}_j\, P^j \\ &= \nabla_j\, P^j + 2\, T_j\, P^j \\ &= P^j_j + (2-n)\, T_k\, P^k \\ &= \nabla f - (n-2)\, g(T, \nabla f) \end{aligned} \tag{3.37}$$

we obtain

$$\frac{\tilde{R}-R}{2(n-1)}=\nabla f+\frac{(n-2)}{2}\,[|\nabla f|^2-2\,g(T,\nabla f)], \tag{3.38}$$

where $|\nabla f|=g^{ij}\,P_i\,P_j$ and $\dot{\nabla}_j\,P^j$ denote the length of P and the generalized divergence of P^j, respectively. □

Suppose that the Weyl space $W_n(g,T)$ and the flat Weyl space $\tilde{W}_n(\tilde{g},\tilde{T})$ are related by a conformal mapping preserving the Einstein tensor. Then, we have

$$\tilde{R}^h_{ijk}=0, \tag{3.39}$$

which implies that

$$\tilde{R}_{ij}=0, \quad \tilde{R}=0, \quad \tilde{E}_{ij}=0. \tag{3.40}$$

Since $E_{ij}=\tilde{E}_{ij}$,

$$E_{ij}=0. \tag{3.41}$$

Similarly, if the flat Weyl space $W_n(g,T)$ changes to the Weyl space $\tilde{W}_n(\tilde{g},\tilde{T})$ by a conformal mapping preserving the Einstein tensor, $\tilde{E}_{ij}$ becomes zero.

Thus, we get that a Weyl manifold and a flat Weyl manifold, which are conformal correspondent preserving the Einstein tensor are Einstein-Weyl manifold. So, we proved that

Theorem 3.5. *If the conformal mapping of the Weyl manifold $W_n(g,T)$ onto a flat Weyl manifold $\tilde{W}_n(\tilde{g},\tilde{T})$ preserves the Einstein tensor. Then both Weyl manifolds are Einstein-Weyl manifolds.*

4. **Isotropic Weyl manifolds**

Let p be any point of $W_n(g,T)$ and $T_p(W_n)$ be the tangent space of W_n. The scalar defined by [6]

$$K(\Pi)=\frac{R_{ijkl}X^iY^jX^kY^l}{(g_{ik}g_{jl}-g_{il}g_{jk})X^iY^jX^kY^l}, \tag{4.1}$$

is called the sectional curvature of $W_n(g,T)$ at p with respect to the plane spanned by two linearly independent vectors $X,Y\in T_p(W_n)$, where X^i and Y^i are the components of X and Y [9].

If at each point, the sectional curvature K of $W_n(g,T)$ is independent of the 2-plane chosen, then, $W_n(g,T)$ is named as an isotropic manifold respectively [6].

Lemma 4.1. *Suppose that S is any 4-covariant tensor and that X and Y are two arbitrary linearly independent vectors. If, for all X and Y*

$$S_{ijlk}X^iY^jX^kY^l = 0, \tag{4.2}$$

then

$$S_{ijkl} + S_{klij} + S_{ilkj} + S_{kjil} = 0, \tag{4.3}$$

where X^i and Y^j are respectively the components of X and Y [2, 6].

Lemma 4.2. *An isotropic Weyl manifold is an Einstein-Weyl manifold with zero scalar curvature.*

Proof. Suppose that $W_n(g,T)$ is an isotropic Weyl manifold, then S_{ijkl} can be defined as,

$$S_{ijkl} = R_{ijkl} - R(g_{ik}g_{jk} - g_{il}g_{jk}). \tag{4.4}$$

By considering S_{ijkl}, S_{klij}, S_{ilkj} and S_{kjil} and using Lemma 4.1, we get

$$\begin{aligned} &R_{ijkl} + R_{klij} + R_{ilkj} + R_{kjil} \\ &R(4g_{ik}g_{lj} - 2g_{il}g_{jk} - 2g_{ij}g_{lk}) = 0. \end{aligned} \tag{4.5}$$

Transvecting (4.5) by g^{ik} and then using the property $R_{ijkl} = -R_{ijlk}$ of R_{ijkl} we obtain

$$2(R_{jl} + R_{lj}) + 4(n-1)g_{jl}R = 0, \tag{4.6}$$

which implies that

$$R_{(jl)} = \lambda\, g_{jl}, \tag{4.7}$$

where $\lambda = (1-n)R$.

Hence the symmetric part of the Ricci tensor is proportional to the conformal metric tensor g, $W_n(g,T)$ is an Einstein-Weyl manifold and by virtue of (4.7), we have

$$R = 0 \tag{4.8}$$

which completes the proof.

□

From Lemma 4.2 it is clear that, for an isotropic Weyl manifold the Einstein tensor

$$E_{ij} = 0. \tag{4.9}$$

Thus, we have

Theorem 4.3. *If the Weyl manifold $W_n(g,T)$ is locally conformal to the isotropic Weyl manifold $\tilde{W}_n(\tilde{g},\tilde{T})$ under the mapping preserving the Einstein tensor, then both manifolds are Einstein-Weyl manifolds.*

Combining Corollary 3.4 and Theorem 4.3 we can state the following corollary.

Corollary 4.4. *Let $\tau : W_n(g,T) \to \tilde{W}_n(\tilde{g},\tilde{T})$, be a conformal mapping preserving the Einstein tensor of the Weyl manifold $W_n(g,T)$. If the concircular or the projective curvature tensors are preserved by τ then the covector field P of the mapping satisfies the following differential equation for any scalar function $f \in C^2(W_n)$*

$$\nabla f + \frac{(n-2)}{2}\left[|\nabla f|^2 - 2\, g(T, \nabla f)\right] = 0. \tag{4.10}$$

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